Can. J. Math., Vol. XXXVIII, No. 2, 1986, pp. 277-285

A NOTE ON ORBITS OF SUBGROUPS OF THE PERMUTATION GROUPS

R. D. LEITCH

In [**2**] we studied Milgram's Complex, $C(n - 1)$, which was first defined in [**3**], in the following manner. Let S_n, the permutation group of n symbols, act on $\mathbf{R}^n$ in the obvious manner; put $\alpha(\mathbf{x}) = (\mathbf{y})$, where $y_i = x_{\alpha^{-1}(i)}$. Let $\mathbf{s} = (1, 2, \dots, n)$, then $C(n - 1)$ is the convex hull of the points $\alpha(\mathbf{s})$, $\alpha \in S_n$. Here we shall generalise this construction as follows. Let G be a subgroup of S_n, and let $\mathbf{v} \in \mathbf{R}^n$. Then $C(G, \mathbf{v})$ is the convex hull of $\alpha(\mathbf{v})$, $\alpha \in G$. We prove invariance over $\mathbf{v}$ subject to certain restrictions, give counter-examples to shew lack of invariance if we alter G, discuss how we may describe $C(G, \mathbf{v})$, shew that the only "nice" case is essentially when G is S_n, and lastly give some examples.

We use the definition of a polygonal cell (which we call a p-cell) found in [**1**]; a p-cell is an intersection of a finite number of closed half spaces. We can talk of the faces of a p-cell, in particular the vertices and the k-skeleton X_k of the p-cell X. A p-map $f:X \to Y$ between two p-cells X and Y is a map taking vertices to vertices and which is linear, up to a translation, on the 1-faces of X. If f is one to one it defines a p-inclusion, if one to one and onto, and hence with an inverse, we say that it is a p-isomorphism, and write

$$X \underset{p}{\simeq} Y.$$

Let $\mathbf{x}$, $\mathbf{y}$ be vertices of the same 1-cell in the boundary of a p-cell X. Then we say $\mathbf{x}$ is adjacent to $\mathbf{y}$, and write $\mathbf{x} \sim \mathbf{y}$.

THEOREM 1. *Let $V(X)$, $V(Y)$ be the vertices of the p-cells X, Y respectively, and let $f: V(X) \to V(Y)$ be a map which is one to one, onto and is such that $\mathbf{x} \sim \mathbf{y}$ if and only if $f(\mathbf{x}) \sim f(\mathbf{y})$. Then $X \underset{p}{\simeq} Y$.*

Proof. We can extend f to $f_1:X_1 \to Y_1$, and $f_k:X_k \to Y_k$ to f_{k+1} inductively over k, f_k being an extension of f_{k-1}, and unique up to homotopy. Then eventually $X_k = X$, and as a p-cell is the convex hull of its vertices, we have the theorem.

COROLLARY 2. *Let $F(X)$, $F(Y)$ be the faces of the p-cells, X, Y respectively, and let*

Received August 29, 1980 and in revised form June 1, 1983.

https://doi.org/10.4153/CJM-1986-013-5 Published online by Cambridge University Press

$$h: F(X) \to F(Y)$$

be a map which is one to one, onto and preserves dimension and incidence. Then $X \underset{p}{\simeq} Y$.

Proof. $h|V(X)$ satisfies the conditions of 1.

Let $\mathbf{x}$ be a vertex of X, and let H be a closed half space containing $\mathbf{x}$ in its interior, but containing no other vertices of X. $H \cap X$ we call a p-nbd of $\mathbf{x}$. All the p-nbds of $\mathbf{x}$ are p-isomorphic.

Definition 3. We say the vertices $\mathbf{x}$, $\mathbf{y}$ of the p-cells X, Y are *p-isomorphic* if their p-nbds are p-isomorphic. If, further, we can find p-nbds X', Y', and a distance preserving map $f:X' \to Y'$ such that $f(\mathbf{x}) = f(\mathbf{y})$, we say $\mathbf{x}$, $\mathbf{y}$ are *m-isomorphic*.

(For example, if D is the 2-simplex with vertices at (0, 0), (1, 0), (0, 1), then the three vertices are p-isomorphic, but the vertex at the origin is not m-isomorphic to the other two, which are in the same m-isomorphism class.)

Definition 4. If the vertices of the p-cell X are p-isomorphic, call X *V-regular*. Similarly, if the vertices of a p-cell are m-isomorphic, call X *M-regular*.

THEOREM 5. *Let* $f:X \to Y$ *be a p-inclusion, and suppose* X, Y *are M-regular, and their vertices are m-isomorphic. Then* X, Y *are p-isomorphic.*

We leave the proof to the reader. Observe that the vertices must be m-isomorphic (not just p-isomorphic), for we can find, for example, a distance preserving map taking a regular tetrahedron into a regular cube.

THEOREM 6. $C(G, \mathbf{v})$ *is a polygonal cell of dimension at most* $n - 1$.

G acts on $C(G, \mathbf{v})$ on the left giving

THEOREM 7. $C(G, \mathbf{v})$ *is M-regular.*

For $\alpha \in S_n$ let $f_\alpha:S_n \to S_n$ be the inner automorphism $\beta \to \alpha^{-1}\beta\alpha$, and let G_α be the image of G under f_α. Now recall the following definition from [**2**]. Suppose

$$p_1 + p_2 + \ldots + p_k = n.$$

Then we include $S_{p_1} \times S_{p_2} \times \ldots \times S_{p_k}$ in S_n by having S_{p_1} act on the first p_1 elements, S_{p_2} act on the next p_2 elements, and so on, S_{p_k} acting on the last p_k. We call this inclusion the canonical inclusion, and its image $S(\mathbf{p})$. If $G = S(\mathbf{p})_\alpha$ for some $\alpha \in S_n$, call G a c-subgroup of S_n, or simply a

c-group. Clearly G is a c-group if and only if it is generated by 2-cycles. If G is the c-group $S(\mathbf{p})$, then S_{p_i} acts on the symbols $n_1, n_2, \ldots, n_{p_i}$ (in ascending order). Consider the subgroup of G which acts as S_m on the symbols $n_1, n_2, \ldots, n_m$ and as S_{p_i-m} on $n_{m+1}, n_{m+2}, \ldots, n_{p_i}$, and as G otherwise. Denote it $G_{i,m}$. Clearly $G_{i,m}$ is a c-group, we say it is properly included (or p-included) in G. We extend this term to include a subgroup G_r or G such that

$$G_r \subset G_{r-1} \subset \ldots \subset G_1 \subset G_0 = G,$$

where G_j is p-included in G_{j-1}, $j = 1, 2, \ldots, r$. If H is p-included in G, call H a p-subgroup of G. Then we can generalise 2.2 of [**2**].

THEOREM 8. *If G is a c-group, all the entries of $\mathbf{v}$ are distinct and in ascending order, then the cells of the boundary $C(G, \mathbf{v})$ are in one to one correspondence with the left cosets of the p-subgroups of G. Further, two cells have a proportion of their boundary in common if and only if the corresponding cosets have a non-empty intersection, and two cells are incident, i.e., one is contained in the boundary of the other, if and only if the coset corresponding to the one is contained in the coset corresponding to the other.*

Proof. If $G = S(\mathbf{p})_\alpha$, then as in [**2**] it will be sufficient to show that $C(G_{i,m}, \mathbf{v})$ lies in the hyperplane $H_{i,m}$, and $C(G, \mathbf{v})$ lies entirely to one side of it, $H_{i,m}$ being the hyperplane

$$\sum a_j x_j = 0,$$

where

$$\begin{aligned} a_j &= v_{n_{m+1}} + v_{n_{m+2}} + \ldots + v_{n_{p_1}} = A, \text{ if } j = n_1, n_2, \ldots, n_m \\ &= -(v_{n_1} + v_{n_2} + \ldots + v_{n_m}) = -B, \text{ if } j = n_{m+1}, n_{m+2}, \ldots, n_{p_i} \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then clearly,

$$\sum a_j v_{\alpha^{-1}(j)} = 0 \quad \text{if } \alpha \in H$$

(so $\alpha(\mathbf{v}) \in H_{i,m}$ if $\alpha \in H$)

$$= a_j v_{\alpha^{-1}(j)} - \sum a_j v_j$$

otherwise

$$= A \sum_{j \leqq m} (v_{\alpha^{-1}(n_j)} - v_{n_j}) - B \sum_{j > m} (v_{\alpha^{-1}(n_j)} - v_{n_j}).$$

But clearly,

$$\left.\sum_{j \leqq m} (v_{\alpha^{-1}(n_j)} - v_{n_j}) \begin{array}{l} = 0 \quad \alpha \in G_{i,m} \\ > 0 \text{ otherwise} \end{array}\right\}$$

$$\sum_{j>m} (v_{\alpha^{-1}(n_j)} - v_{n_j}) \begin{array}{l} = 0 \quad \alpha \in G_{i,m} \\ > 0 \text{ otherwise} \end{array} \Bigg\}.$$

as the entries of $\mathbf{v}$ are in ascending order. Hence

$$\sum a_j v_{\alpha^{-1}(j)} > 0 \quad \text{if } \alpha \notin G_{i,m}$$

i.e., $C(G, \mathbf{v})$ lies entirely to one side of the hyperplane, and so $C(H, \mathbf{v})$ is a face of $C(G, \mathbf{v})$ and we have proved the theorem.

THEOREM 9. *If the entries of* $\mathbf{v}$ *are not in ascending order, suppose* $\mathbf{v} = \alpha(\mathbf{u})$, *and* $u_i < u_{i+1}$, *then*

$$C(G, \mathbf{v}) \underset{p}{\simeq} C(G_\alpha, \mathbf{u}).$$

Proof. The set $\{G\alpha(\mathbf{u})\} = \{G\mathbf{v}\}$, and α^{-1} acting on the left is a p-isomorphism (as it preserves distance).

While we have these isomorphisms if we vary $\mathbf{v}$ (up to a simple restriction), if we vary G, up to isomorphism, we need not produce isomorphic p-cells, as the following examples show.

Example 1. Let

$G = \{I, (1\ 2)(3\ 5)(4\ 6), (1\ 3)(2\ 6)(4\ 5), (1\ 4)(2\ 5)(3\ 6), (1\ 6\ 5)(2\ 3\ 4), (1\ 5\ 6)(2\ 4\ 3)\}$,

where I is the identity. Then $G \simeq S_3$.

The vertices of $C(G, \mathbf{s})$ are (1, 2, 3, 4, 5, 6)(2, 1, 5, 6, 3, 4)(3, 6, 1, 5, 4, 2)(4, 5, 6, 1, 2, 3)(5, 4, 3, 2, 6, 1)(6, 3, 4, 2, 1, 5). If we subtract $\mathbf{s}$ where $s = (1, 2, 3, 4, 5, 6)$ from each of the other vertices we find the rank of these five vectors is five, so $C(G, \mathbf{s})$ must have dimension 5, and so is isomorphic to Δ_5.

Example 2. Let

$H = \{(I, (1, 2, 3)(4\ 6\ 5), (1\ 3\ 2)(4\ 5\ 6), (1\ 4)(2\ 5)(3\ 6), (1\ 5)(2\ 6)(3\ 4), (1\ 6)(2\ 4)(3\ 5)\}$.

Then $H \simeq S_3$. The dimension of $C(H, \mathbf{s})$ is 3, by similar reasoning to that used in Example 1, and in fact $C(H, \mathbf{s})$ is an octahedron.

Observe that not only are H and G isomorphic subgroups of S_6, but the elements of the one are obtained from those of the other by interchanging the roles of 1 and 4, i.e., $H = G_{(1\ 4)}$. The effect of the automorphism is to collapse the $\Delta_5 \underset{p}{\simeq} C(G, \mathbf{s})$ into 3 dimensions to give an octahedron. This means that there is no canonical isomorphism

$$C(S(\mathbf{p}), \mathbf{v}) \simeq C(S(p)_\alpha, \mathbf{v}).$$

We do however have the following cases.

THEOREM 10. *Let $G \subset S_p$, $H \subset S_q$ are subgroups, then we can include $G \times H$ in S_{p+q} in a canonical manner.*

$$G \times H \subset S_p \times S_q \subset S_{p+q}$$

and we have

$$C(G \times G, \mathbf{s}) \underset{p}{\simeq} C(G, \mathbf{s}) \times C(H, \mathbf{s}).$$

Proof. Consider the map

$$f:\mathbf{R}^p \times \mathbf{R}^q \times \mathbf{R}^{p+q}$$

given by

$$(\mathbf{x}, \mathbf{y}) \rightarrow (x_1, x_2, \dots, x_p, y_1 + p, y_2 + p, \dots, y_q + p).$$

Then $f|C(G, \mathbf{s}) \times C(H, \mathbf{s})$ maps onto $C(G \times H, \mathbf{s})$ and is clearly an isomorphism.

COROLLARY 11. *If all the entries of* **u** *are distinct,*

$$C(S(p_i), \mathbf{u}) \underset{p}{\simeq} \prod C(p_i - 1).$$

8 does not generalise to an arbitrary group G, as the following counter-example shows.

Example 3. Let

$$G = \{I, (1\ 2\ 3), (1\ 3\ 2)\} \subset S_3,$$

then $G \simeq Z_3 \cdot C(G, \mathbf{s})$ is the triangle with vertices (1, 2, 3), (3, 1, 2), (2, 3, 1). The subsets of G corresponding to the faces of $C(G)$ are $\{I, (1\ 2\ 3)\}$ and $\{I, (1\ 3\ 2)\}$. The third face corresponds to

$$\{(1\ 2\ 3), (1\ 3\ 2)\} = (1\ 3\ 2)\{I, (1\ 2\ 3)\} = (1\ 2\ 3)\{I, (1\ 3\ 2)\}.$$

We have, in fact, the worst possible case, in the following sense.

THEOREM 12. *Suppose $G \subset S_n$, and the entries of* **u** *are distinct. Then the faces of $C(G)$ are in a one to one, incidence preserving, correspondence, in the sense of Theorem 8, with the left cosets of a collection of subgroups G_i of G if and only if G is a c-group.*

Proof. Theorem 8 is one half of our proof.

Let L be a one face of $C(G, \mathbf{u})$ such that **u** is a vertex of L, and suppose $\alpha(u)$ is the other vertex of L. Then clearly $\{I, \alpha\}$ is a group (as it is a coset and contains the identity). Then $\alpha^2 = I$, so α is a 2-cycle. Let H be the subgroup of G generated by all such α. Then H is a c-group.

Consider the inclusion

$$I:C(H, \mathbf{u}) \subset C(G, \mathbf{u})$$

given by $I(\alpha(\mathbf{u})) = \alpha(\mathbf{u})$ on the vertices of H, and extended linearly over all of H as in 1. Then this map satisfies the condition of 5, and so

$$C(H, \mathbf{u}) \underset{p}{\simeq} C(G, u).$$

In particular, they have the same number of vertices, and so H and G have the same number of elements, and as $H \subset G$, then $H = G$.

We now give some further examples of $C(G, \mathbf{s})$. In **[2]** and **[3]**

$$C(m - 1) = C(S_m, \mathbf{s})$$

is examined. Now $(1\ 2 \ldots n) \in S_n$ generates $Z_n \subset S_n$, and we have in this case.

Example 4. Consider the embedding

$$E:Z_n \subset S_n, \quad E(g) = (1\ 2\ 3 \ldots n)$$

where g is a generator of Z_n. Then

$$C(Z_n, \mathbf{s}) \underset{p}{\simeq} \Delta_{n-1}.$$

Clearly the vertices of $C(G, \mathbf{s})$ are $(k, k + 1, k + 2, \ldots, n, 1, \ldots, k - 1)$, $k = 1, 2, \ldots, n$, and as this gives a linearly independent set, $C(G, \mathbf{s})$ must be $n - 1$ dimensional, and as it has n vertices can only be Δ_{n-1}.

We now make two observations; the first that $C(G, \mathbf{s})$ is not entirely regular, as the edges are not all the same length. The other is that we identified Z_n with a particular embedding. If we put

$$Z_{n+m} \simeq Z_n \times Z_m \subset S_n \times S_m \subset S_{n+m}$$

(assume n, m coprime) then

$$C(Z_{n+m}, \mathbf{s}) \underset{p}{\simeq} \Delta_{n-1} \times \Delta_{m-1}.$$

Example 5. Let Δ_{n-1} be the regular n-simplex with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that

$$\sum \mathbf{v}_i = \mathbf{0}$$

(i.e., the centroid is at the origin). Then $-\mathbf{v}_1, -\mathbf{v}_2, \ldots, -\mathbf{v}_n$ is a second such simplex. Put Δ'_{n-1} the convex hull of $\mathbf{v}_1, \ldots, \mathbf{v}_n, -\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Consider the subgroup of S_n generated by $(1\ 2\ 3 \ldots n)$ and $(1\ n)(2\ n - 1)(\ldots)\ (ab)$ where the last two-cycle in the second element is $\left(\frac{n}{2}\,\frac{n+2}{2}\right)$ if n is even $\left(\frac{n-1}{2}\,\frac{n+2}{2}\right)$ for n odd. It is not difficult to see that this subgroup is isomorphic with D_n, the dihedral group. Call it $D_{n,1}$. Then

THEOREM 13.

$$C(D_{n,1}, \mathbf{s}) \underset{p}{\simeq} \Delta'_{n-1}.$$

Proof. The vertices of $C(D_{n,1}, \mathbf{s})$ consist of the vectors $\mathbf{u}_k$, $\mathbf{u}'_k$, where

$$\mathbf{u}_k = (k, k + 1, \ldots, k - 1)$$

$$\mathbf{u}'_k = (n - k + 1, n - k, \ldots, n - k + 2)$$

put

$$\mathbf{c} = \frac{\sum \mathbf{u}_k}{n} = \frac{\sum \mathbf{u}'_k}{n} = \left(\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2}\right)$$

$$\mathbf{u}_k - \mathbf{c} = \tfrac{1}{2}(2k - n - 1, 2k - n + 1, 2k - n + 3, \ldots, 2k - n - 3)$$

$$c - \mathbf{u}'_k = \tfrac{1}{2}(2k - n - 1, 2k - n + 1, 2k - n + 3, \ldots, 2k - n - 3)$$

so clearly the map $\mathbf{u}_k \leftrightarrow \mathbf{v}_k$, $\mathbf{u}'_k \leftrightarrow -\mathbf{v}_k$, induces a p-isomorphism between $C(D_{n,1}, \mathbf{s})$ and Δ'_{n-1}.

Example 6. Let $D_{n,2}$ be the subgroup of S_{2n} generated by

$$\alpha = (1\ 2\ 3 \ldots n)(n + 1\ 2n\ 2n - 1 \ldots n + 2)$$

$$\beta = (1\ n + 1)(2\ n + 2)(\ldots)(n\ 2n).$$

Then once more, $D_{n,2} \simeq D_n$, and this time

$$C(D_{n,2}, \mathbf{s}) \underset{p}{\simeq} \Delta_{n-1} \times I$$

for the vertices of $C(D_{n,2}, \mathbf{s})$ consist of the vectors $\mathbf{u}_k$, $\mathbf{v}_k$, where

$$\mathbf{u}_k = (k, k + 1, \ldots, \underset{\underset{n-k+1}{\uparrow}}{n}, 1, \ldots, \underset{\underset{n}{\uparrow}}{k - 1}, 2n - k + 2, 2n - k + 3, \ldots \underset{\underset{n+k-1}{\uparrow}}{2n}, n + 1, \ldots, 2n - k + 1)$$

$$\mathbf{v}_k = (n + k, n + k + 1, \dots, \underset{\substack{\uparrow \\ n-k+1}}{2n}, n + 1, \dots, n + \underset{\substack{\uparrow \\ n}}{k} - 1,$$

$$n - k + 2, n - k + 3, \dots, \underset{\substack{\uparrow \\ n+k-1}}{n}, 1, \dots, n - k + 1,$$

$$k = 1, 2, \dots, m$$

$$\mathbf{u}_k - \mathbf{u}_1 = (k - 1, k - 1, \dots, \underset{\substack{\uparrow \\ n-k+1}}{k} - 1, k - 1 - n, \dots, k - \underset{\substack{\uparrow \\ n}}{1},$$

$$n - k + 1, \dots, n - \underset{\substack{\uparrow \\ n+k-1}}{k} + 1, 1 - k, \dots, 1 - k)$$

$$\mathbf{v}_k - \mathbf{v}_1 = \mathbf{u}_k - \mathbf{u}_1$$

so that

$$U \underset{p}{\simeq} V,$$

where U, V are the convex hull of $\{\mathbf{u}_i\}$, $\{\mathbf{v}_i\}$ respectively. $\mathbf{u}_k - \mathbf{u}_1$ are linearly independent, $k = 2, 3, \dots, n$, and so $U \simeq \Delta_{n-1}$.

$$\mathbf{u}_k - \mathbf{v}_k = (-n, -n, \dots, -\underset{\substack{\uparrow \\ n}}{n}, n, n, \dots, n)$$

$$\mathbf{u}_s - \mathbf{u}_t = (s - t, s - t, \dots, s - \underset{\substack{\uparrow \\ n-s+t}}{t}, s - t - n, \dots, s - \underset{\substack{\uparrow \\ n-t+1}}{t} - n,$$

$$s - t, \dots, s \underset{\substack{\uparrow \\ n}}{-} t, t - s, \dots, t \underset{\substack{\uparrow \\ n+t-1}}{-} s, n + t - s, \dots,$$

$$n + \underset{\substack{\uparrow \\ n+s-1}}{t} - s, t - s, \dots, t - s)$$

for $s > t$.

https://doi.org/10.4153/CJM-1986-013-5 Published online by Cambridge University Press

Then

$$(\mathbf{u}_k - \mathbf{v}_k) \cdot (\mathbf{u}_s - \mathbf{u}_t) = 0$$

and similarly

$$(\mathbf{u}_k - \mathbf{v}_k) \cdot (\mathbf{v}_s - \mathbf{v}_t) = 0$$

and so $(u_k - v_k)$ is perpendicular to U and V, $C(D_{m,2}, \mathbf{s})$ then is a prism, and hence the result.

References

1. S. S. Cairns, *Introductory topology* (Ronald Press Co., N.Y., 1961).
2. R. D. Leitch, *The homotopy commutative cube*, J. Lon. Math. Soc. (2)(1974), 23-29.
3. R. J. Milgram, *Iterated loop spaces*, Ann. Math *84* (1966), 386-403.

Royal Military College of Science,
Shrivenham, England