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OSCILLATION THEOREMS FOR SECOND ORDER SUPERLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract

Some oscillation criteria for solutions of a general second order ordinary superlinear differential equation

$$(a(t)x'(t))' + p(t)x'(t) + g(t)f(x(t)) = 0,$$

with alternating coefficients are given. The results generalize and complement some existing results in the literature.

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1. Introduction

This paper deals with the question of oscillation of all solutions of second order ordinary differential equations with alternating coefficients of the form

$$(1) \qquad (a(t)x'(t))' + p(t)x'(t) + q(t)f(x(t)) = 0,$$

where a, p, q: $[t_0, \infty) \to R = (-\infty, \infty)$, f: $R \to R$ are continuous and $a(t) > 0$ for $t \geq t_0 > 0$.

We assume that

$$(2) \qquad xf(x) > 0 \quad \text{and} \quad f'(x) \geq 0 \quad \text{for } x \neq 0.$$

Also, we suppose that equation (1) is strongly superlinear in the sense that

$$(3) \qquad \int^{\infty} \frac{du}{f(u)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{du}{f(u)} < \infty.$$

In the absence of damping, there is a large body of literature concerning the equations

$$(*) \qquad x''(t) + q(t)f(x(t)) = 0$$

and

$$(**) \qquad (a(t)x'(t))' + q(t)f(x(t)) = 0$$

Althouogh (1) can be easily transformed to the forms $(*)$ and $(**)$, there are advantages in obtaining direct oscillation theorems for (1); besides the obvious practical advantages of eliminating the need for finding an integrating factor, there is an incentive, as remarked by Yan [12], in developing methods which will generalize to more general equations.

Throughout this paper we restrict our attention only to the solutions of equation (1) which exist on some ray $[t_0, \infty)$, $t_0 > 0$. Such a solution is said to be oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. Equation (1) is called oscillatory if all of its solutions are oscillatory.

It is of interest to discuss conditions on the alternating coefficients p and q which are sufficient for (1) to be oscillatory. The use of average functions in the study of oscillation has been made extensively, most recently, by Butler [1], Grace and Lalli [2, 4], Kwong and Wong [6], Philos [8], Wong [11] and Yan [12]. In this study we deal with the possibility of averaging techniques for studying the oscillatory behavior of the superlinear differential equation (1).

J. S. W. Wong [11] proved the following oscillation theorem for the Emden-Fowler equation

$$(4) \qquad x''(t) + q(t)|x(t)|^{\lambda} \operatorname{sgn} x(t) = 0, \qquad \lambda > 0,$$

where q is a continuous function on $[t_0, \infty)$.

THEOREM W. *If*

$$(5) \qquad \liminf_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > 0$$

and

$$(6) \qquad \limsup_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\,ds = \infty \quad \textit{for some integer } n > 2,$$

then equation (4) *is oscillatory for all* $\lambda > 0$.

Recently, Philos [9] extended Theorem W to more general second order differential equations of the type

$$(7) \qquad x''(t) + q(t)f(x(t)) = 0,$$

where the functions q and f are defined as in equation (1), and obtained the following oscillation results.

THEOREM P. *Let conditions* (2) *and* (3) *hold, and suppose*

$$(8)\qquad \int^{\infty} \frac{\sqrt{f'(u)}}{f(u)}du < \infty \quad \text{and} \quad \int^{-\infty} \frac{\sqrt{f'(u)}}{f(u)}du < \infty$$

and

$$(9)\quad \min\left\{\inf_{u>0}\sqrt{f'(u)}\int_u^{\infty}\frac{\sqrt{f'(z)}}{f(z)}dz\,,\ \inf_{u<0}\sqrt{f'(u)}\int_u^{-\infty}\frac{\sqrt{f'(z)}}{f(z)}dz\right\} > 0.$$

Moreover, suppose that there exists a continuously differentiable function

$$\rho\colon\ [t_0\,,\infty) \to (0\,,\infty)$$

such that ρ' *is nonnegative and decreasing on* $[t_0\,,\infty)$. *Then equation* (7) *is oscillatory if*

$$(10)\qquad \liminf_{t\to\infty}\int_{t_0}^{t}\rho(s)q(s)\,dx > -\infty\,,$$

$$(11)\qquad \limsup_{t\to\infty}\frac{1}{t^2}\int_{t_0}^{t}\rho(s)\left(\int_{t_0}^{s}\frac{du}{\rho(u)}\right)ds < \infty$$

and

$$(12)\quad \limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_0}^{t}(t-s)^{n-1}\rho(s)q(s)\,ds = \infty \quad \textit{for some integer } n > 2.$$

Of particular interest, therefore, is the problem of finding oscillation criteria when condition (6), (11) or (12) is not satisfied. The answer to this problem for equation (1) with $f(x) = x$ was given by Yan [12], while for equation (4) with $0 < \lambda < 1$ and equation (7) with $\int_{+0}(1/f(u))/du < \infty$ was given by Kwong and Wong [6] and Philos [8] respectively.

In this paper we establish a new oscillation criterion for the superlinear equation (1) when condition (12) is not satisfied. This work complements the works of Kwong and Wong [6], Philos [8] and Yan [12]. Some of the earlier results of the authors are also extended.

Main results

THEOREM 1. *Let conditions* (2), (3), (8) *and* (9) *hold, and suppose that there exists a differentiable function* $\rho\colon\ [t_0\,,\infty) \to (0\,,\infty)$ *such that condition*

(10) *is satisfied and that,*

$$a(t)\rho'(t) - p(t)\rho(t) = \gamma(t) \geq 0 \quad \textit{and} \quad \gamma'(t) \leq 0 \quad \textit{for } t \geq t_0, \tag{13}$$

and

$$\int^{\infty} \xi(s)\,ds = \infty, \quad \textit{where } \xi(t) = \frac{1}{a(t)\rho(t)}. \tag{14}$$

If there exists an integer $\beta > 1$ and a continuous function ψ: $[t_0, \infty] \to R$ such that for any constant $c^* > 0$

$$\liminf_{t\to\infty} \frac{1}{t^\beta} \int_s^t (t-u)^{\beta-2} \times \left[(t-u)^2\rho(u)q(u) - \frac{c^*}{4V(u)}[\gamma(u)\xi(u)(t-u) - \beta]^2\right] du \geq \psi(s), \tag{15}$$

and

$$\lim_{t\to\infty} \int_{t_0}^t V(s)\psi_+^2(s)\,ds = \infty, \tag{16}$$

where $V(t) = \xi(t)/\int_{t_0}^t \xi(s)\,ds$ and $\psi_+(t) = \max\{\psi(t), 0\}$, then equation (1) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1). Without loss of generality we assume that $x(t) \neq 0$ for $t \geq t_0 > 0$. Furthermore, we suppose that $x(t) > 0$ for $t \geq t_0$, since the substitution $y = -x$ transforms equation (1) into an equation of the same form subject to the assumptions of the theorem.

Let W be defined by

$$W(t) = \rho(t)\frac{a(t)x'(t)}{f(x(t))}.$$

Then for $t \geq t_0$, we have

$$W'(t) = -\rho(t)q(t) + \gamma(t)\frac{x'(t)}{f(x(t))} - \frac{1}{a(t)\rho(t)}W^2(t)f'(x(t)), \tag{17}$$

and consequently

(18)

$$W(t) = W(t_0) - \int_{t_0}^t \rho(s)q(s)\,ds + \int_{t_0}^t \gamma(s)\frac{x'(s)}{f(x(s))}\,ds - \int_{t_0}^t \xi(s)W^2(s)f'(x(s))\,ds.$$

By the Bonnet theorem, for a fixed $t \geq t_0$ and for some $\eta \in [t_0, t]$

$$\begin{aligned}\int_{t_0}^{t} \gamma(s)\frac{x'(s)}{f(x(s))}\,ds &= \gamma(t_0)\int_{t_0}^{\eta} \frac{x'(s)}{f(x(s))}\,ds \\ &= \gamma(t_0)\int_{x(t_0)}^{x(\eta)} \frac{du}{f(u)} \qquad (19) \\ &\leq \gamma(t_0)\int_{x(t_0)}^{\infty} \frac{du}{f(u)} = K.\end{aligned}$$

Therefore, by taking into account (19), we conclude that

$$W(t) \leq L - \int_{t_0}^{t} \rho(s)q(s)\,ds - \int_{t_0}^{t} \xi(s)W^2(s)f'(x(s))\,ds\,, \qquad t \geq t_0, \tag{20}$$

where $L = K + W(t_0)$.

We consider the following two cases.

CASE 1. The integral

$$\int_{t_0}^{\infty} \xi(s)W^2(s)f'(x(s))\,ds$$

is finite. There exists a positive constant K_1 so that

$$\int_{t_0}^{t} \xi(s)W^2(s)f'(x(s))\,ds \leq K_1 \quad \text{for } t \geq t_0. \tag{21}$$

Furthermore, by using the Schwarz inequality, for $t \geq t_0$, we get

$$\begin{aligned}\left|\int_{t_0}^{t} \frac{x'(s)}{f(x(s))}\sqrt{f'(x(s))}\,ds\right|^2 &= \left|\int_{t_0}^{t} \sqrt{\xi(s)}\left(\sqrt{\xi(s)}W(s)\sqrt{f'(x(s))}\right)\,ds\right|^2 \\ &\leq \left(\int_{t_0}^{t} \xi(s)\,ds\right)\left(\int_{t_0}^{t} \xi(s)W^2(s)f'(x(s))\,ds\right).\end{aligned}$$

So, in view of (21) we have

$$\left|\int_{t_0}^{t} \frac{x'(s)}{f(x(s))}\sqrt{f'(x(s))}\,ds\right| \leq K_1\int_{t_0}^{t} \xi(s)\,ds \quad \text{for } t \geq t_0. \tag{22}$$

From (9), we obtain

$$\sqrt{f'(x(t))}\int_{x(t)}^{\infty} \frac{\sqrt{f'(u)}}{f(u)}\,du \geq M \quad \text{for } t \geq t_0, \tag{23}$$

where M is a positive constant. Next, we put

$$M_1 = \int_{x(t_0)}^{\infty} \frac{\sqrt{f'(u)}}{f(u)}\,du > 0$$

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and by (23), we have

$$\begin{aligned} f'(x(t)) &\geq M^2\left[\int_{x(t)}^{\infty}\frac{\sqrt{f'(u)}}{f(u)}\,du\right]^{-2} \\ &= M^2\left[M_1-\int_{x(t_0)}^{x(t)}\frac{\sqrt{f'(u)}}{f(u)}\,du\right]^{-2} \\ &= M^2\left[M_1-\int_{t_0}^{t}\frac{x'(s)}{f(x(s))}\sqrt{f'(x(s))}\,ds\right]^{-2} \\ &\geq M^2\left[M_1+\left|\int_{t_0}^{t}\frac{x'(s)}{f(x(s))}\sqrt{f'(x(s))}\,ds\right|\right]^{-2}. \end{aligned}$$

Thus, by (23), we get

$$f'(x(t)) \geq M^2\left[M_1+\left(K_1\int_{t_0}^{t}\xi(s)\,ds\right)^{1/2}\right]^{-2}.$$

There exists a positive constant c (depending on the constants M, M_1 and K_1) and a $T > t_0$ so that

$$f'(x(t)) \geq \frac{c}{\int_{t_0}^{t}\xi(s)\,ds} \quad \text{for } t \geq T. \tag{24}$$

Using (24) in equation (17) we obtain

$$W'(t) \leq -\rho(t)q(t)+\gamma(t)\xi(t)W(t)-c\frac{\xi(t)}{\int_{t_0}^{t}\xi(s)\,ds}W^2(t), \qquad t \geq T,$$

or

$$W'(t) \leq -\rho(t)q(t)+\gamma(t)\xi(t)W(t)-cV(t)W^2(t) \quad \text{for } t \geq T \tag{25}$$

and consequently, for $t > s \geq T$

$$\begin{aligned} \int_s^t (t-u)^\beta W'(u)\,du \leq &-\int_s^t (t-u)^\beta \rho(u)q(u)\,du \\ &-\int_s^t (t-u)^\beta [cV(u)W^2(u)-\gamma(u)\xi(u)W(u)]\,du. \end{aligned}$$

Since

$$\int_s^t (t-u)^\beta W'(u)\,du = -(t-s)^\beta W(s)+\beta\int_s^t (t-u)^{\beta-1}W(u)\,du,$$

we obtain that

(26)

$$\begin{aligned} &\int_s^t (t-u)^\beta \rho(u)q(u)\,du \leq (t-s)^\beta W(s) \\ &\quad -\int_s^t [(t-u)^\beta cV(u)W^2(u)-(t-u)^{\beta-1}[\gamma(u)\xi(u)(t-u)-\beta]W(u)]\,du \end{aligned}$$

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and hence

$$
\begin{aligned}
&\int_s^t \left[(t-u)^\beta \rho(u)q(u) - \frac{1}{4c}\frac{(t-u)^{\beta-2}}{V(u)}[\gamma(u)\xi(u)(t-u)-\beta]^2\right] du \\
&\quad \le (t-s)^\beta W(s) - \int_s^t \left[\sqrt{cV(u)}(t-u)^{\beta/2} W(u) \right. \\
&\qquad \left. - \frac{1}{2\sqrt{cV(u)}}(t-u)^{\beta/2}[\gamma(u)\xi(u)(t-u)-\beta]\right]^2 du \qquad (27)\\
&\quad \le (t-s)^\beta W(s) \quad \text{for } s \ge T.
\end{aligned}
$$

Dividing (27) by t^β and taking the lower limit as $t \to \infty$, we obtain

$$\psi(s) \le W(s), \qquad s \ge T,$$

which implies that

$$\psi_+^2(s) \le W^2(s). \tag{28}$$

In view of (12) and (24) we have

$$
\begin{aligned}
\lim_{t\to\infty} \int_T^t cV(s)\psi_+^2(s)\,ds &\le \lim_{t\to\infty} \int_T^t cV(s)W^2(s)\,ds \\
&\le \lim_{t\to\infty} \int_T^t \xi(s)W^2(s)f'(x(s))\,ds < \infty.
\end{aligned}
$$

This contradicts condition (16).

CASE 2. The integral

$$\int_{t_0}^t \xi(s)W^2(s)f'(x(s))\,ds$$

is infinite. By (10), it follows from (20) that for some constant σ

$$-W(t) \ge \sigma + \int_{t_0}^t \xi(s)W^2(s)f'(x(s))\,ds \quad \text{for } t \ge t_0. \tag{29}$$

We choose a $T^* \ge t_0$ so that

$$\theta \equiv \sigma + \int_{t_0}^{T^*} \xi(s)W^2(s)f'(x(s))\,ds > 0.$$

Then (29) ensures that W is negative on $[T^*, \infty)$. Now, (39) gives

$$\frac{\xi(t)W^2(t)f'(x(t))}{\sigma + \int_{t_0}^t \xi(s)W^2(s)f'(x(s))\,ds} \ge -\frac{x'(t)f'(x(t))}{f(x(t))}, \qquad t \ge T^*,$$

and consequently for all $t \ge T^*$

$$\log\frac{1}{\theta}\left[\sigma + \int_{t_0}^t \xi(s)W^2(s)f'(x(s))\,ds\right] \ge \log\frac{f(x(T^*))}{f(x(t))}.$$

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Hence

$$\sigma + \int_{t_0}^{t} \xi(s)W^2(s)f'(x(s))\,ds \geq \theta \frac{f(x(T^*))}{f(x(t)} \quad \text{for } t \geq T^*.$$

So, (29) yields

$$x'(t) \leq -\theta f(x(T^*))\xi(t) < 0 \quad \text{for } t \geq T^*.$$

Thus, we get

$$x(t) \leq x(T^*) - \theta f(x(T^*)) \int_{T^*}^{t} \xi(s)\,ds \quad \text{for } t \geq T^*$$

which, in view of (14), leads to the contradiction

$$\lim_{t\to\infty} x(t) = -\infty.$$

This completes the proof.

The following theorem extends and improves [9, Theorem 2] and [11, Theorem 1].

THEOREM 2. *Let conditions* (2), (3), (8) *and* (9) *hold and suppose that there exists a differentiable function* ρ: $[t_0, \infty) \to (0, \infty)$ *such that conditions* (9), (13) *and* (14) *are satisfied. If there exists an integer* $\beta > 1$ *such that for any constant* $c^* > 0$

(30)

$$\limsup_{t\to\infty} \frac{1}{t^\beta} \int_{t_0}^{t} (t-u)^{\beta-2}\left[(t-u)^2\rho(u)q(u) - \frac{c^*}{4V(u)}[\gamma(u)\xi(u)(t-u)-\beta]^2\right] du = \infty,$$

where ξ *and* V *are defined as in Theorem 1, then equation (1) is oscillatory.*

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \geq t_0 > 0$. As in the proof of Theorem 1 (Case 1), we obtain (27). Dividing (27) by t^β and taking the upper limit as $t \to \infty$, we get a contradiction to condition (30). The rest of the proof is similar to the proof of Theorem 1 (Case 2) and hence is omitted.

REMARKS. 1. The results of this paper are new even when specialized to equations (4) with $\lambda > 1$ and (7).

2. If $f(x) = |x|^\lambda \operatorname{sgn} x$, $\lambda > 1$, then conditions (2), (3), (8) and (9) are disregarded.

3. Theorem 1 is a complement of the results obtained by Kwong and Wong [6] for equation (4) with $0 < \lambda < 1$, by Philos for equation (7) (where f satisfies the condition $\int_{\pm 0}(1/f(u))du < \infty$), and by Yan [12] for equation (1) with $f(x) = x$.

4. Theorem 2 extends and improves some of the results in [1], [9, Theorem 2] and [11, Theorem 1]. It complements the work of Grace [2], Grace and Lalli [4], Kamenev [5], Yan [12] and Yeh [13] for equation (1) and/or related equations with $f(x) = x$ or where f satisfies the condition: $f'(x) \geq K > 0$ for $x \neq 0$; and Grace and Lalli [3] for equation (1) with $a(t) = 1$ and $f(x) = |x|^\lambda \operatorname{sgn} x$, $0 < \lambda < 1$.

5. One can easily deduce many corollaries from Theorems 1 and 2. We omit the details.

For illustration we consider the following example.

EXAMPLE 1. Consider the second order equation

$$x''(t)+(\cos t)|x(t)|^\lambda[\delta^*+\sin\log(1+|x(t)|)]\operatorname{sgn} x(t) = 0 \quad \text{for } t \geq t_0 > 0, \tag{31}$$

where $\lambda > 1$ and $\delta^* > 1 + (1/\lambda)$. It is easy to check that conditions (2), (3), (8) and (9) are satisfied (see [7]).

Taking $\rho(t) = 1$ and $\beta = 2$, we get

$$\gamma(t) = 0, \quad V(t) = \frac{1}{t - t_0}, \qquad t > t_0 > 0,$$

$$\liminf_{t\to\infty} \int_{t_0}^{t} \cos s\, ds > -\infty,$$

$$\limsup_{t\to\infty} \frac{1}{t^2} \int_{t_0}^{t} (t-u)^2 \cos u\, du = -\sin t_0 < \infty,$$

$$\liminf_{t\to\infty} \frac{1}{t^2} \int_{s}^{t} \left[(t-u)^2 \cos u - \frac{c^*}{u - t_0}\right] du \geq -\sin s - K,$$

where c^* and K are positive constants and K is small. Set

$$\psi(s) = -\sin s - K.$$

Next, we consider an integer N such that $(2N+1)\pi + (\pi/4) > t_0$. Then for all integers $n \geq N$ and

$$(2n+1)\pi + \frac{\pi}{4} \leq s \leq 2(n+1)\pi - \frac{\pi}{4}, \; \psi(s) = -\sin s - K \geq \delta s,$$

where δ is a small constant. Thus,

$$\lim_{t\to\infty} \int_{t_0}^{t} V(s)\psi_+^2(s)\, ds \geq \sum_{N=n}^{\infty} \delta^2 \int_{(2n+1)\pi+\frac{\pi}{4}}^{2(n+1)\pi-\frac{\pi}{4}} s\, ds = \infty.$$

All the conditions of Theorem 1 are satisfied and hence equation (31) is oscillatory.

We note that Theorems P and W are not applicable to equation (31). Moreover, none of the known oscillation criteria can cover this result.

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