J. Appl. Prob. **43,** 634–651 (2006)
Printed in Israel

DERIVATIVES PRICING VIA p-OPTIMAL MARTINGALE MEASURES: SOME EXTREME CASES

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Abstract

In an incomplete financial market in which the dynamics of the asset prices is driven by a d-dimensional continuous semimartingale X, we consider the problem of pricing European contingent claims embedded in a power utility framework. This problem reduces to identifying the p-optimal martingale measure, which can be given in terms of the solution to a semimartingale backward equation. We use this characterization to examine two extreme cases. In particular, we find a necessary and sufficient condition, written in terms of the mean–variance trade-off, for the p-optimal martingale measure to coincide with the minimal martingale measure. Moreover, if and only if an exponential function of the mean–variance trade-off is a martingale strongly orthogonal to the asset price process, the p-optimal martingale measure can be simply expressed in terms of a Doléans-Dade exponential involving X.

Keywords: Semimartingale backward equation; p-optimal martingale measure; incomplete market; diffusion

2000 Mathematics Subject Classification: Primary 91B28
Secondary 60H30; 90C39

1. Introduction and main results

We consider the problem of evaluating a derivative in a market in which the dynamics of the discounted prices of traded assets is described by an $\mathbb{R}^d$-valued continuous semimartingale $X = (X_t,\ t \in [0, T])$. We assume all processes to be defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, equipped with a filtration $F = (F_t,\ t \in [0, T])$ satisfying the usual conditions, where $\mathcal{F} = F_T$ and $T < \infty$ is a fixed time horizon. The filtration represents the information flows available to the agents operating in the market and any European contingent claim will be a random value η that will be observed only at the exercise time T. To avoid arbitrage in the market, the price process X has to satisfy the structure condition; this means that it admits the decomposition

$$X_t = X_0 + \int_0^t \mathrm{d}\langle M\rangle_s \lambda_s + M_t, \tag{1}$$

where M is a continuous local martingale and λ a predictable, $\mathbb{R}^d$-valued process such that $K_T = \int_0^T \lambda_s^\top \,\mathrm{d}\langle M\rangle_s \lambda_s < \infty$ almost surely (a.s.), '$^\top$' denoting transposition. The so-called mean–variance trade-off process of X, $K = \int_0 \lambda_s^\top \,\mathrm{d}\langle M\rangle_s \lambda_s$, turns out to be the key quantity: it can be seen as the integrated squared market price of risk associated with X and, for instance, in the well-known Black–Scholes model with drift b, volatility σ, and riskless interest rate r it coincides with $K_t = ((b - r)/\sigma)^2 t$. Furthermore, the process K measures the extent to which the price process deviates from being a martingale. In fact, the process X, admitting the

Received 19 October 2005; revision received 2 May 2006.
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decomposition (1), is a martingale if and only if $K_T = 0$, a.s. (see Schweizer (1994) for further details).

The question is: How does an agent evaluate the random pay-off η? An 'admissible' price can be the maximum value at which the agent agrees to buy or the minimum value at which she agrees to sell, completely hedging the risk. The existence of such prices of general contingent claims (and their dynamical counterpart) was proved in El Karoui and Quenez (1995), under the hypothesis of Brownian dynamics of the price process. In this paper it is shown that the maximal price of a contingent claim at time t can be expressed as a portfolio with an initial value (the initial wealth of an investor) plus a stochastic integral with respect to X, which models the gains or losses that she accumulates up to time t, and an increasing optional process modelling her consumption. In Kramkov (1996), this representation, which is not obvious, was named 'the optional decomposition theorem' and extended to a rather general setting in which the underlying assets are locally bounded semimartingales.

This approach to pricing leads us to determine an interval of arbitrage-free prices; in the range of admissible prices the choice will depend on the risk aversion or utility preference.

The most significant contribution since the 1970s is the well-known Black–Scholes theory of pricing, which provides the price as the expected value of the random pay-off with respect to the unique martingale measure, i.e. a probability measure equivalent to P on F_T and under which X is a martingale. This price is independent of agents' preferences but the theory's main arguments, i.e. replicating claims and no arbitrage, are in general not realistic.

In fact, in general, markets are incomplete: mathematically, this corresponds to the fact that the martingale measure is no longer unique. Instead, we have a set, $\mathcal{M}^{\mathrm{e}}$, of probability measures Q equivalent to P such that X is a Q-local martingale. Hence, any martingale measure leads to a different arbitrage-free contingent claim price.

In this paper we concentrate on p-optimal martingale measures. These measures include the variance-optimal martingale measure, which corresponds to $p = 2$, and the minimal-entropy martingale measure, which arises as the limit as p tends to 1 (Grandits and Rheinländer (2002), Santacroce (2005)). While the variance-optimal martingale measure plays a role in determining the mean–variance hedging strategy, the latter is used to solve problems of exponential hedging. In general, p-optimal martingale measures are related to power-law utility maximization problems (see, e.g. Goll and Rüschendorf (2001) and Frittelli (2000)). The case $p < 0$ corresponds to standard utility functions: functions strictly increasing, concave, and defined on $\mathbb{R}^+$. Nevertheless, the choice of p-optimal martingale measures makes sense for any $p \neq 0, 1$. Note that the case $p < 1$ ($p \neq 0$) can be studied as in Santacroce (2005).

For any measure Q, let Z_t^{Q} be the density process of Q relative to the basic measure P. For any $\mathrm{Q} \in \mathcal{M}^{\mathrm{e}}$, there is a P-local martingale M^{Q} such that

$$Z^{\mathrm{Q}} = \mathcal{E}(M^{\mathrm{Q}}) = (\mathcal{E}_t(M^{\mathrm{Q}}),\ t \in [0, T]),$$

where $\mathcal{E}(M)$ is the Doléans-Dade exponential of M. If the local martingale

$$\hat{Z} = (\mathcal{E}_t(-\lambda \cdot M),\ t \in [0, T])$$

is a strictly positive martingale, then $\mathrm{dQ}^{\min} / \mathrm{dP} = \hat{Z}_T$ defines an equivalent probability measure, $\mathrm{Q}^{\min}$, called the minimal martingale measure for X. We use the notation $\lambda \cdot M$ to denote the stochastic integral with respect to M. Let

$$\mathcal{M}_p^{\mathrm{e}} = \left\{\mathrm{Q} \in \mathcal{M}^{\mathrm{e}} \colon \mathrm{E}\left(\frac{\mathrm{dQ}}{\mathrm{dP}}\right)^p < \infty\right\}, \qquad p > 1.$$

https://doi.org/10.1239/jap/1158784935 Published online by Cambridge University Press

Throughout the paper, we make the following assumptions.

Assumption 1. *The minimal martingale measure* $\mathrm{Q}^{\min}$ *exists and satisfies the reverse Hölder inequality* $R_p(\mathrm{P})$, *i.e. there is a constant* C *such that*

$$\mathrm{E}(\mathcal{E}^p_{\tau T}(-\lambda \cdot M) \mid F_\tau) \le C$$

for any stopping time τ. *We use the notation*

$$\mathcal{E}_{\tau T}(M) = \frac{\mathcal{E}_T(M)}{\mathcal{E}_\tau(M)} = \mathcal{E}_T(M - M_{\cdot\wedge\tau}).$$

Assumption 2. *All* P*-local martingales are continuous.*

Note both that Assumption 1 implies the existence of an equivalent martingale measure ($\mathcal{M}^{\mathrm{e}}_p$ is not empty) and that, since X is continuous, the structure condition is automatically satisfied. When X is continuous and $\mathcal{M}^{\mathrm{e}}_p$ is not empty, Grandits and Krawczyk (1998) showed that the p-optimal martingale measure is equivalent to P; for $p = 2$ this fact was previously proved by Delbaen and Schachermayer (1996).

Therefore, we consider the optimization problem

$$\min_{\mathrm{Q}\in\mathcal{M}^{\mathrm{e}}_p} \mathrm{E}(\mathcal{E}^p_T(M^{\mathrm{Q}})), \qquad p > 1,$$

and define the related value process as

$$V_t = \operatorname*{ess\,inf}_{\mathrm{Q}\in\mathcal{M}^{\mathrm{e}}_p} \mathrm{E}(\mathcal{E}^p_{tT}(M^{\mathrm{Q}}) \mid F_t), \qquad p > 1. \tag{2}$$

As stated before, for $p = 2$ the solution to the problem leads to the variance-optimal martingale measure (see, e.g. Delbaen and Schachermayer (1996), Schweizer (1996), Gourieroux *et al.* (1998), Laurent and Pham (1999), and Pham *et al.* (1998)). In Laurent and Pham (1999), a characterization of the variance-optimal martingale measure was given in terms of the value function related to a stochastic control problem in the case of Brownian filtration. In Mania *et al.* (2002), a description of the p-optimal martingale measure was provided in terms of the value process (2), and it was shown that V uniquely solves a semimartingale backward equation. Moreover, Pham *et al.* (1998) stated a sufficient condition under which the minimal martingale measure and the variance-optimal martingale measure coincide, and observed that this condition typically fails if the mean–variance trade-off is not deterministic and includes exogenous randomness not induced by X. Laurent and Pham (1999) explicitly characterized the variance-optimal martingale measure in two opposite cases. In the first case the so-called market price of risk does not depend on the exogenous randomness, which is represented in their model by an untraded asset price process Y. In the opposite case the market price of risk does not depend on the asset price process X. In their paper, Biagini *et al.* (2000) came to similar conclusions for the variance-optimal martingale measure by exploiting their equation involving exponential martingales.

The main contributions of this work are described in Theorems 1 and 2 and represent a generalization to the semimartingale setting of the results obtained by Pham *et al.* (1998), Laurent and Pham (1999), and Biagini *et al.* (2000). In particular, we give necessary and sufficient conditions under which the p-optimal martingale measure can be expressed in two specific forms, which in some sense represent two extreme cases. It is worth remarking that in the papers quoted above the respective authors supplied sufficient conditions, whereas we give necessary and sufficient conditions for both cases.

We find that, under Assumptions 1 and 2, the p-optimal martingale measure coincides with the minimal martingale measure if and only if

$$\exp\left\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_T\right\} = c + \int_0^T \phi_s^\top \,\mathrm{d}X_s(p),$$

where c is a constant, $X(p) = M + p\langle M\rangle\cdot\lambda$, and ϕ is a $X(p)$-integrable process such that $(c + \int_0^t \phi_s^\top \,\mathrm{d}X_s(p),\ t\in[0,T])$ is a martingale with respect to the measure $\mathrm{Q}^{p\ \min}$, whose density, $\mathcal{E}_T(M^{\mathrm{Q}^{p\ \min}})$, on F_T is defined by $\mathrm{d}(\mathrm{Q}^{p\ \min}) = \mathcal{E}_T(-p\lambda\cdot M)\,\mathrm{dP}$. Moreover, still under Assumptions 1 and 2, the p-optimal martingale measure, Q^*, satisfies

$$\mathcal{E}_T^{p-1}(M^{\mathrm{Q}^*}) = V_0\mathcal{E}_T((1-p)\lambda\cdot X)$$

if and only if

$$\exp\left\{-\frac{p}{2}\langle\lambda\cdot M\rangle_T\right\} = c + \hat{m}_T,$$

where $\hat{m}$ is a martingale strongly orthogonal to M, i.e. $\langle\hat{m}, M\rangle = 0$.

In Section 2 the main results are presented in detail, while the diffusion case and the stochastic volatility models are studied in Section 3 and Section 4, respectively. The results given in this paper rely heavily upon the theory of backward stochastic differential equations. Here we just mention that they were introduced by Bismut (1973) for the linear case, and by Chitashvili (1983) and Pardoux and Peng (1990) for more general generators.

For all unexplained notation concerning martingale theory we refer to Jacod (1979), Dellacherie and Meyer (1980), and Liptser and Shiryaev (1989). For information about *BMO*-martingales and reverse Hölder conditions, see Doléans-Dade and Meyer (1979) or Kazamaki (1994).

2. Backward semimartingale equation for the value process

We now recall the definition of *BMO*-martingales and the reverse Hölder condition.

The square-integrable, continuous martingale M belongs to the class *BMO* if there is a constant $C > 0$ such that

$$\mathrm{E}^{1/2}(\langle M\rangle_T - \langle M\rangle_\tau \mid F_\tau) \le C \quad \mathrm{P}\text{-a.s.}$$

for every stopping time τ. The *BMO* norm of M, denoted by $\|M\|_{BMO}$, is the smallest constant with this property (and takes the value $+\infty$ if no such constant exists).

A strictly positive adapted process Z satisfies the reverse Hölder inequality $R_p(\mathrm{P})$, $1 \le p < \infty$, if there is a constant C such that

$$\mathrm{E}\left(\left(\frac{Z_T}{Z_\tau}\right)^p \,\middle|\, F_\tau\right) \le C \quad \mathrm{P}\text{-a.s.}$$

for every stopping time τ.

We remark that, since X is continuous, any element Q of $\mathcal{M}^{\mathrm{e}}$ is given by the density $Z_t(\mathrm{Q})$, which is expressed as an exponential martingale of the form

$$\mathcal{E}_t(-\lambda\cdot M + N),$$

https://doi.org/10.1239/jap/1158784935 Published online by Cambridge University Press

where N is a local martingale strongly orthogonal to M. We observe that the problem of finding the p-optimal martingale measure is in fact the problem of identifying the optimal martingale, N^*, in a proper subclass of local martingales N such that the corresponding martingale measure is in $\mathcal{M}_p^{\mathrm{e}}$.

Since $\mathcal{M}_p^{\mathrm{e}} \neq \varnothing$, the process V is a special semimartingale with respect to the measure P. Let

$$V_t = V_0 + m_t + A_t, \qquad m \in M_{\mathrm{loc}}^2,\ A \in \mathcal{A}_{\mathrm{loc}}, \tag{3}$$

where M_{loc}^2 denotes the set of locally square-integrable martingales and $\mathcal{A}_{\mathrm{loc}}$ the set of processes of locally bounded variation, be the canonical decomposition of V, and let

$$m_t = \int_0^t \phi_s^\top \,\mathrm{d}M_s + \tilde{m}_t, \qquad \langle \tilde{m}, M\rangle = 0, \tag{4}$$

be the Galtchouk–Kunita–Watanabe decomposition of m with respect to M. Any P-special semimartingale Y admits a decomposition (similar to (3) and (4))

$$Y_t = Y_0 + L_t + B_t \quad \text{with } L_t = \int_0^t \psi_s^\top \,\mathrm{d}M_s + \tilde{L}_t,$$

where $\tilde{L}, L \in M_{\mathrm{loc}}^2$, $\langle \tilde{L}, M\rangle = 0$, and $B \in \mathcal{A}_{\mathrm{loc}}$ is predictable.

Here, we restate the main result of Mania *et al.* (2002) (namely their Theorem 1(b) and Corollary 2) in the form suitable for our purposes in which by the use of dynamic programming techniques the value process V is characterized as the unique solution to a semimartingale backward equation.

Proposition 1. *If Assumptions 1 and 2 hold, then the value process V is the unique solution to the semimartingale backward equation*

$$\begin{aligned} Y_t = Y_0 &- \frac{p(p-1)}{2}\int_0^t Y_s \,\mathrm{d}\langle \lambda\cdot M\rangle_s + p\langle \lambda\cdot M, \psi\cdot M\rangle_t \\ &+ \frac{p}{2(p-1)}\int_0^t \frac{1}{Y_s}\,\mathrm{d}\langle \tilde{L}\rangle_s + \int_0^t \psi_s^\top \,\mathrm{d}M_s + \tilde{L}_t, \qquad t < T, \end{aligned} \tag{5}$$

with the boundary condition

$$Y_T = 1, \tag{6}$$

in the class of semimartingales Y satisfying the two-sided inequality

$$c \le Y_t \le C \quad \textit{for all } t \in [0, T], \textit{a.s.}, \tag{7}$$

for some constants $c < 1$ and $C > 1$.

Moreover, the martingale measure Q^* is p-optimal if and only if it is given by the density $\mathrm{dQ}^* = \mathcal{E}_T(M^{\mathrm{Q}^*})\,\mathrm{dP}$, *where*

$$M_t^{\mathrm{Q}^*} = -\int_0^t \lambda_s^\top \,\mathrm{d}M_s - \frac{1}{p-1}\int_0^t \frac{1}{V_s}\,\mathrm{d}\tilde{m}_s. \tag{8}$$

Corollary 1. *The martingale measure Q^* is p-optimal if and only if its density $Z^* = \mathcal{E}_T(M^{\mathrm{Q}^*})$ is expressed as*

$$\mathcal{E}_T^{p-1}(M^{\mathrm{Q}^*}) = V_0\biggl(1 + \int_0^T \mathcal{E}_s\biggl(\biggl[\frac{\psi}{V} + (1-p)\lambda\biggr]\cdot X\biggr)\biggl[\frac{\psi_s^\top}{V_s} + (1-p)\lambda_s^\top\biggr]\mathrm{d}X_s\biggr).$$

Let us introduce the process $X(p) = M + p\langle M\rangle \cdot \lambda$. It is not difficult to verify that Proposition 1 can be formulated in exponential form as follows.

Proposition 2. *Equations (5) and (6) are equivalent to the equation*

$$\frac{\mathcal{E}_T(\bar{\psi}\cdot X(p))}{\mathcal{E}_T^{p-1}(\bar{L})} = \bar{c}\exp\left\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_T\right\}, \tag{9}$$

i.e. if Y is a solution to (5) and (6), then the triple $(\bar{c}, \bar{\psi}, \bar{L})$, where

$$\bar{c} = \frac{1}{Y_0}, \qquad \bar{\psi} = \frac{\psi}{Y}, \qquad \bar{L} = -\frac{1}{p-1}\int_0^t \frac{1}{Y_s}\,\mathrm{d}\tilde{L}_s,$$

will be a solution to (9). Conversely, if $(\bar{c}, \bar{\psi}, \bar{L})$ solves (9), then Y defined by

$$Y_t = \frac{1}{\bar{c}}\exp\left\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_t\right\}\mathcal{E}_t(\bar{\psi}\cdot M + p\langle\lambda\cdot M, \overline{\psi}\cdot M\rangle)\mathcal{E}_t^{1-p}(\bar{L})$$

satisfies (5) and (6).

Remark 1. Note that, for $p = 2$, (9) coincides with Equation (2.2) of Biagini *et al.* (2000).

Let us call $\mathrm{Q}^{p\ \min}$ the measure whose density $\mathcal{E}(M^{\mathrm{Q}^{p\ \min}})$ is defined by $\mathrm{d}(\mathrm{Q}^{p\ \min}) = \mathcal{E}_T(-p\lambda\cdot M)\,\mathrm{dP}$. It is evident that, for any $p > 1$, we have

$$\mathcal{E}_t(-p\lambda\cdot M) \le \mathrm{E}(\mathcal{E}_T^p(-\lambda\cdot M) \mid F_t).$$

Therefore, if $\mathrm{Q}^{\min}$ is in $\mathcal{M}_p^{\mathrm{e}}$ then $\mathrm{Q}^{p\ \min}$ is a probability measure equivalent to P. In fact, $\mathcal{E}_t(-p\lambda\cdot M) > 0$ and

$$\mathrm{E}(\mathcal{E}_T^p(-\lambda\cdot M)) < \infty,$$

which implies that $(\mathcal{E}_t(-p\lambda\cdot M),\ t\in[0,T])$ is a uniformly integrable martingale, since it is bounded from above by a uniformly integrable martingale.

In the next two theorems we use the previous characterization of the value process to examine two opposite cases. In each we give a necessary and sufficient condition for the p-optimal martingale measure to assume two special forms.

Theorem 1. *Let Assumptions 1 and 2 hold. Then the minimal martingale measure is p-optimal if and only if*

$$\exp\left\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_T\right\} = c + \int_0^T \varphi_s^\top\,\mathrm{d}X_s(p) \tag{10}$$

for an $X(p)$-integrable process φ, such that the process $(c + \int_0^t \varphi_s^\top\,\mathrm{d}X_s(p),\ t\in[0,T])$ is a $\mathrm{Q}^{p\ \min}$-martingale, and a constant c.

Proof. Let us start by proving the sufficiency. We define the process Y by

$$Y_t = \mathrm{E}(\mathcal{E}_{tT}^p(-\lambda\cdot M) \mid F_t) = \mathrm{E}^{\mathrm{Q}^{p\ \min}}\left(\exp\left\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_{tT}\right\}\,\middle|\, F_t\right).$$

The above-defined process Y satisfies the two-sided inequality $1 \le Y_t \le C$; one side follows from Jensen's inequality and the other from Assumption 1, i.e. because the minimal martingale measure satisfies the reverse Hölder inequality $R_p(\mathrm{P})$. It is easy to see that Y can be represented as the product of a decreasing process J and a $\mathrm{Q}^{p\ \min}$-martingale $\bar{\bar{m}}$:

$$Y_t = J_t \bar{\bar{m}}_t = \exp\left\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_t\right\} \mathrm{E}^{\mathrm{Q}^{p\ \min}}\left(\exp\left\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_T\right\} \,\middle|\, F_t\right).$$

Keeping (10) in mind, we have

$$Y_t = \exp\left\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_t\right\}\left(c + \int_0^t \varphi_s^\top \,\mathrm{d}X_s(p)\right).$$

If we write the Itô formula for the product $J_t\bar{\bar{m}}_t$, we find that

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \exp\left\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_s\right\}\varphi_s^\top \,\mathrm{d}X_s(p) - \frac{p(p-1)}{2}\int_0^t Y_s \,\mathrm{d}\langle\lambda\cdot M\rangle_s \\ &= Y_0 + \int_0^t \psi_s^\top \,\mathrm{d}M_s + p\langle\psi\cdot M, \lambda\cdot M\rangle_t - \frac{p(p-1)}{2}\int_0^t Y_s \,\mathrm{d}\langle\lambda\cdot M\rangle_s, \qquad (11) \end{aligned}$$

where we denote $\exp\{-[p(p-1)/2]\langle\lambda\cdot M\rangle\}\varphi$ by ψ. Observe that (11) coincides with (5) when $\tilde{L} = 0$ and, moreover, that $1 \le Y_t \le C$ and $Y_T = 1$. Thus, Y is a solution to (5) satisfying the two-sided inequality (7). Proposition 1 implies that $Y = V$ and, therefore, that $\mathrm{Q}^* = \mathrm{Q}^{\min}$.

On the other hand, if the p-optimal martingale measure coincides with the minimal martingale measure, from (8) we have

$$\bar{m}_t = -\frac{1}{p-1}\int_0^t \frac{1}{V_s}\,\mathrm{d}\tilde{m}_s = 0$$

and, from (9), we have

$$\exp\left\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_T\right\} = V_0\mathcal{E}_T(\bar{\psi}\cdot X(p)) = c + \int_0^T \varphi_s^\top \,\mathrm{d}X_s(p),$$

where $c = V_0$ and $\varphi = V_0\mathcal{E}(\bar{\psi}\cdot X(p))\bar{\psi}$. Thus, (10) holds.

We will show now that the process $(c + \int_0^t \varphi_s^\top \,\mathrm{d}X_s(p),\ t\in[0,T])$ is a $\mathrm{Q}^{p\ \min}$-martingale or, equivalently, that $(\mathcal{E}_t(\bar{\psi}\cdot X(p)),\ t\in[0,T])$ is a $\mathrm{Q}^{p\ \min}$-martingale.

According to Theorem 2.3 of Kazamaki (1994), it will be enough to prove that $\bar{\psi}\cdot X(p)$ is in $BMO(\mathrm{Q}^{p\ \min})$. For this purpose, it is sufficient that $\bar{\psi}\cdot M$ is in $BMO(\mathrm{P})$. In fact, from Theorem 3.6 of Kazamaki (1994), if $M^{\mathrm{Q}^{p\ \min}}$ is in $BMO(\mathrm{P})$, as in our case, then the Girsanov transformation (see, e.g. Kazamaki (1994, paragraph 3.3)) is an isomorphism of $BMO(\mathrm{P})$ onto $BMO(\mathrm{Q}^{p\ \min})$. We should see now that $M^{\mathrm{Q}^{p\ \min}}$ and $\bar{\psi}\cdot M$ are in $BMO(\mathrm{P})$; we recall that $\bar{\psi} = \psi/Y$ and that $Y \ge 1$. On the one hand, we have

$$\begin{aligned} \mathrm{E}(\langle M^{\mathrm{Q}^{p\ \min}}\rangle_T - \langle M^{\mathrm{Q}^{p\ \min}}\rangle_\tau \mid F_\tau) &= p^2\,\mathrm{E}(\langle\lambda\cdot M\rangle_T - \langle\lambda\cdot M\rangle_\tau \mid F_\tau) \\ &\le p^2\|\lambda\cdot M\|_{BMO}^2 \qquad (12) \end{aligned}$$

and, since, by Assumption 1, the minimal martingale measure satisfies the reverse Hölder inequality $R_p(\mathrm{P})$, the minimal martingale measure is, by Theorem 3.4 of Kazamaki (1994), in $BMO(\mathrm{P})$. On the other hand,

$$\begin{aligned} \mathrm{E}(\langle\bar{\psi}\cdot M\rangle_T - \langle\bar{\psi}\cdot M\rangle_\tau \mid F_\tau) &\le \mathrm{E}(\langle\psi\cdot M\rangle_T - \langle\psi\cdot M\rangle_\tau \mid F_\tau) \\ &\le \|L\|_{BMO}^2 \qquad (13) \end{aligned}$$

and, as has been shown in the proof of Proposition 1 (Theorem 1(b) of Mania *et al.* (2002)), L is actually in $BMO(\mathrm{P})$. Thus, (12) and (13) complete the proof.

Theorem 2. *Let Assumptions 1 and 2 hold. Then the p-optimal martingale measure satisfies*

$$\mathcal{E}_T^{p-1}(M^{\mathrm{Q}^*}) = V_0 \mathcal{E}_T((1-p)\lambda \cdot X) \tag{14}$$

if and only if

$$\exp\left\{-\frac{p}{2}\langle \lambda \cdot M\rangle_T\right\} = c + \hat{m}_T, \tag{15}$$

where $\hat{m}$ is a martingale strongly orthogonal to M, i.e. such that $\langle \hat{m}, M\rangle = 0$.

Proof. Let the p-optimal martingale measure satisfy (14). From (8) and (14) we have

$$\mathcal{E}_T^{p-1}(-\lambda \cdot M)\mathcal{E}_T^{p-1}(\bar{m}) = V_0 \mathcal{E}_T((1-p)\lambda \cdot X), \tag{16}$$

where, recall, $\bar{m} = -[1/(p-1)]\int_0^t (1/V_s)\,\mathrm{d}\tilde{m}_s$. Since

$$\mathcal{E}_T^{p-1}(-\lambda \cdot M) = \mathcal{E}_T((1-p)\lambda \cdot X)\exp\left\{\frac{p(p-1)}{2}\langle \lambda \cdot M\rangle_T\right\},$$

from (16), it follows that

$$\exp\left\{-\frac{p(p-1)}{2}\langle \lambda \cdot M\rangle_T\right\} = \frac{1}{V_0}\mathcal{E}_T^{p-1}(\bar{m}). \tag{17}$$

Now, by taking both sides of (17) to the power $1/(p-1)$, we obtain

$$\exp\left\{-\frac{p}{2}\langle \lambda \cdot M\rangle_T\right\} = c\mathcal{E}_T(\bar{m}) = c + \hat{m}_T \tag{18}$$

with $c = 1/(V_0)^{1/(p-1)}$. Since $\bar{m} \in BMO$ (see the proof of Theorem 1 of Mania *et al.* (2002)), the processes $\mathcal{E}_t(\bar{m})$ and, hence, $\hat{m}_t = c\int_0^t \mathcal{E}_s(\bar{m})\,\mathrm{d}\bar{m}_s$ are martingales. From (18) it should now be clear that (15) holds.

Let us now prove the converse. Consider the martingale $Z_t = \mathrm{E}(Z_T \mid F_t)$, where $Z_T = \bar{c}\mathcal{E}_T(-\lambda \cdot M - (p/2)\langle \lambda \cdot M\rangle)$, i.e. $Z_T^{p-1} = \bar{c}^{(p-1)}\mathcal{E}_T((1-p)\lambda \cdot X)$. Therefore, Z_t has the following expression, where the last equality is due to (15):

$$\begin{aligned} Z_t &= \bar{c}\mathcal{E}_t(-\lambda \cdot M)\,\mathrm{E}\left(\mathcal{E}_{tT}(-\lambda \cdot M)\exp\left\{-\frac{p}{2}\langle \lambda \cdot M\rangle_T\right\} \,\middle|\, F_t\right) \\ &= \bar{c}\mathcal{E}_t(-\lambda \cdot M)E^{\mathrm{Q}^{\min}}\left(\exp\left\{-\frac{p}{2}\langle \lambda \cdot M\rangle_T\right\} \,\middle|\, F_t\right) \\ &= \bar{c}\mathcal{E}_t(-\lambda \cdot M)(c + \hat{m}_t). \end{aligned}$$

In fact, since $\hat{m}$ is a P-martingale strongly orthogonal to M (i.e. $\langle \hat{m}, M\rangle = 0$), it follows from Girsanov's theorem that $\hat{m}$ is a $\mathrm{Q}^{\min}$-martingale.

It is easy to see that

$$\frac{Z_T}{Z_t} = \mathcal{E}_{tT}(-\lambda \cdot M)\frac{\exp\{-(p/2)\langle \lambda \cdot M\rangle_{tT}\}}{\mathrm{E}(\exp\{-(p/2)\langle \lambda \cdot M\rangle_{tT}\} \mid F_t)}.$$

Now let us define the process Y, via $Y_t = \mathrm{E}((Z_T/Z_t)^p \mid F_t)$. We will show that $Y = V$ and, therefore, that the optimal martingale measure has density satisfying (14). For this purpose let us first prove that

$$Y_t = \mathrm{E}^{1-p}\left(\exp\left\{-\frac{p}{2}\langle \lambda \cdot M\rangle_{tT}\right\} \,\middle|\, F_t\right). \tag{19}$$

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It follows from the expression for Z_T/Z_t that

$$\begin{aligned} Y_t &= \mathrm{E}\biggl(\frac{\mathcal{E}^p_{tT}(-\lambda\cdot M)\exp\{-(p^2/2)\langle\lambda\cdot M\rangle_{tT}\}}{\mathrm{E}^p(\exp\{-(p/2)\langle\lambda\cdot M\rangle_{tT}\}\mid F_t)}\biggr) \\ &= \frac{\mathrm{E}(\mathcal{E}_{tT}(-p\lambda\cdot M)\exp\{-(p/2)\langle\lambda\cdot M\rangle_{tT}\}\mid F_t)}{\mathrm{E}^p(\exp\{-(p/2)\langle\lambda\cdot M\rangle_{tT}\}\mid F_t)} \\ &= \mathrm{E}^{1-p}(\exp\{-(p/2)\langle\lambda\cdot M\rangle_{tT}\}\mid F_t), \end{aligned}$$

since

$$\mathrm{E}^{\mathrm{Q}^{p\ \min}}\biggl(\exp\biggl\{-\frac{p}{2}\langle\lambda\cdot M\rangle_{tT}\biggr\}\biggm| F_t\biggr) = \mathrm{E}\biggl(\exp\biggl\{-\frac{p}{2}\langle\lambda\cdot M\rangle_{tT}\biggr\}\biggm| F_t\biggr). \tag{20}$$

Under Assumption 1, the process Y is bounded, since, for any stopping time τ,

$$\mathrm{E}(\mathcal{E}^p_{\tau T}(-\lambda\cdot M)\mid F_\tau)\le C$$

and

$$\begin{aligned} \mathrm{E}(\mathcal{E}^p_{tT}(-\lambda\cdot M)\mid F_t) &= \mathrm{E}^{\mathrm{Q}^{p\ \min}}\biggl(\exp\biggl\{\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_{tT}\biggr\}\biggm| F_t\biggr) \\ &= \mathrm{E}\biggl(\exp\biggl\{-\frac{p}{2}\langle\lambda\cdot M\rangle_{tT}\biggr\}^{1-p}\biggm| F_t\biggr) \\ &\ge \mathrm{E}^{1-p}\biggl(\exp\biggl\{-\frac{p}{2}\langle\lambda\cdot M\rangle_{tT}\biggr\}\biggm| F_t\biggr), \end{aligned}$$

where we use (20) and Jensen's inequality. Furthermore, $Y_t = \mathrm{E}((Z_T/Z_t)^p \mid F_t)$ and by Jensen's inequality it is evident that $Y_t \ge 1$ for all $t \in [0,T]$. Therefore, Y satisfies the two-sided inequality (7).

Now, from (19) we can express Y as follows, where $D_t = (c+\hat{m}_t)^{1-p}$:

$$Y_t = \exp\biggl\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_t\biggr\}(c+\hat{m}_t)^{1-p} = J_t D_t.$$

Writing Itô's formula for this product, we have

$$\begin{aligned} Y_t = (c+\hat{m}_0)^{1-p} &- \frac{p(p-1)}{2}\int_0^t Y_s\,\mathrm{d}\langle\lambda\cdot M\rangle_s \\ &+ (1-p)\int_0^t \exp\biggl\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_s\biggr\}(c+\hat{m}_s)^{-p}\,\mathrm{d}\hat{m}_s \\ &+ \frac{p(p-1)}{2}\int_0^t \exp\biggl\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_s\biggr\}(c+\hat{m}_s)^{-p-1}\,\mathrm{d}\langle\hat{m}\rangle_s, \end{aligned}$$

which coincides with (5) when $\psi = 0$ and

$$\tilde{L}_t = (1-p)\int_0^t \exp\biggl\{-\frac{p(p-1)}{2}\langle\lambda\cdot M\rangle_s\biggr\}(c+\hat{m}_s)^{-p}\,\mathrm{d}\hat{m}_s.$$

The process Y coincides with the value process V, by Proposition 1, because it satisfies (5) and $Y_T = 1$. Therefore, since $\psi = 0$, (14) follows immediately from Corollary 1.

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In the case $p = 2$, the next result has already been pointed out in Laurent and Pham (1999) and in Biagini *et al.* (2000); we obtain it as a corollary to Theorems 1 and 2.

Corollary 2. *The p-optimal martingale measure coincides with the minimal martingale measure and $\psi = 0$ if and only if the mean–variance trade-off $\langle \lambda \cdot M \rangle_T$ is deterministic.*

In fact, if the mean–variance trade-off $\langle \lambda \cdot M \rangle_T$ is deterministic, then (10) and (15) are satisfied.

3. Diffusion case

We consider a diffusion model for the financial market already considered in Karatzas *et al.* (1991) and Laurent and Pham (1999). Let $W = (W^1, \dots, W^n)$ be an n-dimensional standard Brownian motion defined on a complete probability space, $(\Omega, \mathcal{F}, \mathrm{P})$, equipped with the P-augmented filtration generated by W, namely $F = (F_t,\ t \in [0, T])$. We denote by $W^l = (W^1, \dots, W^d)$ and $W^\perp = (W^{d+1}, \dots, W^n)$ the d-dimensional and $(n-d)$-dimensional Brownian motions, respectively.

Assume that there are d risky assets (stocks) and a bond traded on the market. For simplicity, the bond price is assumed to be 1 at all times and the stock price dynamics is given by

$$\mathrm{d}X_t = \mathrm{diag}(X_t)(\mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t^l), \qquad t \in [0, T],$$

where $\mathrm{diag}(X)$ denotes the diagonal $d \times d$ matrix with diagonal elements $(X^1, \dots, X^d)$.

With reference to the market coefficients, we assume that the d-dimensional vector process, μ, of stock appreciation rates and the $d \times d$ volatility matrix, σ, are progressively measurable with respect to F. We also require that, for any $t \in [0, T]$, the volatility matrix is nonsingular almost surely. We take $n > d$, so that there are more sources of uncertainty than there are stocks available for trading and the market is incomplete in the Harrison and Pliska (1981) sense.

Straightforward calculations yield, in this case, $\lambda = \mathrm{diag}(X^{-1})(\sigma\sigma^\top)^{-1}\mu$, $\int_0^t \lambda_s^\top\,\mathrm{d}M_s = \int_0^t \theta_s^\top\,\mathrm{d}W_s^l$, $\langle \lambda \cdot M \rangle_t = \int_0^t \|\theta_s\|^2\,\mathrm{d}s$ (which is the mean–variance trade-off) and $\theta = \sigma^{-1}\mu$ (the market price of risk). As before, we denote by $\mathcal{M}^{\mathrm{e}}$ the set of equivalent martingale measures of X. Let $\mathcal{K}(\sigma)$ be the class of F-predictable $\mathbb{R}^{n-d}$-valued processes ν such that $\int_0^T \|\nu_t\|^2\,\mathrm{d}t < \infty$ a.s.

Since σ is nonsingular, by the Itô representation theorem any local martingale N that is strongly orthogonal to $M = \mathrm{diag}(X)\sigma \cdot W^l$ admits an integral representation $N_t = \int_0^t \nu_s^\top\,\mathrm{d}W_s^\perp$ for some $\nu \in \mathcal{K}(\sigma)$. Therefore, the density of any martingale measure can be expressed as

$$Z_t^\nu = \mathcal{E}_t\biggl(-\int_0^\cdot \theta_s^\top\,\mathrm{d}W_s^l + \int_0^\cdot \nu_s^\top\,\mathrm{d}W_s^\perp\biggr), \qquad t \in [0, T],$$

for some $\nu \in \mathcal{K}(\sigma)$. Let $\mathcal{K}_p(\sigma) = \{\nu \in \mathcal{K}(\sigma) \colon \mathrm{E}(Z_T^\nu) = 1,\ \mathrm{E}(Z_T^\nu)^p < \infty\}$. Then the subclass $\mathcal{M}_p^{\mathrm{e}}$ of equivalent martingale measures is given by

$$\mathcal{M}_p^{\mathrm{e}} = \{\mathrm{P}^\nu \colon \mathrm{dP}^\nu/\mathrm{dP} = Z_T^\nu,\ \nu \in \mathcal{K}_p(\sigma)\}$$

and Assumption 1 ensures that $\mathcal{K}_p(\sigma) \neq \varnothing$.

We now make a further assumption.

Assumption 3. *The mean–variance trade-off is bounded, i.e. $\int_0^T \|\theta_s\|^2\,\mathrm{d}s \le c$ a.s. for some $c > 0$.*

Remark 2. Assumption 3 implies Assumption 1, i.e. that the minimal martingale measure exists and satisfies the reverse Hölder inequality $R_p(\mathrm{P})$, since, for any stopping time τ,

$$\mathrm{E}(\mathcal{E}^p_{\tau T}(-\theta\cdot W^l)\mid F_\tau)=\mathrm{E}^{\mathrm{Q}^{p\ \min}}\biggl(\exp\biggl\{\frac{p(p-1)}{2}\int_\tau^T\|\theta_s\|^2\,\mathrm{d}s\biggr\}\biggm| F_\tau\biggr).$$

Recall that the measure $\mathrm{Q}^{p\ \min}$ is defined by $\mathrm{d}(\mathrm{Q}^{p\ \min})=\mathcal{E}_T(-p\theta\cdot W^l)\,\mathrm{dP}$.

By the martingale representation theorem, the martingale part of the value process is expressed as a stochastic integral,

$$m_t=\int_0^t\varphi_s^\top\,\mathrm{d}W_s^l+\int_0^t(\varphi_s^\perp)^\top\,\mathrm{d}W_s^\perp.$$

Now, since Assumption 3 implies that the minimal martingale measure satisfies the inequality $R_p(\mathrm{P})$, and the filtration F is continuous, the following statement follows from Proposition 1 as a corollary.

Theorem 3. *Let Assumption 3 hold. Then the value process V is the unique bounded, positive solution to the backward stochastic differential equation*

$$\begin{aligned}Y_t=Y_0&-\int_0^t\biggl[\frac{p(p-1)}{2}Y_s\|\theta_s\|^2-p\theta_s^\top\psi_s-\frac{p}{2(p-1)Y_s}\|\psi_s^\perp\|^2\biggr]\mathrm{d}s\\&+\int_0^t\psi_s^\top\,\mathrm{d}W_s^l+\int_0^t(\psi_s^\perp)^\top\,\mathrm{d}W_s^\perp,\qquad Y_T=1.\end{aligned}$$

Moreover, ν^* is optimal if and only if

$$\nu_t^*=-\frac{1}{(p-1)V_t}\varphi_t^\perp\quad(\mathrm{d}t\times\mathrm{dP})\text{-almost everywhere},$$

i.e. if and only if the p-optimal martingale measure is given by the density

$$Z_T^{\nu^*}=\mathcal{E}_T\biggl(-\int_0^\cdot\theta_s^\top\,\mathrm{d}W_s^l-\frac{1}{p-1}\int_0^\cdot\frac{1}{V_s}(\varphi_s^\perp)^\top\,\mathrm{d}W_s^\perp\biggr).$$

Remark 3. Let us introduce $R_t=\ln V_t$. Under Assumption 3, R_t is the unique bounded, nonnegative solution to

$$\begin{aligned}R_t=R_0&-\int_0^t\biggl[\frac{p(p-1)}{2}\|\theta_s\|^2-p\theta_s^\top\bar\psi_s-\frac{1}{2(p-1)}\|\bar\psi_s^\perp\|^2+\frac12\|\bar\psi_s\|^2\biggr]\mathrm{d}s\\&+\int_0^t\bar\psi_s^\top\,\mathrm{d}W_s^l+\int_0^t(\bar\psi_s^\perp)^\top\,\mathrm{d}W_s^\perp,\qquad R_T=0,\end{aligned}\tag{21}$$

where $\bar\psi_s=\psi_s/Y_s$ and $\bar\psi_s^\perp=\psi_s^\perp/Y_s$. Furthermore, the martingale part of R_t is in *BMO*.

As $\mathrm{Q}^{\min}$ is in $\mathcal{M}_p^{\mathrm{e}}$, $\mathrm{Q}^{p\ \min}$ is a probability measure equivalent to P. Thus, by Girsanov's theorem, the process $\hat W^l$ defined via

$$\hat W_t^l=p\int_0^t\theta(s,W^l)\,\mathrm{d}s+W_t^l$$

is the Brownian motion with respect to the measure $\mathrm{Q}^{p\ \min}$.

The following corollaries respectively follow from Theorems 1 and 2.

Corollary 3. *Let Assumption 3 hold. Then the minimal martingale measure is p-optimal, i.e.* $\nu^* = 0$ *and* $Z_T^{Q^*} = \mathcal{E}_T(-\theta \cdot W^l)$, *if and only if*

$$\exp\left\{\frac{p(p-1)}{2}\int_0^T \|\theta_s\|^2\,\mathrm{d}s\right\} = c + \int_0^T \hat{\psi}_s^\top\,\mathrm{d}\hat{W}_s^l \tag{22}$$

for a constant c *and some* $\hat{W}^l$*-integrable,* F*-predictable process* $\hat{\psi}$ *such that the defined stochastic integral is a* $Q^{p\,\min}$*-martingale.*

Remark 4. This condition is satisfied in the 'almost-complete' diffusion model, where the market price of risk is adapted to the filtration, F^l, generated by the Brownian motion W^l, i.e. where $\theta = \theta(t, W^l)$, $t \in [0, T]$ (see Pham *et al.* (1998) and Laurent and Pham (1999)).

In fact, (22) is ensured to hold by the integral representation theorem (see, e.g. Theorem 7.12 of Liptser and Shiryaev (1977)), according to which any $Q^{p\,\min}$-local martingale adapted to F^l can be represented as a stochastic integral. Note that Assumption 3 implies that $\int_0^t \hat{\psi}_s^\top\,\mathrm{d}\hat{W}_s^l \in BMO$.

Corollary 4. *Let Assumption 3 hold. Then the p-optimal martingale measure satisfies*

$$\mathcal{E}_T^{p-1}(M^{Q^*}) = Y_0\mathcal{E}_T\left((1-p)\left(\theta\cdot W^l + \int_0^T \|\theta_s\|^2\,\mathrm{d}s\right)\right) \tag{23}$$

if and only if

$$\exp\left\{-\frac{p}{2}\int_0^T \|\theta_s\|^2\,\mathrm{d}s\right\} = c + \int_0^T \tilde{\psi}_s^\top\,\mathrm{d}W_s^\perp. \tag{24}$$

Equation (23) implies that ν^* *is such that* $\mathcal{E}_T(\nu^*\cdot W^\perp) = 1 + c^{-1}\int_0^T \tilde{\psi}_s^\top\,\mathrm{d}W_s^\perp$, *where* c *and* $\tilde{\psi}$ *are respectively the constant and the* F*-predictable process appearing in (24).*

Condition 4 is satisfied in the case in which the market price of risk is adapted to the filtration, $F^\perp$, generated by the Brownian motion $W^\perp$, i.e. in which $\theta = \theta(t, W^\perp)$, $t \in [0, T]$.

4. Stochastic volatility model

In this, last, section we study a stochastic volatility model. We will assume that the underlying dynamics are Markov, in which setting, provided that mild conditions on the coefficients are satisfied, we can express the solution to the problem of finding the p-optimal martingale measure in terms of a classical Bellman equation.

We start by considering a stochastic volatility model similar to that of Pham *et al.* (1998). We will assume that the dynamics of the asset price process is determined by the following system of stochastic differential equations:

$$\mathrm{d}X_t = \mathrm{diag}(X_t)(\mu(t, X_t, Y_t)\,\mathrm{d}t + \sigma^l(t, X_t, Y_t)\,\mathrm{d}W_t^l), \tag{25}$$

$$\mathrm{d}Y_t = b(t, X_t, Y_t)\,\mathrm{d}t + \delta(t, X_t, Y_t)\,\mathrm{d}W_t^l + \sigma^\perp(t, X_t, Y_t)\,\mathrm{d}W_t^\perp. \tag{26}$$

Assumption 4. *We assume that*

(i) *the coefficients* μ, b, δ, σ^l, *and* $\sigma^\perp$ *are measurable and bounded;*

(ii) *the* $n \times n$ *matrix function* $\sigma\sigma^\top$ *is uniformly elliptic, i.e. there is a constant* $c > 0$ *such that*

$$(\sigma(t,x,y)\lambda, \sigma(t,x,y)\lambda) \geq c|\lambda|^2$$

for all $t \in [0,T]$, $x \in R_+^d$, $y \in R^{n-d}$, *and* $\lambda \in R^n$, *where* σ *is defined by*

$$\sigma(t,x,y) = \begin{pmatrix} \sigma^l(t,x,y) & 0 \\ \delta(t,x,y) & \sigma^\perp(t,x,y), \end{pmatrix};$$

and

(iii) *the system (25), (26) admits a unique strong solution.*

Let us introduce the logarithm of the value function, namely

$$R(t,x,y) = \ln \inf_{\nu \in \mathcal{K}_p^M(\sigma)} \mathrm{E}\bigg(\mathcal{E}_{tT}^p\bigg(-\int_0^\cdot \theta^\top(s,X_s,Y_s)\,\mathrm{d}W_s^l + \int_0^\cdot \nu^\top(s,X_s,Y_s)\,\mathrm{d}W_s^\perp\bigg) \biggm| X_t = x,\ Y_t = y\bigg),$$

where $\theta = \sigma^{l^{-1}}\mu$ and $\mathcal{K}_p^M(\sigma)$ is the class of feedback controls, i.e. controls from $\mathcal{K}_p(\sigma)$ expressed in the form $\nu(t, X_t, Y_t)$ for some measurable function $\nu(t,x,y)$, $t \in [0,T]$, $x \in R_+^d$, $y \in R^{n-d}$.

In the following, we will use Proposition 3 of Mania *et al.* (2004) and the fact that the process R_t satisfies (21). In essence, Proposition 3, applied to R_t, says that the latter can be represented as a space transformation of an asset price process by the logarithm of the value function, which admits a generalized L-operator and all first-order generalized derivatives. Hence, let us recall the definition of generalized derivative together with Proposition 3 of Mania *et al.* (2004).

Let $p(t,x,y) \equiv p(0,(x_0,y_0),t,(x,y))$ be the transition density of the Markov process that is the unique strong solution to (25) and (26) for the fixed initial conditions $X_0 = x_0$ and $Y_0 = y_0$, and introduce the measure ρ on the space $([0,T] \times R_+^d \times R^{n-d}, \mathcal{B}([0,T] \times R_+^d \times R^{n-d}))$:

$$\rho(\mathrm{d}s,\ \mathrm{d}x,\ \mathrm{d}y) = p(s,x,y)\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y.$$

We recall that, for functions f in $C^{1,2}$, continuously differentiable at t on $[0,T]$ and twice differentiable at x, y on $R_+^d \times R^{n-d}$, the L-operator is defined as

$$\begin{aligned} Lf = {} & f_t + \mathrm{tr}(\tfrac{1}{2}\,\mathrm{diag}(x)\sigma^l(\sigma^l)^\top\,\mathrm{diag}(x) f_{xx}) + \mathrm{tr}(\delta(\sigma^l)^\top\,\mathrm{diag}(x) f_{xy}) \\ & + \mathrm{tr}(\tfrac{1}{2}(\delta\delta^\top + \sigma^\perp(\sigma^\perp)^\top) f_{yy}), \end{aligned}$$

where f_t, f_{xx}, f_{xy}, and f_{yy} are partial derivatives of the function f, for which we use the matrix notation.

Definition 1. We shall say that a function

$$f \equiv (f(t,x,y),\ t \geq 0,\ x \in R_+^d,\ y \in R^{n-d})$$

belongs to the class V_ρ^L if there exist a sequence $(f^n,\ n \ge 1)$ of functions in $C^{1,2}$ and measurable ρ-integrable functions f_{x_i} $(i \le d)$, f_{y_j} $(d < j \le n)$, and (Lf) such that, as $n \to \infty$,

$$\mathrm{E}\Big(\sup_{s\le T} |f^n(s, X_s, Y_s) - f(s, X_s, Y_s)|\Big) \to 0,$$

$$\iint_{[0,T]\times R_+^d \times R^{n-d}} (f_{x_i}^n(s,x,y) - f_{x_i}(s,x,y))^2 x_i^2 \rho(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) \to 0, \qquad i \le d,$$

$$\iint_{[0,T]\times R_+^d \times R^{n-d}} (f_{y_j}^n(s,x,y) - f_{y_j}(s,x,y))^2 \rho(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) \to 0, \qquad d < j \le n,$$

$$\iint_{[0,T]\times R_+^d \times R^{n-d}} |Lf^n(s,x,y) - (Lf)(s,x,y)| \rho(\mathrm{d}s, \mathrm{d}x, \mathrm{d}y) \to 0.$$

Proposition 3. (Proposition A of Mania *et al.* (2004).) *Let Assumptions 4(i) and 4(ii) hold and let $f(t, X_t, Y_t)$ be a bounded process. Then the process $(f(t, X_t, Y_t),\ t \in [0,T])$ is an Itô process of the form*

$$f(t, X_t, Y_t) = f(0, X_0, Y_0) + \int_0^t g(s,\omega)\,\mathrm{d}W_s + \int_0^t a(s,\omega)\,\mathrm{d}s, \quad a.s.,$$

with

$$\mathrm{E}\bigg(\int_0^t g^2(s,\omega)\,\mathrm{d}s\bigg) < \infty, \qquad \mathrm{E}\bigg(\int_0^t |a(s,\omega)|\,\mathrm{d}s\bigg) < \infty,$$

if and only if f belongs to V_ρ^L. Moreover, the process $f(t, X_t, Y_t)$ admits the decomposition

$$\begin{aligned} f(t, X_t, Y_t) = f(0, X_0, Y_0) &+ \sum_{i=1}^d \int_0^t f_{x_i}(s, X_s, Y_s)\,\mathrm{d}X_s^i \\ &+ \sum_{j=d+1}^n \int_0^t f_{y_j}(s, X_s, Y_s)\,\mathrm{d}Y_s^j + \int_0^t (Lf)(s, X_s, Y_s)\,\mathrm{d}s. \end{aligned}$$

Remark 5. For continuous functions $f \in V_\rho^L$, the first relation displayed in Definition 1 can be replaced with the condition

$$\sup_{(s,x,y)\in D} |f^n(s,x,y) - f(s,x,y)| \to 0 \quad \text{as } n \to \infty,$$

where D is any compact subset of $[0,T] \times R_+^d \times R^{n-d}$.

Theorem 4. *If Assumptions 4(i), 4(ii), and 4(iii) hold, then the logarithm $R(t,x,y)$ admits all first-order generalized derivatives R_x and R_y, a generalized L-operator LR, and is the unique bounded solution to*

$$\begin{aligned} 0 = {} & LR(t,x,y) + (1-p)\mu^\top(t,x,y)\operatorname{diag}(x) R_x(t,x,y) \\ & + (b^\top(t,x,y) - p\theta_t^\top \delta^\top(t,x,y)) R_y(t,x,y) + \frac{p(p-1)}{2}\|\theta_t\|^2 \\ & - \frac{1}{2(p-1)} \|(\sigma^\perp)^\top(t,x,y) R_y(t,x,y)\|^2 \\ & + \tfrac{1}{2}\|(\sigma^l)^\top(t,x,y)\operatorname{diag}(x) R_x(t,x,y) + \delta^\top(t,x,y) R_y(t,x,y)\|^2 \quad (\mathrm{d}t \times \mathrm{d}x \times \mathrm{d}y)\text{-}a.s. \end{aligned} \tag{27}$$

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with the boundary condition

$$R(T, x, y) = 0. \tag{28}$$

Moreover, $\nu^* = -[1/(p-1)](\sigma^{\perp})^{\top} R_y$ *and the density of the* p*-optimal martingale measure is of the form*

$$Z_T^* = \mathcal{E}_T\biggl(-\int_0^{\cdot} \theta^{\top}(s, X_s, Y_s)\,\mathrm{d}W_s^l - \frac{1}{p-1}\int_0^{\cdot} ((\sigma^{\perp})^{\top} R_y)^{\top}(s, X_s, Y_s)\,\mathrm{d}W_s^{\perp}\biggr).$$

Proof. We give the proof of the existence of a solution (in a certain sense) to the Bellman equation and of the differentiability (in a generalized sense) of the solution. Since (X, Y) is a Markov process, the feedback controls are sufficient and $R_t = R(t, X_t, Y_t)$ a.s. (this can be shown as in Chitashvili and Mania (1996)). As it satisfies (21), the process R_t is an Itô process. Under Assumptions 4(i) and 4(ii), we know that R_t is bounded and that its martingale part is in *BMO* (see Remark 3), and these facts ensure that the finite-variation part of R_t is of integrable variation. We can apply Proposition 3, which implies that the function $R(t, x, y)$ admits a generalized L-operator and all first-order generalized derivatives, and that R_t can be represented as

$$\begin{aligned}
&R(t, X_t, Y_t)\\
&\quad = R_0 + \int_0^t (R_x^{\top}(s, X_s, Y_s)\operatorname{diag}(X_s)\sigma^l(s, X_s, Y_s) + R_y^{\top}(s, X_s, Y_s)\delta(s, X_s, Y_s))\,\mathrm{d}W_s^l\\
&\qquad + \int_0^t R_y^{\top}(s, X_s, Y_s)\sigma^{\perp}(s, X_s, Y_s)\,\mathrm{d}W_s^{\perp} + \int_0^t LR(s, X_s, Y_s)\,\mathrm{d}s\\
&\qquad + \int_0^t (R_x^{\top}(s, X_s, Y_s)\operatorname{diag}(X_s)\mu(s, X_s, Y_s) + R_y^{\top}(s, X_s, Y_s)b(s, X_s, Y_s))\,\mathrm{d}s.
\end{aligned} \tag{29}$$

Moreover, the process R_t is a solution to (21) and, by the uniqueness of the canonical decomposition of semimartingales, comparing the martingale parts of (29) and (21) yields, $\mathrm{d}t \times \mathrm{dP}$-almost everywhere,

$$\bar{\psi}_t = (\sigma^l)^{\top}(t, X_t, Y_t)\operatorname{diag}(X_t)R_x(t, X_t, Y_t) + \delta^{\top}(t, X_t, Y_t)R_y(t, X_t, Y_t), \tag{30}$$

$$(\bar{\psi}_t^{\perp})^{\top} = (\sigma^{\perp})^{\top}(t, X_t, Y_t)R_y(t, X_t, Y_t). \tag{31}$$

Then, by equating the processes of bounded variation of (29) and (21) and taking (30) and (31) into account, we obtain

$$\begin{aligned}
0 = \int_0^t \biggl[&LR(s, X_s, Y_s) + (1-p)R_x^{\top}(s, X_s, Y_s)\operatorname{diag}(X_s)\mu(s, X_s, Y_s)\\
&+ R_y^{\top}(s, X_s, Y_s)b(s, X_s, Y_s) + \frac{p(p-1)}{2}\|\theta_s\|^2 - p\theta_s^{\top}\delta^{\top}(s, X_s, Y_s)R_y(s, X_s, Y_s)\\
&- \frac{1}{2(p-1)}\|(\sigma^{\perp})^{\top}(s, X_s, Y_s)R_y(s, X_s, Y_s)\|^2\\
&+ \tfrac{1}{2}\|(\sigma^l)^{\top}(s, X_s, Y_s)\operatorname{diag}(X_s)R_x(s, X_s, Y_s) + \delta^{\top}(s, X_s, Y_s)R_y(s, X_s, Y_s)\|^2\biggr]\,\mathrm{d}s.
\end{aligned}$$

It follows that $R(t, x, y)$ solves the Bellman equation (27).

We now prove the uniqueness of the solution. If we use the generalized Itô formula (see Proposition 3) with any bounded, nonnegative solution to the equations (27) and (28) from the

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class V_ρ^L, we see that $R(t, X_t, Y_t)$ solves (21). The solution to (21) is unique, so it has to coincide with R_t. This implies that even the solution to (27) and (28) is unique $(\mathrm{d}t \times \mathrm{d}x \times \mathrm{d}y)$-a.s.

Remark 6. Observe that only R_y enters in the construction of the p-optimal martingale measure. Note also that we must make only Assumption 4(iii) to obtain the financial interpretation of the model, and that, for the validity of Theorem 4 (and Proposition 3), the existence of a unique weak solution to (25) and (26), guaranteed by Assumptions 4(i) and 4(ii), is sufficient.

Now we consider the two particular cases.

Case 1. We suppose that the coefficients of (25) depend only on the asset price X, i.e. they take the forms $\mu(t, X_t)$ and $\sigma^l(t, X_t)$. In addition, we suppose that σ^l satisfies the uniform ellipticity condition, and that μ and σ^l are bounded, measurable, and such that (25) admits a unique strong solution. Then $F_t^l = F_t^X$ and the market price of risk, $\theta(t, X_t)$, is F_t^l-measurable.

By the integral representation theorem, any $\mathrm{Q}^{p\,\min}$-local martingale adapted to F^l can be represented as a stochastic integral; hence, condition (22) is satisfied, and by Corollary 3, the p-optimal martingale measure coincides with the minimal martingale measure $\mathrm{Q}^{\min}$.

We can verify that

$$R_t = \ln \mathrm{E}^{\mathrm{Q}^{p\,\min}}\biggl(\exp\biggl\{\frac{p(p-1)}{2}\int_t^T \|\theta_s\|^2\,\mathrm{d}s\biggr\}\;\bigg|\; F_t^l\biggr)$$

characterizes the unique solution to (21), with $\bar\psi_t = \hat\psi_t / \int_0^t \hat\psi_s^\top\,\mathrm{d}\hat W_s^l$ and $\bar\psi_t^\perp = 0$, where $\hat\psi$ is the F^l-predictable process appearing in (22). The Markov property of X implies that $R_t = R(t, X_t)$ a.s., where

$$R(t,x) = \ln \mathrm{E}^{\mathrm{Q}^{p\,\min}}\biggl(\exp\biggl\{\frac{p(p-1)}{2}\int_t^T \|\theta_s\|^2\,\mathrm{d}s\biggr\}\;\bigg|\; X_t = x\biggr).$$

Since the conditions of Theorem 4 are satisfied, $R(t, x)$ is the unique bounded solution to the equation

$$\begin{aligned} 0 &= LR(t,x) + (1-p)\mu^\top(t,x)\,\mathrm{diag}(x)R_x(t,x) + \frac{p(p-1)}{2}\|\theta(t,x)\|^2 \\ &\quad + \tfrac{1}{2}\|(\sigma^l)^\top(t,x)\,\mathrm{diag}(x)R_x(t,x)\|^2 \qquad (\mathrm{d}t \times \mathrm{d}x \times \mathrm{d}y)\text{-a.s.}, \end{aligned} \tag{32}$$

with boundary condition $R(T, x) = 0$, in the class V_ρ^L.

Under suitable regularity conditions on μ and σ^l (see, e.g. Friedman (1975)), the value function $R(t, x)$ is the unique bounded solution to (32) from the class $C^{1,2}$, and

$$LR = \frac{\partial R}{\partial t} + \frac{1}{2}\,\mathrm{tr}(\mathrm{diag}(x)\sigma^l(\sigma^l)^\top\,\mathrm{diag}(x)R_{xx}).$$

Case 2. Now let us suppose that the coefficients in (25) and (26) depend only on Y, where Y is the solution to the autonomous equation (26). Let Assumptions 4(i), 4(ii), and 4(iii) hold. Then $F^\perp = F^Y$ and the market price of risk, $\theta(t, Y_t)$, is $F_t^\perp$-adapted. By the integral representation theorem there exists an $F^\perp$-adapted process $\tilde\psi$ satisfying (24), and such that $\tilde\psi \cdot W^\perp$ is a bounded martingale.

Using the Itô formula, we can see that the process

$$J_t = \ln \mathrm{E}^{1-p}\biggl(\exp\biggl\{-\frac{p}{2}\int_t^T \|\theta_s\|^2\,\mathrm{d}s\biggr\}\;\bigg|\; F_t^\perp\biggr)$$

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is the unique solution to (21), with $\bar{\psi} = 0$ and

$$\bar{\psi}_t^{\perp} = (1-p)\frac{\tilde{\psi}_t}{c + \int_0^t \tilde{\psi}_s^{\top}\,\mathrm{d}W_s^{\perp}},$$

where c and $\tilde{\psi}$ are respectively the constant and the $F_t^{\perp}$-predictable process in (24). Therefore, the process J_t coincides with the logarithm of the value process R_t and, hence, the logarithm of the value function does not depend on x, i.e. $R(t,x,y) = R(t,y)$. Thus, by Theorem 4, $R(t,y)$ is the unique bounded solution to

$$\begin{aligned} 0 &= LR(t,y) + \frac{p(p-1)}{2}\|\theta(t,y)\|^2 + R_y^{\top}(t,y)(b(t,y) - p\delta(t,y)\theta(t,y)) \\ &\quad - \frac{1}{2(p-1)}\|(\sigma^{\perp})^{\top}(t,y)R_y(t,y)\|^2 + \frac{1}{2}\|\delta^{\top}(t,y)R_y(t,y)\|^2 \quad (\mathrm{d}t \times \mathrm{d}x \times \mathrm{d}y)\text{-a.s.} \end{aligned}$$

with boundary condition $R(T,y) = 0$, in the class V_{ρ}^{L}.

Acknowledgements

The author is indebted to Michael Mania for providing countless suggestions during the course of this work. Special thanks are due also to an anonymous referee, whose comments led to a substantial improvement of the paper. This research was partially supported by the Italian Ministry of Education, University and Research, research project 'Stochastic Methods in Finance'.

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