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GENERALIZED HARDY–CESÀRO OPERATORS BETWEEN WEIGHTED SPACES

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Abstract. We characterize those non-negative, measurable functions ψ on [0, 1] and positive, continuous functions ω_1 and ω_2 on $\mathbb{R}^+$ for which the generalized Hardy–Cesàro operator

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t)\,dt$$

defines a bounded operator $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$. This generalizes a result of Xiao [**7**] to weighted spaces. Furthermore, we extend U_ψ to a bounded operator on $M(\omega_1)$ with range in $L^1(\omega_2) \oplus \mathbb{C}\delta_0$, where $M(\omega_1)$ is the weighted space of locally finite, complex Borel measures on $\mathbb{R}^+$. Finally, we show that the zero operator is the only weakly compact generalized Hardy–Cesàro operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

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1. Introduction. A classical result of Hardy [**5**] shows that the Hardy–Cesàro operator

$$(Uf)(x) = \frac{1}{x}\int_0^x f(s)\,ds$$

defines a bounded linear operator on $L^p(\mathbb{R}^+)$ with $\|U\| = p/(p-1)$ for $p > 1$. Clearly, U is not bounded on $L^1(\mathbb{R}^+)$. Hardy's result has been generalized in various ways, of which we will mention some, which have inspired this paper.

For $1 \leq p \leq q \leq \infty$ and non-negative measurable functions u and v on $\mathbb{R}^+$, Muckenhoupt [**6**] and Bradley [**3**] gave a necessary and sufficient condition for the existence of a constant C such that

$$\left(\int_0^\infty \left(u(x)\int_0^x f(t)\,dt\right)^q dx\right)^{1/q} \leq C\left(\int_0^\infty (v(x)f(x))^p\,dx\right)^{1/p}$$

for every positive, measurable function f on $\mathbb{R}^+$. This can be rephrased as a characterization of the weighted L^p and L^q spaces on $\mathbb{R}^+$ between which the Hardy–Cesàro operator U is bounded.

In a different direction, for a non-negative measurable funtion ψ on [0, 1], Xiao [7] considered the generalized Hardy–Cesàro operators

$$(U_\psi f)(x) = \int_0^1 f(tx)\psi(t)\,dt$$

for measurable functions f on $\mathbb{R}^n$. We remark that

$$(U_\psi f)(x) = \frac{1}{x}\int_0^x f(s)\psi(s/x)\,ds$$

for measurable functions f on $\mathbb{R}$. Xiao proved that U_ψ defines a bounded operator on $L^p(\mathbb{R}^n)$ (for $p \geq 1$) if and only if

$$\int_0^1 \frac{\psi(t)}{t^{n/p}}\,dt < \infty.$$

Xiao's result is the main motivation for this paper.

Finally, we mention that Albanese, Bonet and Ricker in a recent series of papers (see, for instance, [1] and [2]) have considered the spectrum, compactness and other properties of the Hardy–Cesàro operator on various spaces of continuous functions and discrete spaces.

In this paper, we will study the generalized Hardy–Cesàro operators between weighted spaces of integrable functions, and we will obtain a generalization of Xiao's result in this context. Let ω be a positive, continuous function on $\mathbb{R}^+$ and let $L^1(\omega)$ be the Banach space of (equivalence classes of) measurable functions f on $\mathbb{R}^+$ for which

$$\|f\|_{L^1(\omega)} = \int_0^\infty |f(t)|\omega(t)\,dt < \infty.$$

In the usual way, we identify the dual space of $L^1(\omega)$ with the space $L^\infty(1/\omega)$ of measurable functions h on $\mathbb{R}^+$ for which

$$\|h\|_{L^\infty(1/\omega)} = \operatorname{ess\,sup}_{t\in\mathbb{R}^+} |h(t)|/\omega(t) < \infty.$$

We denote by $C_0(1/\omega)$ the closed subspace of $L^\infty(1/\omega)$ consisting of the continuous functions g in $L^\infty(1/\omega)$ for which g/ω vanishes at infinity. Finally, we identify the dual space of $C_0(1/\omega)$ with the space $M(\omega)$ of locally finite, complex Borel measures μ on $\mathbb{R}^+$ for which

$$\|\mu\|_{M(\omega)} = \int_{\mathbb{R}^+} \omega(t)\,d|\mu|(t) < \infty.$$

We consider the space $L^1(\omega)$ as a closed subspace of $M(\omega)$.

In Section 2, we characterize those functions ψ, ω_1 and ω_2 for which U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. These operators are extended to bounded operators on $M(\omega_1)$ in Section 3, where we also obtain results about their ranges. Finally, in Section 4, we show that there are no non-zero weakly compact generalized Hardy–Cesàro operators from $L^1(\omega_1)$ to $L^1(\omega_2)$.

2. A characterization of the generalized Hardy–Cesàro operators. For a non-negative, measurable function ψ on $[0, 1]$ and positive, continuous functions ω_1 and ω_2 on $\mathbb{R}^+$, we say that condition (C) is satisfied if there exists a constant C such that

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt \le C\omega_1(s)$$

for every $s \in \mathbb{R}^+$.

THEOREM 2.1. *Let ψ be a non-negative, measurable function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Then, U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if condition (C) is satisfied.*

Proof. Assume that condition (C) is satisfied and let $f \in L^1(\omega_1)$. Then,

$$\int_0^\infty \int_0^1 |f(s)|\,\frac{\psi(t)}{t}\,\omega_2(s/t)\,dt\,ds \le C\int_0^\infty |f(s)|\omega_1(s)\,ds = C\|f\|_{L^1(\omega_1)} < \infty,$$

so it follows from Fubini's theorem that

$$\int_0^1 \int_0^\infty |f(tx)|\psi(t)\omega_2(x)\,dx\,dt = \int_0^1 \int_0^\infty |f(s)|\,\frac{\psi(t)}{t}\,\omega_2(s/t)\,ds\,dt \le C\|f\|_{L^1(\omega_1)} < \infty.$$

Another application of Fubini's theorem thus shows that $(U_\psi f)(x)$ is defined for almost all $x \in \mathbb{R}^+$ with

$$\begin{aligned}\|U_\psi f\|_{L^1(\omega_2)} &= \int_0^\infty |(U_\psi f)(x)|\omega_2(x)\,dx \le \int_0^\infty \int_0^1 |f(tx)|\psi(t)\omega_2(x)\,dt\,dx\\ &= \int_0^1 \int_0^\infty |f(tx)|\psi(t)\omega_2(x)\,dx\,dt \le C\|f\|_{L^1(\omega_1)} < \infty.\end{aligned}$$

Hence, U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

Conversely, assume that U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Since $L^1(\omega_2)$ is a closed subspace of $M(\omega_2)$ which we identify with the dual space of $C_0(1/\omega_2)$, it follows from [**4**, Theorem VI.8.6] that there exists a map ρ from $\mathbb{R}^+$ to $M(\omega_2)$ for which the map $s \mapsto \langle g, \rho(s)\rangle = \int_{\mathbb{R}^+} g(x)\,d\rho(s)(x)$ is measurable and essentially bounded on $\mathbb{R}^+$ for every $g \in C_0(1/\omega_2)$ with $\|U_\psi\| = \operatorname{ess\,sup}_{s\in\mathbb{R}^+}\|\rho(s)\|_{M(\omega_2)}$ and such that

$$\langle g, U_\psi f\rangle = \int_0^\infty \langle g, \rho(s)\rangle f(s)\omega_1(s)\,ds = \int_0^\infty \int_{\mathbb{R}^+} g(x)\,d\rho(s)(x)f(s)\omega_1(s)\,ds$$

for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$. On the other hand,

$$\begin{aligned}\langle g, U_\psi f\rangle &= \int_0^\infty g(x)(U_\psi f)(x)\,dx\\ &= \int_0^\infty \int_0^x \frac{g(x)}{x} f(s)\psi(s/x)\,ds\,dx\\ &= \int_0^\infty \frac{1}{\omega_1(s)} \int_s^\infty \frac{g(x)}{x}\,\psi(s/x)\,dx\,f(s)\omega_1(s)\,ds\end{aligned}$$

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for every $g \in C_0(1/\omega_2)$ and $f \in L^1(\omega_1)$, so it follows that

$$\int_{\mathbb{R}^+} g(x)\,d\rho(s)(x) = \frac{1}{\omega_1(s)}\int_s^\infty \frac{g(x)}{x}\,\psi(s/x)\,dx$$

for almost all $s \in \mathbb{R}^+$ and every $g \in C_0(1/\omega_2)$ (considering both sides as elements of $L^\infty(\mathbb{R}^+)$). Considered as elements of $M(\omega_2)$, we thus have

$$d\rho(s)(x) = \frac{1}{\omega_1(s)}\,\frac{1}{x}\,\psi(s/x)\,1_{x\geq s}\,dx$$

for almost all $s, x \in \mathbb{R}^+$. Hence, $\rho(s) \in L^1(\omega_2)$ with

$$\begin{aligned}\|\rho(s)\|_{L^1(\omega_2)} &= \int_0^\infty \omega_2(x)\,d\rho(s)(x)\\ &= \frac{1}{\omega_1(s)}\int_0^\infty \frac{1}{x}\,\psi(s/x)\,1_{x\geq s}\,\omega_2(x)\,dx\\ &= \frac{1}{\omega_1(s)}\int_s^\infty \frac{1}{x}\,\psi(s/x)\omega_2(x)\,dx\\ &= \frac{1}{\omega_1(s)}\int_0^1 \frac{\psi(t)}{t}\,\omega_2(s/t)\,dt\end{aligned}$$

for almost all $s \in \mathbb{R}^+$. Therefore,

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt = \|\rho(s)\|_{L^1(\omega_2)}\omega_1(s) \leq \|U_\psi\|\omega_1(s)$$

for almost all $s \in \mathbb{R}^+$. Since both sides of the inequality are continuous functions of s, the inequality holds for every $s \in \mathbb{R}^+$, so condition (C) holds. □

Letting $s = 0$ in condition (C), we see that Xiao's condition is necessary in our situation.

COROLLARY 2.2. *Let ψ be a non-negative, measurable function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. If U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then*

$$\int_0^1 \frac{\psi(t)}{t}\,dt < \infty.$$

The following straightforward consequences can be deduced from Theorem 2.1.

COROLLARY 2.3. *Let ψ be a non-negative, measurable function on* $[0, 1]$

(*a*) *Let ω be a decreasing, positive, continuous function on $\mathbb{R}^+$, and assume that $\int_0^1 \psi(t)/t\,dt < \infty$. Then, U_ψ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$.*
(*b*) *Let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$, and assume that ω_2 is increasing. If U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$, then there exists a constant C such that $\omega_2(s) \leq C\omega_1(s)$ for every $s \in \mathbb{R}^+$.*

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(c) *Let ω be an increasing, positive, continuous function on $\mathbb{R}^+$, and assume that there exists $a < 1$ and $K > 0$ such that $\psi(t) \geq K$ almost everywhere on $[a, 1]$. If U_ψ defines a bounded operator from $L^1(\omega)$ to $L^1(\omega)$, then there exist positive constants C_1 and C_2 such that*

$$C_1\omega(s) \leq \int_0^1 \omega(s/t)\,\frac{\psi(t)}{t}\,dt \leq C_2\omega(s)$$

for every $s \in \mathbb{R}^+$.

Proof.

(a) We have

$$\int_0^1 \omega(s/t)\,\frac{\psi(t)}{t}\,dt \leq \int_0^1 \frac{\psi(t)}{t}\,dt\,\omega(s)$$

for every $s \in \mathbb{R}^+$, so condition (C) is satisfied with $\omega_1 = \omega_2 = \omega$ and the result follows.

(b) We have

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt \geq \int_0^1 \frac{\psi(t)}{t}\,dt\,\omega_2(s)$$

for every $s \in \mathbb{R}^+$. Since condition (C) is satisfied, the result follows.

(c) We have

$$\int_0^1 \omega(s/t)\,\frac{\psi(t)}{t}\,dt \geq K\int_a^1 \omega(s/t)\,dt \geq K(1-a)\omega(s)$$

for every $s \in \mathbb{R}^+$. The other inequality is just condition (C) with $\omega_1 = \omega_2 = \omega$.

□

We finish the section with some examples of functions ψ, ω_1 and ω_2 for which U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$.

EXAMPLE 2.4.

(a) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, for $\beta_1, \beta_2 \in \mathbb{R}$, let $\omega_i(x) = (1 + x)^{\beta_i}$ for $x \in \mathbb{R}^+$ and $i = 1, 2$. Then, U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$ if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b) For $\alpha > 0$, let $\psi(t) = t^\alpha$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{-x}/(1 + x)$ and $\omega_2(x) = e^{-x}$ for $x \in \mathbb{R}^+$. Then, U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c) Let $\psi(t) = e^{-1/t^2}$ for $t \in [0, 1]$. Also, let $\omega_1(x) = e^{x^2/4}/x$ and $\omega_2(x) = e^x$ for $x \in \mathbb{R}^+$. Then, U_ψ defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. Moreover, it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity.

Proof.

(a) For $s \geq 1$ and $t \in [0, 1]$, we have $s/t < 1 + s/t \leq 2s/t$, so

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1}\,dt \simeq s^{\beta_2} \int_0^1 t^{\alpha-\beta_2-1}\,dt \simeq s^{\beta_2}$$

for $s \geq 1$ if $\beta_2 < \alpha$ (where $F(s) \simeq G(s)$ for positive functions F and G on $[1, \infty)$ indicates the existence of positive constants C_1 and C_2 such that $C_1 F(s) \leq G(s) \leq C_2 F(s)$ for all $s \in [1, \infty)$), whereas the integrals diverge if $\beta_2 \geq \alpha$. Moreover, the expression

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt = \int_0^1 \left(1 + \frac{s}{t}\right)^{\beta_2} t^{\alpha-1}\,dt$$

defines a positive, continuous function of s on $\mathbb{R}^+$, so it follows that condition (C) is satisfied if and only if $\beta_2 \leq \beta_1$ and $\beta_2 < \alpha$.

(b) For $s \geq 1$, we have

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt = \int_s^\infty \frac{\omega_2(x)}{x}\,\psi(s/x)\,dx = \int_s^\infty \frac{e^{-x}}{x}\,\frac{s^\alpha}{x^\alpha}\,dx \leq \int_s^\infty \frac{e^{-x}}{x}\,dx \leq \frac{e^{-s}}{s}.$$

Moreover,

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt \leq \int_0^1 \frac{\psi(t)}{t}\,dt < \infty$$

for all $s \in \mathbb{R}^+$, so condition (C) is satisfied and U_ψ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. On the other hand, since

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt \geq \int_s^{2s} \frac{e^{-x}}{x}\,\frac{s^\alpha}{x^\alpha}\,dx \geq \frac{1}{2^{\alpha+1}s} \int_s^{2s} e^{-x}\,dx \geq \frac{1}{2^{\alpha+2}}\,\frac{e^{-s}}{s}$$

for $s \geq 1$, it is not possible to replace $\omega_1(x)$ by a function tending faster to zero at infinity.

(c) For $s \in \mathbb{R}^+$, we have

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt = \int_s^\infty \frac{\omega_2(x)}{x}\,\psi(s/x)\,dx = \int_s^\infty \frac{e^{x-x^2/s^2}}{x}\,dx = \int_1^\infty \frac{e^{sy-y^2}}{y}\,dy.$$

Moreover, for $s \geq 4$

$$\int_{s/4}^\infty \frac{e^{sy-y^2}}{y}\,dy \leq \frac{4}{s} \int_{s/4}^\infty e^{-(y-s/2)^2+s^2/4}\,dy = 4 \int_{-s/4}^\infty e^{-u^2}\,du\,\frac{e^{s^2/4}}{s}$$

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and

$$\int_1^{s/4} \frac{e^{sy-y^2}}{y}\,dy \le \int_1^{s/4} e^{sy}\,dy \le \frac{e^{s^2/4}}{s},$$

so condition (C) is satisfied and U_ψ thus defines a bounded operator from $L^1(\omega_1)$ to $L^1(\omega_2)$. On the other hand, the estimate

$$\int_0^1 \omega_2(s/t)\,\frac{\psi(t)}{t}\,dt = \int_1^{\infty} \frac{e^{sy-y^2}}{y}\,dy \ge \frac{1}{s}\int_{s/2}^{s/2+1} e^{-(y-s/2)^2+s^2/4}\,dy = \int_0^1 e^{-u^2}\,du\,\frac{e^{s^2/4}}{s}$$

for $s \ge 2$ shows that it is not possible to replace $\omega_1(x)$ by a function tending slower to infinity at infinity.

□

In Example 2.4(b), we have $\omega_2(x)/\omega_1(x) \to \infty$ as $x \to \infty$, which should be compared to the conclusion in Corollary 2.3(b). Conversely, Example 2.4(c) shows an example where we need $\omega_2(x)/\omega_1(x) \to 0$ rapidly as $x \to \infty$ in order for U_ψ to be defined.

3. Extensions to weighted spaces of measures. Identifying the dual space of $L^1(\omega)$ with $L^\infty(1/\omega)$ as in the introduction, we have the following result about the adjoint of U_ψ.

PROPOSITION 3.1. *Let ψ be a non-negative, measurable function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, and consider the adjoint operator $U_\psi^* : L^\infty(1/\omega_2) \to L^\infty(1/\omega_1)$.*

(*a*) *For $h \in L^\infty(1/\omega_2)$, we have*

$$(U_\psi^* h)(x) = \int_0^1 h(x/t)\,\frac{\psi(t)}{t}\,dt$$

for almost all $x \in \mathbb{R}^+$.

(*b*) *U_ψ^* maps $C_0(1/\omega_2)$ into $C_0(1/\omega_1)$.*

Proof.

(a) Let $h \in L^\infty(1/\omega_2)$. Since $|h(x/t)| \le \|h\|_{L^\infty(1/\omega_2)}\omega_2(x/t)$ for almost all $x, t \in \mathbb{R}^+$, it follows from condition (C) that $\int_0^1 h(x/t)\psi(t)/t\,dt$ is defined and satisfies

$$\left|\int_0^1 h(x/t)\,\frac{\psi(t)}{t}\,dt\right| \le \|h\|_{L^\infty(1/\omega_2)} \int_0^1 \omega_2(x/t)\,\frac{\psi(t)}{t}\,dt \le C\|h\|_{L^\infty(1/\omega_2)}\omega_1(x)$$

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for almost all $x \in \mathbb{R}^+$. Hence, the function $x \mapsto \int_0^1 h(x/t)\psi(t)/t\,dt$ belongs to $L^\infty(1/\omega_1)$. Also, for $f \in L^1(\omega_1)$ we have

$$\begin{aligned}\langle f, U_\psi^* h\rangle = \langle U_\psi f, h\rangle &= \int_0^\infty (U_\psi f)(s)h(s)\,ds \\ &= \int_0^\infty \int_0^s \frac{1}{s} f(x)\psi(x/s)h(s)\,dx\,ds \\ &= \int_0^\infty \int_x^\infty \frac{h(s)}{s}\,\psi(x/s)\,ds\,f(x)\,dx\end{aligned}$$

from which it follows that

$$(U_\psi^* h)(x) = \int_x^\infty \frac{h(s)}{s}\,\psi(x/s)\,ds = \int_0^1 h(x/t)\,\frac{\psi(t)}{t}\,dt$$

for almost all $x \in \mathbb{R}^+$.

(b) It suffices to show that U_ψ^* maps $C_c(\mathbb{R}^+)$ (the continuous functions on $\mathbb{R}^+$ with compact support) into $C_0(1/\omega_1)$. Let $g \in C_c(\mathbb{R}^+)$, let $x_0 \in \mathbb{R}^+$ and let (x_n) be a sequence in $\mathbb{R}^+$ with $x_n \to x_0$ as $n \to \infty$. Then,

$$(U_\psi^* g)(x_n) - (U_\psi^* g)(x_0) = \int_0^1 (g(x_n/t) - g(x_0/t))\,\frac{\psi(t)}{t}\,dt$$

for $n \in \mathbb{N}$. Since g is bounded on $\mathbb{R}^+$ and since $\int_0^1 \psi(t)/t\,dt < \infty$ by Corollary 2.2, it follows from Lebesgue's dominated convergence theorem that $(U_\psi^* g)(x_n) \to (U_\psi^* g)(x_0)$ as $n \to \infty$. Hence, $U_\psi^* g$ is continuous on $\mathbb{R}^+$. Finally, from the expression

$$(U_\psi^* g)(x) = \int_x^\infty \frac{g(s)}{s}\,\psi(x/s)\,ds$$

it follows that $\operatorname{supp} U_\psi^* g \subseteq \operatorname{supp} g$, so we conclude that $U_\psi^* g \in C_c(\mathbb{R}^+) \subseteq C_0(1/\omega_1)$. □

Let V_ψ be the restriction of U_ψ^* to $C_0(1/\omega_2)$ considered as a map into $C_0(1/\omega_1)$. We then immediately have the following result.

Corollary 3.2. *Let ψ be a non-negative, measurable function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. The bounded operator $\overline{U}_\psi = V_\psi^*$ from $M(\omega_1)$ to $M(\omega_2)$ is an extension of U_ψ.*

Let ψ be a non-negative, continuous function on $[0, 1]$ with $\psi(0) = 0$. For $\mu \in M(\omega_1)$ and $x > 0$ let

$$(W_\psi \mu)(x) = \frac{1}{x} \int_{(0,x)} \psi(s/x)\,d\mu(s).$$

Proposition 3.3. *Let ψ be a non-negative, continuous function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so*

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that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ *is a bounded operator. Then,* $W_\psi \mu \in L^1(\omega_2)$ *and*

$$\overline{U}_\psi \mu = W_\psi \mu + \int_0^1 \frac{\psi(t)}{t}\, dt \cdot \mu(\{0\})\delta_0$$

for $\mu \in M(\omega_1)$*. In particular,* $\operatorname{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ *and* $\overline{U}_\psi$ *maps* $M((0,\infty), \omega_1)$ *into* $L^1(\omega_2)$*.*

Proof. By Corollary 2.2, we have $\int_0^1 \psi(t)/t\, dt < \infty$, so it follows that $\psi(0) = 0$. Let $\mu \in M(\omega_1)$ with $\mu(\{0\}) = 0$. By condition (C), we have

$$\begin{aligned}\int_{(0,\infty)} \int_s^\infty \frac{1}{x}\, \psi(s/x)\omega_2(x)\, dx\, d|\mu|(s) &= \int_{(0,\infty)} \int_0^1 \omega_2(s/t)\, \frac{\psi(t)}{t}\, dt\, d|\mu|(s) \\ &\leq C \int_{(0,\infty)} \omega_1(s)\, d|\mu|(s) = C\|\mu\|_{M(\omega_1)} < \infty,\end{aligned}$$

so it follows from Fubini's theorem that

$$\int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x)\, d|\mu|(s)\, \omega_2(x)\, dx < \infty.$$

Hence, $W_\psi \mu \in L^1(\omega_2)$. Moreover, for $g \in C_0(1/\omega_2)$, we have

$$\begin{aligned}\langle g, \overline{U}_\psi \mu\rangle = \langle V_\psi g, \mu\rangle &= \int_{(0,\infty)} \int_0^1 g(s/t)\, \frac{\psi(t)}{t}\, dt\, d\mu(s) \\ &= \int_{(0,\infty)} \int_s^\infty \frac{g(x)}{x}\, \psi(s/x)\, dx\, d\mu(s) \\ &= \int_0^\infty \frac{1}{x} \int_{(0,x)} \psi(s/x)\, d\mu(s)\, g(x)\, dx \\ &= \int_0^\infty (W_\psi \mu)(x) g(x)\, dx = \langle g, W_\psi \mu\rangle,\end{aligned}$$

so we conclude that $\overline{U}_\psi \mu = W_\psi \mu$. Finally, for $g \in C_0(1/\omega_2)$, we have

$$\langle g, \overline{U}_\psi \delta_0\rangle = \langle V_\psi g, \delta_0\rangle = (V_\psi g)(0) = g(0) \int_0^1 \frac{\psi(t)}{t}\, dt = \langle g, \int_0^1 \frac{\psi(t)}{t}\, dt \cdot \delta_0\rangle.$$

Since $W_\psi \delta_0 = 0$ this finishes the proof. □

The conclusion about the range of $\overline{U}_\psi$ can be generalized to the case, where ψ is not assumed to be continuous.

PROPOSITION 3.4. *Let* ψ *be a non-negative, measurable function on* $[0, 1]$ *and let* ω_1 *and* ω_2 *be positive, continuous functions on* $\mathbb{R}^+$*. Assume that condition (C) is satisfied so that* $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ *is a bounded operator. Then,* $\operatorname{ran} \overline{U}_\psi \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$*.*

Proof. Choose a sequence of non-negative, continuous functions (ψ_n) on $[0, 1]$ with $\psi_n \leq \psi$ and

$$\int_0^1 \frac{\psi(t) - \psi_n(t)}{t}\, dt \to 0 \qquad \text{as } n \to \infty.$$

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For $\mu \in M(\omega_1)$ and $g \in C_0(1/\omega_2)$, we have

$$\begin{aligned} |\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n})\mu\rangle| &= |\langle (V_\psi - V_{\psi_n})g, \mu\rangle| \\ &= \left| \int_{\mathbb{R}^+} \int_0^1 g(x/t)\, \frac{\psi(t) - \psi_n(t)}{t}\, dt\, d\mu(x) \right| \\ &\leq \|g\|_{C_0(1/\omega_2)} \int_{\mathbb{R}^+} \int_0^1 \omega_2(x/t)\, \frac{\psi(t) - \psi_n(t)}{t}\, dt\, d|\mu|(x). \end{aligned}$$

Let

$$p_n(x) = \int_0^1 \omega_2(x/t)\, \frac{\psi(t) - \psi_n(t)}{t}\, dt$$

for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. By condition (C), there exists a constant C such that $p_n(x) \leq C\omega_1(x)$ for every $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$. Moreover, for every $x \in \mathbb{R}^+$, we have $p_n(x) \to 0$ as $n \to \infty$ by Lebesgue's dominated convergence theorem. Hence,

$$\|(\overline{U}_\psi - \overline{U}_{\psi_n})\mu\|_{M(\omega_2)} = \sup_{\|g\|_{C_0(1/\omega_2)} \leq 1} |\langle g, (\overline{U}_\psi - \overline{U}_{\psi_n})\mu\rangle| \leq \int_{\mathbb{R}^+} p_n(x)\, d|\mu|(x) \to 0$$

as $n \to \infty$ again by Lebesgue's dominated convergence theorem. Consequently, $\overline{U}_{\psi_n} \to \overline{U}_\psi$ strongly as $n \to \infty$. Since ran $\overline{U}_{\psi_n} \subseteq L^1(\omega_2) \oplus \mathbb{C}\delta_0$ for $n \in \mathbb{N}$ by Proposition 3.3, the same thus holds for ran $\overline{U}_\psi$. □

COROLLARY 3.5. *Let ψ be a non-negative, measurable function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator. For $s > 0$, we then have $(\overline{U}_\psi \delta_s)(x) = \psi(s/x)/x$ for almost all $x \geq s$ and $(\overline{U}_\psi \delta_s)(x) = 0$ for almost all $x < s$.*

Proof. For ψ continuous, this follows from Proposition 3.3. For general ψ, it follows from the approach in the proof of Proposition 3.4 using $\overline{U}_{\psi_n} \to \overline{U}_\psi$ strongly as $n \to \infty$. □

It follows from Corollary 3.5 that

$$\|\overline{U}_\psi \delta_s\|_{M(\omega_2)} = \int_s^\infty \frac{\omega_2(x)}{x}\, \psi(s/x)\, dx = \int_0^1 \omega_2(s/t)\, \frac{\psi(t)}{t}\, dt,$$

whereas $\|\delta_s\|_{M(\omega_1)} = \omega_1(s)$. Since $\overline{U}_\psi$ is bounded we thus recover condition (C). If we without using Theorem 2.1 could show that if $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ is a bounded operator, then is has a bounded extension $\overline{U}_\psi : M(\omega_1) \to M(\omega_2)$ for which Corollary 3.5 holds, then we would in this way obtain an alternative proof of condition (C).

4. Weakly compact operators. We finish the paper by showing that there are no non-zero, weakly compact generalized Hardy–Cesàro operators between $L^1(\omega_1)$ and $L^1(\omega_2)$.

PROPOSITION 4.1. *Let ψ be a non-negative, measurable function on* $[0, 1]$ *and let ω_1 and ω_2 be positive, continuous functions on $\mathbb{R}^+$. Assume that condition (C) is satisfied*

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so that $U_\psi : L^1(\omega_1) \to L^1(\omega_2)$ *is a bounded operator. If* $\psi \neq 0$*, then* U_ψ *is not weakly compact.*

Proof. For $f \in L^1(\omega_1)$ and $x \in \mathbb{R}^+$, we have

$$(U_\psi f)(x) = \frac{1}{x}\int_0^x f(s)\psi(s/x)\,ds = \int_0^\infty f(s)\rho(s)(x)\omega_1(s)\,ds,$$

where (with a slight change of notation compared to the proof of Theorem 2.1)

$$\rho(s)(x) = \frac{1}{\omega_1(s)}\,\frac{1}{x}\,\psi(s/x)\,1_{x\geq s}$$

for $x, s \in \mathbb{R}^+$. In the proof of Theorem 2.1, we saw that $\rho(s) \in L^1(\omega_2)$ with $\|\rho(s)\|_{L^1(\omega_2)} \leq C$ for a constant C for almost all $s \in \mathbb{R}^+$. It thus follows from [**4**, Theorem VI.8.10] that U_ψ is weakly compact if and only if $\{\rho(s) : s \in \mathbb{R}^+\}$ is contained in a weakly compact set of $L^1(\omega_2)$ (except possibly for s belonging to a null-set). Consider $\rho(s)$ as an element of $C_0(1/\omega_2)^*$ for $s \in \mathbb{R}^+$ and let $g \in C_0(1/\omega_2)$. Then,

$$\begin{aligned}\langle g, \rho(s)\rangle &= \int_0^\infty g(x)\rho(s)(x)\,dx\\ &= \frac{1}{\omega_1(s)}\int_s^\infty \frac{g(x)}{x}\,\psi(s/x)\,dx\\ &= \frac{1}{\omega_1(s)}\int_0^1 g(s/t)\,\frac{\psi(t)}{t}\,dt.\end{aligned}$$

Since $g(s/t) \to g(0)$ as $s \to 0_+$ for all $t > 0$, it follows from Lebesgue's dominated convergence theorem that

$$\langle g, \rho(s)\rangle \to \frac{1}{\omega_1(0)}\,g(0)\int_0^1 \frac{\psi(t)}{t}\,dt$$

as $s \to 0_+$. We, therefore, conclude that

$$\rho(s) \to \frac{1}{\omega_1(0)}\int_0^1 \frac{\psi(t)}{t}\,dt \cdot \delta_0$$

weak-star in $M(\omega_2)$ as $s \to 0_+$. Since $\delta_0 \notin L^1(\omega_2)$, it follows that $\{\rho(s) : s \in \mathbb{R}^+\}$ is not contained in a weakly compact set of $L^1(\omega_2)$ (even excepting null sets), and the result follows. □

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