Statistical Analyses of Hellin's Law

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During the history of research on multiple maternities, Hellin's law has played a central role as a rule of thumb. It is mathematically simple and approximately correct, but shows discrepancies that are difficult to explain or to eliminate. It has been mathematically proven that Hellin's law does not hold as a general rule. Varying improvements to this law have been proposed. In this paper, we consider how Hellin's law can be used and tested in statistical analyses of the rates of multiple maternities. Such studies can never confirm the law, but only identify errors too large to be characterized as random. It is of particular interest to determine why the rates of higher multiple maternities are sometimes too high or too low when Hellin's law is used as a benchmark. Excesses of triplet and quadruplet maternities are particularly unexpected and challenging. Our analyses of triplet and quadruplet rates indicated that triplet rates are closer to Hellin's law than quadruplet rates. According to our analyses of the twinning rate and the transformed triplet rate and quadruplet rate for Sweden (1751–2000), both triplet and quadruplet rates showed excesses after the 1960s. This is mainly caused by artificial fertility-enhancing reproduction technologies. Regression analyses of twinning and triplet rates yield rather good fits, but deficiencies in the triplet rates are commonly present. We introduced measures of concordance between triplet rates with Hellin's law. According to these measures, historic data showed deficiencies in triplet rates, but recent data revealed excesses, especially found among older mothers. The excesses obtained are in good agreement with other studies of recent data.

Keywords: artificial reproduction technologies, confidence intervals, measures of agreements, multiple maternities, regression models

Hellin's law has played a central role in the history of research on multiple maternities. While in the literature authors generally refer to Hellin (1895), the law was already formulated by Strassmann in 1889. The contributions by Zeleny (1921) have resulted in the law also being known as the Hellin-Zeleny law. Fellman and Eriksson (2009) reviewed papers where the genesis of Hellin's law was traced and where the strengths and weaknesses of the law were analysed and improvements suggested. Usually the arguments for Hellin's law are based on stochastic models for multiple fertilizations and fissions of fertilized eggs (Zeleny, 1921; Jenkins, 1927, 1929; Jenkins & Gwin, 1940; Allen & Firschein, 1957). The influence of both multiple fertilizations and fissions of fertilized eggs has inspired scientists to associate the rates of higher multiple maternities with both monozygotic (MZ) and dizygotic (DZ) twinning rates (TWRs; e.g., Bulmer, 1970; Fellman & Eriksson, 2004). Peller (1946) was the first, at least indirectly, to connect Hellin's law to inter-individual variation in mothers' chances for multiple maternities. Later, Eriksson (1973) considered recurrent twin maternities in families on the Åland Islands (Finland) and presented a modified model (in the paper, the law called Fellman's law). When Eriksson applied this law to his Åland data, he obtained better congruence with Hellin's law than had Peller's version been applied. Fellman and Eriksson (1993) have given a mathematical proof that Hellin's law cannot hold as a general rule.

The interest in Hellin's law is mainly the result of its being mathematically simple and approximately correct, but it shows discrepancies that are difficult to explain or eliminate. Statistical studies on empirical rates of multiple maternities can never confirm the law, but only identify errors too large to be characterized as random. It is of particular interest to ask why the rates of higher orders of multiple maternities are sometimes too high and sometimes too low when Hellin's law is used as a benchmark.

Hellin's law presupposes strong correlations between the TWR and the triplet rate (TRR), but even strong correlations do not prove Hellin's law, but only a linear relationship. Fellman and Eriksson (2004) considered the correlation between the TWR and the square root of the TRR in Sweden. After elimination of influential temporal factors, they found that the correlation was positive, but not very strong. This finding indicates that, in general, Hellin's law cannot be exact. One application of Hellin's law is to compare the TWR and the square root of the TRR, the cubic root of the quadruplet rate (QUR), and so on (Fellman & Eriksson, 2006, 2009; Eriksson & Fellman, 2007).

Methods

A problem that complicates the discussion of Hellin's law is that the law is a mathematical rule concerning theoretical rates, but all checks of the law have to be based on empirically obtained rates. In fact, one can only check if the discrepancies are too large or cannot be explained by random errors. In this way, no exact proof to support the law can be obtained.

Standard Deviations and Confidence Intervals

In the following, we consider formulae applicable in the statistical analysis of Hellin's law. Let the theoretical TRR be r. One has different possibilities to study the random errors of the TRR and particularly of the square root of the TRR. The first one is to estimate the standard deviations (SDs) of the TRR and construct confidence intervals (CIs) for r. The square root is a monotone-increasing function, and consequently, one can construct the CI for $\sqrt{r}$ by a square root transformation of the limits of the CI for r. An alternative is to estimate the SD of the square root of the TRR and to use it in order to obtain the CI for $\sqrt{r}$.

Let the observed TRR be $\hat{r}$, Then

$$SD_{\hat{r}} = \sqrt{\frac{r(1-r)}{n}}$$

and the standard CI of r is

$$\left(\hat{r} - k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}, \hat{r} + k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}} \right), \quad (1)$$

where the factor k defines the confidence level. For $\sqrt{r}$, the corresponding CI is (say).

$$\left(\sqrt{\hat{r} - k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}}, \sqrt{\hat{r} + k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}} \right) = (L_1, U_1) \quad (2)$$

If we use the general approximate formula

$$Var(f(z)) \approx \left(\frac{df}{dz}\right)^2 Var(z), \text{ we obtain}$$

$$Var(\sqrt{\hat{r}}) \approx \left(\frac{1}{2\sqrt{r}}\right)^2 Var(\hat{r}) = \frac{1}{4r}\frac{r(1-r)}{n} = \frac{1-r}{4n} \quad (3)$$

and $SD_{\sqrt{\hat{r}}} \approx \sqrt{\frac{1-r}{4n}}$.

Now, the approximate CI for $\sqrt{r}$ is (say).

$$\left(\sqrt{\hat{r}} - k\sqrt{\frac{1-\hat{r}}{4n}}, \sqrt{\hat{r}} + k\sqrt{\frac{1-\hat{r}}{4n}} \right) = (L_2, U_2) \quad (4)$$

Fellman and Eriksson (1993) presented also the transformation $\nu = arcsin(\sqrt{r})$, yielding values close to $\sqrt{r}$

and having the simple variance formula

$$Var(\nu) \approx \frac{1}{4n},$$

which is independent of r. However, in connection with Hellin's law, we prefer the transformation $\sqrt{r}$, which is exactly associated with the law.

Although formula (3) is only approximate, we can prove that for large data sets the difference between the alternative CIs (2) and (4) is minute. Consider the difference between the upper limits of the CIs:

$$U_2 - U_1 = \sqrt{\hat{r}} + k\sqrt{\frac{1-\hat{r}}{4n}} - \sqrt{\hat{r} + k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}} =$$

$$\frac{\left(\sqrt{\hat{r}} + k\sqrt{\frac{1-\hat{r}}{4n}} - \sqrt{\hat{r} + k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}} \right) \times \left(\sqrt{\hat{r}} + k\sqrt{\frac{1-\hat{r}}{4n}} + \sqrt{\hat{r} + k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}} \right)}{U_2 + U_1} =$$

$$\frac{\left(\hat{r} + 2k\sqrt{\hat{r}\frac{1-\hat{r}}{4n}} + \frac{k^2(1-\hat{r})}{4n} - \left(\hat{r} + k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}} \right) \right)}{U_2 + U_1} =$$

$$= \frac{k\sqrt{\hat{r}\frac{1-\hat{r}}{n}} + \frac{k^2(1-\hat{r})}{4n} - k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}}{U_2 + U_1}$$

$$= \frac{k^2(1-\hat{r})}{4n}\left(\frac{1}{U_2 + U_1} \right). \quad (5)$$

In a similar way, we obtain

$$L_2 - L_1 = \sqrt{\hat{r}} - k\sqrt{\frac{1-\hat{r}}{4n}} - \sqrt{\hat{r} - k\sqrt{\frac{\hat{r}(1-\hat{r})}{n}}}$$

$$= \frac{k^2(1-\hat{r})}{4n}\left(\frac{1}{L_2 + L_1} \right). \quad (6)$$

Note that $U_2 > U_1$ and that $L_2 > L_1$. Consequently, the CI in (4) is slightly shifted upwards. When $n \to \infty$, the limits U_1, U_2, L_1, and L_2 remain finite at the same time as the factor

$$\frac{k^2}{4n}$$

converges towards zero. Consequently, for large n, the difference between the alternative CIs is minute and both are good alternatives.

For the QUR, the simplest CI for the cubic root is obtained if the cubic root transformation is applied on

the standard CI for the QUR. If the theoretical QUR is denoted q and the observed QUR $\hat{q}$, then

$$SD_{\hat{q}} = \sqrt{\frac{q(1-q)}{n}}$$

and the CI for the cubic root $\sqrt[3]{q}$ is

$$\left(\sqrt[3]{\hat{q} - k\sqrt{\frac{\hat{q}(1-\hat{q})}{n}}}, \sqrt[3]{\hat{q} + k\sqrt{\frac{\hat{q}(1-\hat{q})}{n}}} \right). \quad (7)$$

In the analyses of the data we use the standard CI for the TWR, the CIs (2) for the TRRs and (7) for the QURs.

Agreement Measures

In this section, we introduce measures to check both Hellin's law and Jenkins's (1927) model. We introduce the ratio

$$HR = \frac{TRR}{(TWR)^2},$$

named Hellin's ratio, and assume that it is a measure of the agreement with respect to Hellin's law. If $HR > 1$, there is an excess, but if $HR < 1$ there is a deficit in the TRR. An alternative measure is based on Jenkins's model

$$TRR = \frac{1}{n} \sum_i (TWR_i)^2 n_i. \quad (8)$$

We introduce the short notation

$$J = \frac{1}{n} \sum_i (TWR_i)^2 n_i$$

and define Jenkins's ratio as,

$$JR = \frac{TRR}{J},$$

where TRR is the total triplet rate. If $JR > 1$, there are excesses and if $JR < 1$ there are deficits in the TRRs. Hellin's ratio can be defined for both age-specific and total rates, but Jenkins's ratio applies only to total rates. In addition, Eq. (8) indicates that JR can be calculated only for data grouped according to maternal age.

According to Schwarz's inequality a comparison between HR for the total set of maternities and JR yields

$$(TWR)^2 = \left(\frac{1}{n} \sum_i (TWR_i) n_i \right)^2 \leq$$

$$\frac{1}{n} \sum_i (TWR_i \sqrt{n_i})^2 \frac{1}{n} \sum_i (\sqrt{n_i})^2$$

$$\leq \frac{1}{n} \sum_i (TWR_i)^2 n_i \frac{1}{n} \sum_i n_i = J$$

Equality is obtained if and only if

$$\frac{TWR_i \sqrt{n_i}}{\sqrt{n_i}} = TWR_i$$

for all i. Consequently,

$$HR = \frac{TRR}{(TWR)^2} \geq \frac{TRR}{J} = JR.$$

This relation can be seen in the last two lines in Table 2.

Results

In the following, we present alternative methods to identify discrepancies with respect to Hellin's law and to try to explain them. The comparisons of TWR and transformed TRR and QUR must observe the effect of random errors.

Excesses and Deficiencies

In Table 1, our comparisons are based on 95% CIs. We present the TWR per 10^3, the TRR per 10^6 and the QUR per 10^9 for the Prussian data presented in Veit (1855), the Wappäus (1859) data and the Swedish data for the period 1801–1850. Also included in the table are the square root of the TRR and the cubic root of the QUR given per 1000, the CIs for the TWR, the square root of the TRR and the cubic root of the QUR per 1000. The square roots of the TRR are slightly too high, but the corresponding CIs cover the CI for the TWR. The HR proposed in this paper indicates good agreement with Hellin's law. The cubic roots of the QURs are too high, indicating marked discrepancies from Hellin's law. In the Discussion section, we stress that these discrepancies are mainly caused by the Hellin transformations of the TRR and the QUR, and an attempt to eliminate these discrepancies is proposed. The corrected rates are included in Table 1 and Figure 1.

Figure 1 presents the TWR and the transformed TRR and QUR for the Prussian, the Wappäus' and the Swedish data presented in Table 1. Included in the figure are the corresponding CIs. To simplify the figure, the relatively short CIs for the TWRs are excluded. The figure indicates that for all data sets the TRRs are in good agreement with Hellin's law, but the QURs are markedly too high. It is a remarkable result that the QURs yield values that are too high rather that too low.

Temporal Trends

Following Eriksson and Fellman (2004, 2007) and Fellman & Eriksson (2004, 2006, 2009), we present in Figure 2 the temporal trends in TWR, the square root of TRR and the cubic root of QUR obtained from the Veit data (1855). Note, that the TRR shows stronger fluctuations than the TWR. However, the confidence bands included indicate that the TWR and the transformed TRR show good agreement for the whole period. The transformed QUR is too high for

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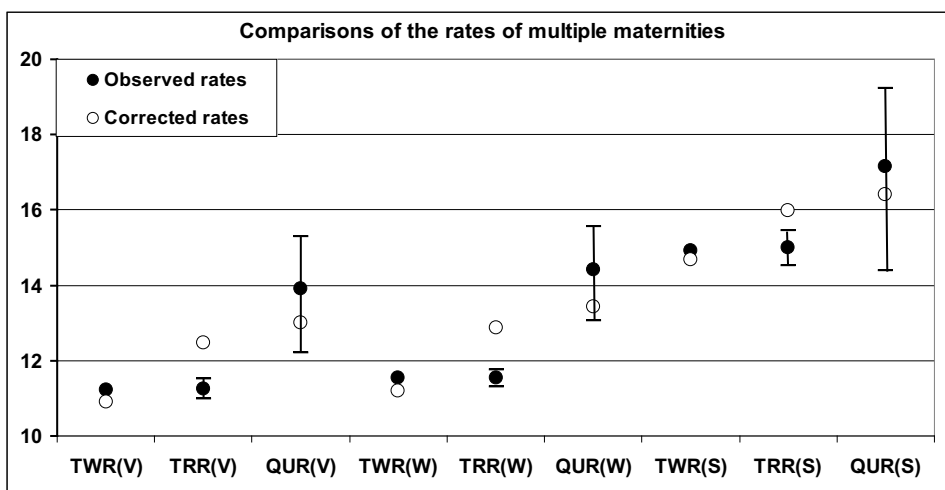
Figure 1

The twinning rate (TWR) and the transformed triplet rate (TRR) and quadruplet rate (QUR) according to Veit (V) (1855), Wappäus (W) (1859) and Sweden (S) (1801–1850).

Note: Included in the figure are the corresponding 95% CIs. To simplify the figure, particularly short CIs for the TWR are excluded. The TRRs support Hellin's law, but the QURs are markedly too high. The corrected rates described in the Discussion section are included.

Table 1

The Twinning Rate (TWR), the Triplet Rate (TRR) and the Quadruplet Rate (QUR) for the Data Presented by Veit (1855) and Wappäus (1859) Compared With Data for Sweden (1801–1850)

	Veit (1855)	Wappäus (1859)	Sweden 1801–1850
N[1]	13360557	19698322	4553518
TWR[2]	11.2244	11.5140	14.9146
CI	(11.168, 11.281)	(11.467, 11.561)	(14,803, 15.026)
TRR[3]	126.417	133.159	224.881
$\sqrt{TRR}$ [2]	11.2435	11.5394	14.9960
CI	(10,972, 11.508)	(11,316, 11.758)	(14.530, 15.448)
	1.003	1.004	1.011
QUR[4]	2694.498	2995.518	5051.040
$\sqrt[3]{TRR}$ [2]	13.9153	14.4148	17.1578
CI	(12,197, 15.290)	(13.067, 15.549)	(14,401, 19.234)
Estimation of corrected rates			
Correction factor	0.2219	0.2294	0.1519
TWR[2]	10.92	11.20	14.67
TRR[2]	12.47	12.87	15.98
QUR[2]	12.99	13.42	16.42

Note: [1] Number of maternities; [2] per 10^3; [3] per 10^6; [4] per 10^9.

Included in the table are $\sqrt{TRR}$ and $\sqrt[3]{QUR}$ and the 95% confidence intervals (CIs) for TWR, $\sqrt{TRR}$ and $\sqrt[3]{QUR}$. For the TRRs, Hellin's ratios (*HRs*) are also included. For all data sets, the *HRs* are close to one, indicating good agreement with Hellin's law.

almost the whole period. This is in good agreement with the result in Table 1 and Figure 1.

In Figure 3, we compare the TWR and the transformed TRR and QUR for Sweden (1751–2000). For the period 1871–1960, there is a deficiency in the TRR. Fellman and Eriksson (2009) discuss this deficiency in more detail. There is almost constantly an excess in the QUR for the whole period. After 1970, both the TRR and QUR show excesses, but this is mainly caused by the influence of the artificial repro-

duction technologies, particularly the use of fertility-enhancing drugs. For references, see Fellman and Eriksson (2006).

In the Gotland county, the TWR is almost always the highest regional TWR in Sweden (Fellman & Eriksson, 2003, 2005). As a comparison, we consider also the county of Älvsborg, known for its low TWR. Figure 4 shows the comparison between the TWR and the transformed TRR for Gotland and Älvsborg for the period 1751–1960. In Gotland, the transformed
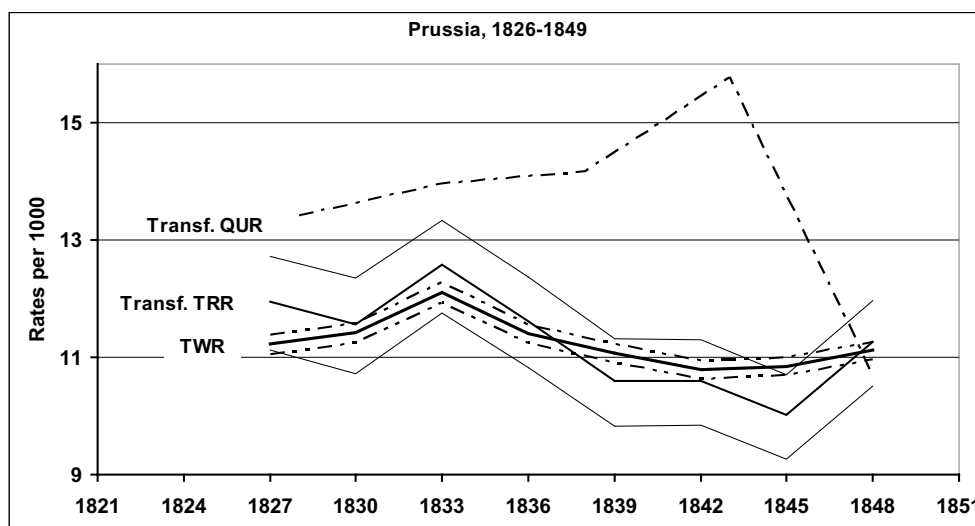
Figure 2

Temporal trends in the twinning rate (TWR) and the transformed triplet rate (TRR) and quadruplet rate (QUR) in Prussia, 1826–1849. The TRR shows stronger fluctuations than the TWR, but the confidence bands indicate that the difference can mainly be ascribed to random errors.

TRR is not as extreme as the TWR, and consequently, there is a better agreement between the TWR and the transformed TRR in Älvsborg than in Gotland.

Eriksson (1973) studied the TWR and the TRR in the south-western part of Finland. In the Åland islands, the TWR was continuously high. For the period 1653–1949, the TWR was 19.21 and the TRR was 375 per 10^6. According to Hellin's law, the expected TRR should have been 369 per 10^6. For the Åland data, $HR = 1.02$, showing a good agreement with Hellin's law. In the Åboland (Turunmaa in Finnish) archipelago, close to the Åland islands, the TWR was also high. For the period 1655–1949, the TWR was 20.90. For the same period the TRR was 252 per 10^6. According to Hellin's law, the expected TRR should be 437 per 10^6, yielding $HR = 0.58$, and consequently, the Åboland archipelago data showed, as did the county of Gotland, a marked deficit in TRR with respect to Hellin's law.

Agreement Measures

We analysed data from Finland 1881–2000, Denmark 1896–1980, USA 1923–1924 and 1927–1936 (Jenkins & Gwin, 1940) and England and Wales 1988–1991 and 1996–2003. For both Denmark and Finland, the series are divided into two half-periods, one early period before 1940 and one late period after 1940. In
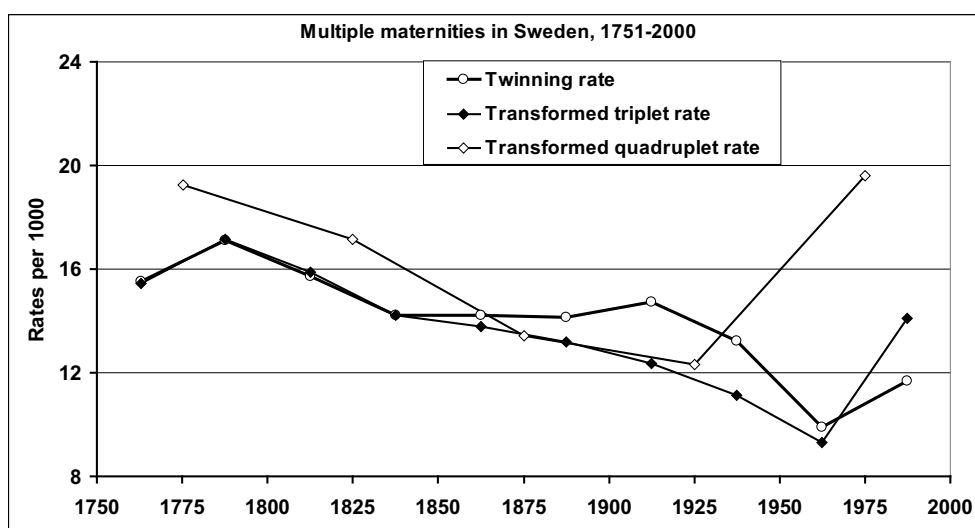
Figure 3

Temporal trends in the twinning rate (TWR) and the transformed triplet rate (TRR) and quadruplet rate (QUR) in Sweden, 1751–2000. For the period 1871–1960, the TRR shows a deficit compared with the TWR, and for the period after 1970 both the TRR and the QUR show excesses.

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Table 2

Hellin's Ratio (*HR*) for the Age-Specific and Total Rates in Different Countries According to Time Periods (The Last Line Contains Jenkins's Ratio [*JR*])

| Maternal age	Finland 1881–1940	Denmark 1896–1940	USA, 1923–24 and 1927–36	Denmark 1941–1980	Finland 1941–2000	England & Wales 1988–1991	England & Wales 1996–2003
under 20	1.0805	0.8775	0.9933	0.9107	1.4041	1.5775	1.5492
20–24	0.9201	0.7710	0.9132	1.4648	1.2956	1.4398	1.2804
25–29	0.6791	0.6426	0.7662	1.3933	0.8427	1.9483	1.8267
30–34	0.7290	0.8549	0.7295	1.5971	0.7861	2.2418	1.8694
35–39	0.7760	0.9290	0.6938	2.6292	0.5683	2.2764	1.6213
40–44	0.9068	1.5568	0.7279	5.1309	0.3722	1.3219	1.6529
over 45	1.0922	2.6506	1.5713	15.3033	. .	9.1882	1.7118
Total	0.8347	0.9287	0.8488	1.9659	0.8683	2.1393	1.9177
JR	0.7730	0.8637	0.7662	1.8159	0.8066	2.0058	1.7414

Note: The absence of excesses in the TRRs for Finland and the decrease in excesses for England and Wales after 1996. Note that the *HR* for the total data sets is always greater than the *JR*. This is proved in the text.

Table 2 and Figure 5, we collected the results obtained for different maternal age groups. We observed that the new measures give similar results. For the earlier data sets, there were deficits, but for the recent data sets there were excesses in triplet maternities. For Denmark (1941–1980) and for England and Wales (1988–1991), marked excesses in TRR were observed, especially among older mothers. These excesses coincided with the introduction of subfertility treatments, mainly ovulation inductions. For England and Wales (1996–2003), the excesses in TRR were still discernible, but the maternal age effects had almost disappeared. Our opinion is that this change is caused by changes in the fertilization policies, especially the reduction in the number of fertilized eggs implemented.

Eriksson and Fellman (2007) compared the rates of twin, triplet and quadruplet maternities in England and Wales for the period 1938–2003. All rates showed increasing trends during the later years. For the QUR, the increase started in 1966–1970, for the TRR in 1971–1975 and for the TWR in 1976–1980. The QUR showed the most marked increase. The start of the strong increases in the triplet and quadruplet rates coincided with the introduction of subfertility treatments, mainly ovulation inductions. After 1991–1995, the QUR and after 1996–2000 the TRR showed decreasing trends. These findings are in good agreement with the behavior in the *HR* and *JR* for the data from England and Wales (cf. Table 2).

Discrepancies obtained during the era of fertility treatments are of less interest when Hellin's law is considered because no natural stochastic model is applicable. Lam and Ho (1999) noted the increase in the number of multiple maternities in Hong Kong in
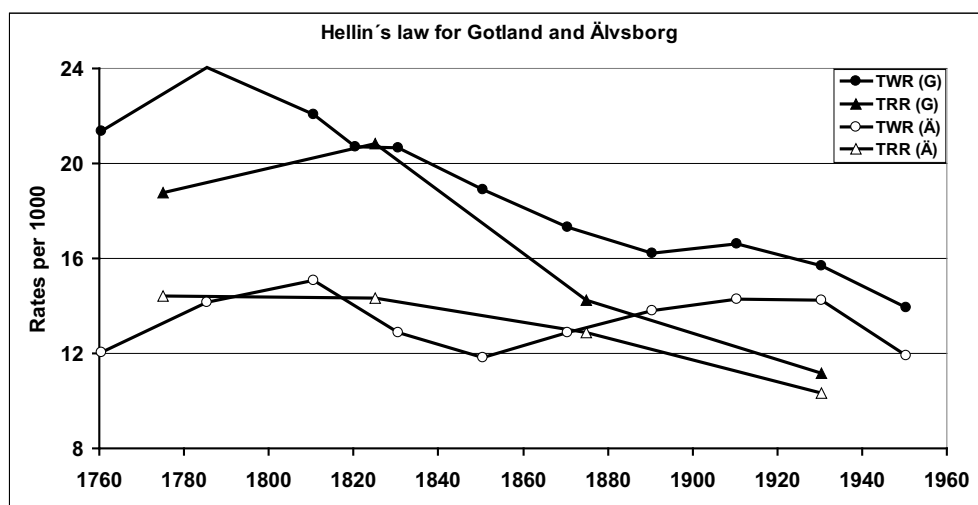
Figure 4

Temporal trends in the twinning rate (TWR) and the transformed triplet rate (TRR) in the Swedish counties Gotland (G), 1751–1869, and Älvsborg (Ä), 1751–1874. For almost the whole period, the TRR shows a deficit in Gotland compared with the TWR. For the whole period, the TRR in Älvsborg shows good agreement with the TWR.
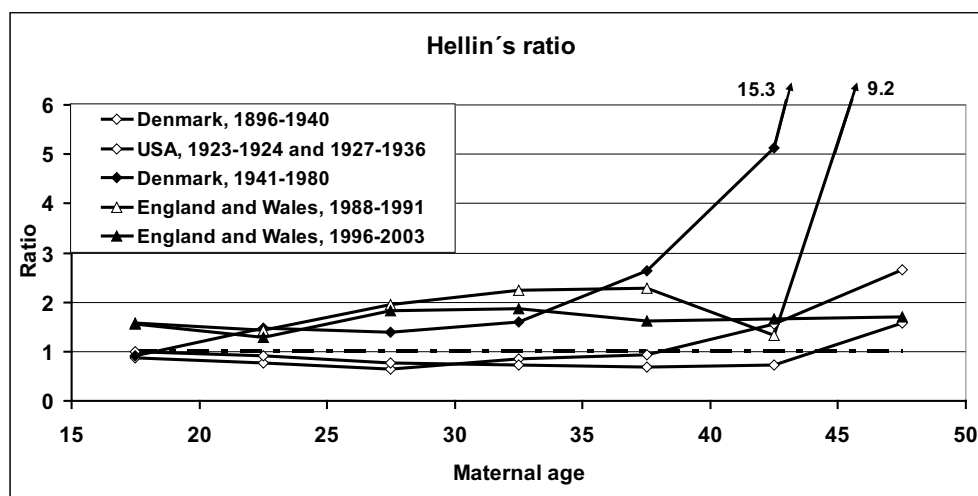
Figure 5

Discrepancies between the twinning rates (TWRs) and the triplet rates (TRRs) measured by Hellin's ratio (defined in the text). Hellin's law holds when the ratios are one. The data are from Denmark in 1896–1940 and 1941–1980, USA in 1923–1924 and 1927–1936 and England and Wales in 1988–1991 and 1996–2003. For Denmark in 1941–1980 and for England and Wales in 1988–1991, Hellin's ratios (HR) for older mothers are extremely high. The data from Finland in 1881–2000 are not included in the figure because they showed no marked excesses or deficits (see Table 2).

1981–1995. They also stressed the marked discrepancy between the observed data and Hellin's law.

Zhang et al. (2002) have observed similar increases in the rates of multiple maternities among older mothers in USA in 1995–1997, and they also attributed this finding to the increased use of assisted reproductive technology. Simmons et al. (2004) noted a dramatic decrease in the proportion of triplet and higher order births since 1998. These decreases were ascribed to changes in the treatment policies discussed above. Finally, no marked excesses in the TRRs for Finland were found.

Linear Models

Using linear curves is the best method for identifying discrepancies from a presumptive model because graphs containing linear curves are easy to interpret. There are two possibilities for checking Hellin's law with linear curves. One is to use graphs with TWR^2 as abscissa and TRR as ordinate, that is, to use the model $TRR = \alpha + \beta\ TWR^2$. An alternative is graphs with TWR as abscissa and $\sqrt{TRR}$ as ordinate. Now, the model is $\sqrt{TRR} = \alpha + \beta\ TWR$.

Jenkins and Gwin (1940) considered US data for the periods 1923–1924 and 1927–1936. They used TWR^2 as abscissa and TRR as ordinate. From their figure, they obtained the linear relation $TRR = 0.000013 + 0.656\ TWR^2$. The intercept indicated that the line did not pass through the origin and the parameter estimate was markedly below the value one, indicating a deficit in triplet sets. When we applied a regression model to the same data set, we obtained the slightly different result $TRR = 0.000039 + 0.584\ TWR^2$. The coefficient of determination is $R^2 = 0.842$ indicating a rather good fit. A deficit in the TRR can be obtained when one tests the parameter estimate against one with a one-sided t test. The $SE_{\hat{\beta}} = 0.113$ yielding $t = -3.7$, and the estimate was significantly below one. The result is presented in Figure 6a. Our analyses confirm the results given by Jenkins and Gwin (1940). The discrepancies between our and their results are mainly caused by two facts; they did not use regression models, but a geometric attempt, and they excluded in their analyses the extreme TRR for the age group 45+. In addition, they did not perform any statistical tests.

As an alternative model, we use TWR as abscissa and $\sqrt{TRR}$ as ordinate. The estimated model is $\sqrt{TRR} = 0.0029 + 0.679\ TWR$ and $R^2 = 0.844$. The $SE_{\hat{\beta}} = 0.130$, $t = -2.5$, and the obtained estimate is significantly below one. The model is presented in Figure 6b.

Both alternatives indicate deficits in the TRR. The parameter estimates are slightly higher for the second model, but the goodness of fit for both models is comparable.

Jenkins and Gwin (1940) also considered data from Finland (1878–1916). They used the data given by Dahlberg (1926). However, our check based on Finnish official registers confirmed our suspicion that Dahlberg's data contained a misprint for the maternal age group 35 to 40. In our analyses, we used the corrected data and present the results in Figure 7. When we applied the linear model to the Finnish data, we obtained the results $TRR = 0.00003 + 0.742\ TWR^2$ and $R^2 = 0.930$. The $SE_{\hat{\beta}} = 0.091$, $t = -2.8$, and the obtained estimate is significantly below one (Figure 7a). The linear relation between $\sqrt{TRR}$ and TWR is $\sqrt{TRR} = 0.0026 + 0.768\ TWR$ with $R^2 = 0.906$. The $SE_{\hat{\beta}} = 0.111$, $t = -2.1$, and the obtained estimate is significantly below one (Figure 7b). All of these results indicate good fit, but deficits in triplet maternities.
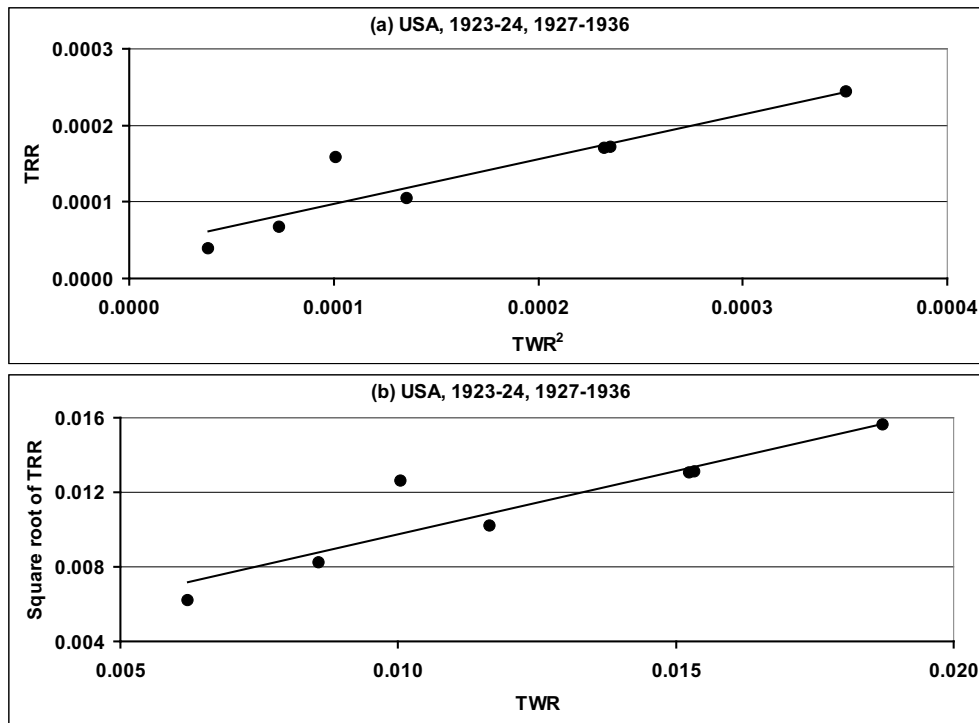
Figure 6

Association between the twinning rate (TWR) and the triplet rate (TRR) in USA (1923–1924, 1927–1936). In (a) the TRR is compared with TWR^2 and in (b) the $\sqrt{TRR}$ with TWR.

Table 2 indicates that the deficit in the TRR holds virtually always.

Discussion

Jenkins (1927, Figure 8) and later Fellman and Eriksson (2009) presented the association between the TWRs and the TRRs in the Prussian data (Veit, 1855). One can observe marked fluctuations for the different annual data, but a good agreement between the TRR and the TWR for the total data set. Our impression is that this finding already noted by Strassmann (1889) was the birth of Hellin's law. Furthermore, Jenkins stressed that Hellin's law is a first approximation. It is generally agreed that the main argument for Hellin's law is that the probabilities of additional ovulations and the fissions of fertilized eggs can be explained by stochastic models. Consequently, in large data sets, the averages could be stable and formulated by a mathematical relation (Hellin's law). A common argument for the discrepancies is that after the fertilizations there is a long process influenced by disturbing factors (intrauterine deaths, spontaneous abortions, etc., of one or more fetuses). Jenkins (1927) and Komai and Fukuoka (1936), for instance, assumed that differential mortality in utero of twins and triplets could be one such factor. Consequently, the final result often shows only a weak resemblance to the outcome of a simple stochastic process associated with the initial conceptions.

Excesses of higher multiple maternities in old birth registers must be considered paradoxical. A probable explanation is that systematic errors in the registers may cause biases in the data. This explanation is less plausible if the data are collected in different countries, as is the case in data of Table 1 and Figure 1.

The following step is a simple analysis of the data to show that the transformations may cause excesses in the transformed triplet and quadruplet rates. We simplify our studies by ignoring any random effects. Assume that after the fertilization and any fissions of the fertilized egg the twinning rate is w_0, the triplet rate is r_0 and the quadruplet rate is q_0, and let us assume that Hellin's law holds for these rates. Consequently, $r_0 = w_0^2$ and $q_0 = w_0^3$. During the pregnancy the rates may decrease and let the relative reductions be c_w, c_r and c_q for the twinning, triplet and quadruplet rates, respectively. An obvious assumption is that $c_w \leq c_r \leq c_q$. At birth, the observed rates are $w = w_0(1 - c_w)$, $r = w_0^2(1 - c_r)$ and $q = w_0^3(1 - c_q)$, and the variables w, r and q do not satisfy Hellin's law. A fundamental question is whether excesses in the transformed rates of triplets and quadruplets are possible. Compare the transformed rates $\sqrt{r} = w_0\sqrt{(1 - c_r)}$ and $\sqrt[3]{q} = w_0\sqrt[3]{(1 - c_q)}$ with $w = w_0(1 - c_w)$. An excess for the triplet rate is obtained if $\sqrt{(1 - c_r)} > (1 - c_w)$, that is, $c_r < 2c_w - c_w^2 \approx 2c_w$. An excess for the quadruplet rate is obtained if $\sqrt[3]{(1 - c_q)} > (1 - c_w)$, that is, $c_q < 3c_w - 3c_w^2 + c_w^3 \approx 3c_w$. These conditions are conceivable, and if the relative reductions in the triplet and quadruplet rates are not too strong, excesses are possible. If one speculates about these results, the extreme excesses observed
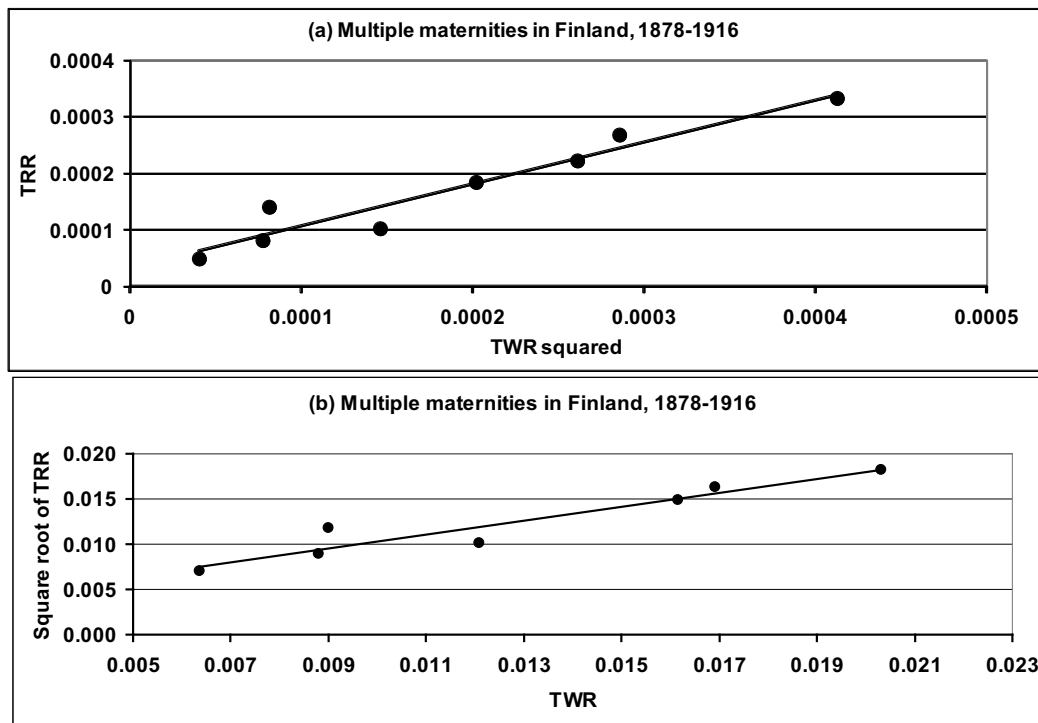
Figure 7

Association between the twinning rate (TWR) and the triplet rate (TRR) in Finland (1878–1916). In (a) the TRR is compared with TWR^2 and in (b) the $\sqrt{TRR}$ with TWR.

for transformed quadruplet rates compared with triplet rates would be explained by the fact that $c_q \leq 3c_w$ is more likely than $c_r \leq 2c_w$.

We consider the data in Table 1 and Figure 1 and assume that $c_w = c_r = c_q$. We can estimate the relative reducing component c_w and the transformed rates w, $\sqrt{r}$. and $\sqrt[3]{q}$. The numerical results obtained are included in Table 1 and Figure 1.

These simple examples show that excesses in the transformed triplet and quadruplet rates can more easily be attributed to the Hellin transformations than to something that happens in quadruplet conceptions and/or gestations that differs from the processes resulting in the deliveries of twins and triplets. Consequently, the transformations should be applied with caution and used only for descriptive purposes, and not for comparisons between the levels of twinning, triplet and quadruplet rates.

Finally, there is common agreement that discrepancies obtained during the era of fertility treatments are of less interest when Hellin's law is considered because no natural stochastic model is applicable.

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