Canad. Math. Bull. Vol. **28** (3), 1985

# GROUPS PRESERVING A CLASS OF BILINEAR FUNCTIONS

BY

W. H. GREUB, J. MALZAN AND J. R. VANSTONE

*Dedicated to the memory of our colleague, Professor R. A. Smith, a scholar and a friend.*

ABSTRACT. Given a finite dimensional Euclidean vector space $V$, $( \ , \ )$ and an involution $\tau$ of $V$, one can form the bilinear function $( \ , \ )_\tau$ defined by $(x, y)_\tau = (\tau(x), y)$, $x, y \in V$.

Let

$$O(\tau) = \{\phi \in GL(V) | (\phi x, \phi y)_\tau = (x, y)_\tau\}.$$

If $\tau$ is self-adjoint the structure of $O(\tau)$ is well known. The purpose of this paper is to detemine the structure of $O(\tau)$ in the general case. This structure is also determined in the complex and quaternionic case, as well as the case when the condition on $\tau$ is replaced by $\tau^2 = \epsilon\iota$, $\epsilon \in \mathbb{R}$.

Let $V$ be a finite dimensional right vector space over $F$ where $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and suppose that $V$ is endowed with a hermitian, sesquilinear inner product $( \ , \ )$ such that $(x, x) > 0$ for all $x \neq 0$ in $V$. Now let $\tau \in L_V$ be such that $\tau^2 = \epsilon\iota$, for some $\epsilon \in \mathbb{R}$. We define a sesquilinear map $( \ , \ )_\tau$ on $V$ by

$$(x, y)_\tau = (\tau(x), y) \text{ for all } x, y \in V.$$

Furthermore, let

$$O(\tau) = \{\phi \in GL(V) | (\phi x, \phi y)_\tau = (x, y)_\tau\}.$$

$O(\tau)$ is clearly a group. The purpose of this note is to obtain the structure of $O(\tau)$ from that of the hermitian and skew-hermitian parts of $\tau$.

The problem has been considered independently by C. Riehm [1] from the standpoint of classifying asymmetric bilinear forms, up to isometry. In principle some of the results given here can be deduced from the proof of his Theorem 5, p. 48; however, the explicit formulae given here should be of interest.

To begin, let $\rho, \sigma$ be the $( \ , \ )$-hermitian and $( \ , \ )$-skew-hermitian parts of $\tau : \rho = \frac{1}{2}(\tau + \tau^*)$, $\sigma = \frac{1}{2}(\tau - \tau^*)$. Then $\tau = \rho + \sigma$ and $\tau^2 = \epsilon\iota$ yield

$$\rho^2 + \sigma^2 = \epsilon\iota \quad \text{and} \tag{1}$$

$$\rho\sigma + \sigma\rho = 0. \tag{2}$$

Since also $(\phi x, \phi y)_\tau = (x, y)_\tau$ for all $x, y \in V$ implies that $(\phi y, \phi x)_{\tau^*} = (y, x)_{\tau^*}$ we

Received by the editors December 2, 1983 and, in revised form, July 6, 1984.

AMS Subject Classification (1980): 15A63, 22E15.





https://doi.org/10.4153/CMB-1985-032-7 Published online by Cambridge University Press

have $O(\tau) = O(\tau^*)$ and in consequence,

$$O(\tau) = O(\rho) \cap O(\sigma). \tag{3}$$

Thus, instead of considering $\tau$ it is sufficient to analyze pairs $(\rho, \sigma)$ satisfying (1) and (2).

We now define four subspaces of $V$:

$$V_0^0 = \ker \rho \cap \ker \sigma \qquad V_0^1 = \ker \rho \cap \operatorname{Im} \sigma$$
$$V_1^0 = \operatorname{Im} \rho \cap \ker \sigma \qquad V_1^1 = \operatorname{Im} \rho \cap \operatorname{Im} \sigma.$$

LEMMA 1.

(i) *$V$ is an orthogonal direct sum $V_0^0 \oplus V_0^1 \oplus V_1^0 \oplus V_1^1$.*

(ii) *$V_0^0 \neq \{0\}$ implies $\epsilon = 0$, $V_0^1 \neq \{0\}$ implies $\epsilon < 0$ and $V_1^0 \neq \{0\}$ implies $\epsilon > 0$.*

(iii) *Each of the spaces $V_i^j$, $i, j = 0, 1$ is $\rho$ and $\sigma$ invariant. $\rho$ and $\sigma$, restricted to $V_1^1$, are non-singular.*

(iv) *Each of the spaces $V_0^0$, $V_0^0 \oplus V_0^1$, $V_0^0 \oplus V_1^0$, $V_0^0 \oplus V_1^1$ is $O(\tau)$-invariant.*

PROOF. Since $\rho$ is hermitian we have $V = \ker \rho \oplus_\perp \operatorname{Im} \rho$, with both spaces $\rho$-invariant and $\rho$, restricted to $\operatorname{Im} \rho$, invertible. We claim that $\ker \rho$ and $\operatorname{Im} \rho$ are both $\sigma$-invariant. Indeed, if $v \in \ker \rho$ then (2) yields

$$\rho\sigma(v) + \sigma\rho(v) = 0 \quad \text{and so}$$
$$\rho\sigma(v) = 0$$

which shows $\sigma(v) \in \ker \rho$. Similarly if $v \in \operatorname{Im} \rho$ we obtain from

$$\sigma(\rho v) = -\rho(\sigma v)$$

that $\sigma(\rho v)$ lies in $\operatorname{Im} \rho$. Since $\rho$ is invertible on $\operatorname{Im} \rho$ this shows that $\operatorname{Im} \rho$ is $\sigma$-invariant. Now restricting $\sigma$ to $\ker \rho$ and $\operatorname{Im} \rho$ in turn and repeating the above argument shows (i) and (iii) of the lemma.

(ii) is a consequence of equation (1). Suppose, for example, that $v$ is a non-zero vector on $V_0^1$. Then $\rho v = 0$ and so

$$\sigma^2 v = \epsilon v.$$

Thus $v$ is an $\epsilon$-eigenvector of $\sigma^2$. But since $\sigma$ is skew-hermitian

$$(\sigma^2 v, v) = \epsilon(v, v) = -(\sigma v, \sigma v)$$

and so we must have $\epsilon < 0$.

Finally, to show (iv), suppose that $v \in \ker \rho$ and that $\phi \in O(\rho)$. For all $w \in V$ we have

$$(\phi v, \phi w)_\rho = (\rho\phi v, \phi w) = (\rho v, w) = 0.$$

Since $\phi$ is invertible $\phi w$ is arbitrary and so $\rho\phi v = 0$ which shows that $\ker \rho$ is $\phi$-invariant. Since also $O(\tau) = O(\rho) \cap O(\sigma)$ we have $\ker \rho$ is $O(\tau)$-invariant, and similarly for $\ker \sigma$. This shows that $V_0^0$, $V_0^0 \oplus V_1^0$ and $V_0^0 \oplus V_1^0$ are $O(\tau)$-invariant.

We have, to finish, to show that $V_0{}^0 + V_1{}^1$ is $O(\tau)$ invariant. If $\epsilon = 0$ this is trivial since then $V_0{}^0 + V_1{}^1 = V$. So suppose (say) $\epsilon > 0$ so that $V = V_1{}^0 + V_1{}^1$. We must show $V_1{}^1$ is $O(\tau)$-invariant. Now for $\phi \in O(\tau) = O(\rho) \cap O(\sigma)$ we have

$$\phi^*\rho\phi = \rho \quad \text{and} \quad \phi^*\sigma\phi = \sigma.$$

Since $\rho$ is invertible we can eliminate $\phi^*$ from this system, and we find

$$(\rho^{-1}\sigma)\phi = \phi(\rho^{-1}\sigma). \tag{4}$$

Thus $\phi$ holds invariant the eigenspaces of $\rho^{-1}\sigma$, which is hermitian. Im $\sigma$ is $\rho$-invariant and so $\mathrm{Im}(\rho^{-1}\sigma) = \mathrm{Im}\,\sigma$. Thus $\mathrm{Im}\,\sigma = V_1{}^1$ is $O(\tau)$-invariant, as required.

NOTATION. *If $\psi$ is a nondegenerate hermitian form on a space $W$ over $F = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ we shall denote by $U_\psi$ the classical Lie group which preserves $\psi$. Thus if $F = \mathbb{R}$, then $U_\psi = 0_{p,q}$ where $(p, q)$ is the signature of $\psi$. If $F = \mathbb{C}$, then $U_\psi = U_{p,q}$ with $p$ and $q$ as above. If $F = \mathbb{H}$, then $U_\psi$ is the corresponding quaternionic group preserving a form of type $(p, q)$.*

*If $\omega$ is a nondegenerate skew-hermitian form on $W$ with $F = \mathbb{R}$ or $\mathbb{H}$ we shall denote by $\mathrm{Sp}(n)$ the corresponding symplectic group preserving $\omega$, acting on a space of dimension $n$.*

Our next lemma yields the structure of $O(\tau)$ on $V_1{}^1$.

LEMMA 2. *Suppose that $\rho$ and $\sigma$ are both invertible. Then*

$$O(\tau) \cong GL_{n_1}(F)x \ldots xGL_{n_r}(F)$$

*where $n_1, \ldots, n_r$ are the multiplicities of the distinct positive eigenvalues of $\rho$.*

PROOF. Since $\rho$ is hermitian $V$ is the orthogonal direct sum of its (real) eigenspaces. Let $V_\lambda$ denote one of these. Equation (2) yields immediately that $\sigma(V_\lambda) = V_{-\lambda}$. Denote by $W_\lambda$ the space $V_\lambda \oplus V_{-\lambda}$, with $\lambda > 0$. We have

$$V = W_{\lambda_1} \oplus \ldots \oplus W_{\lambda_r}, \qquad \lambda_i > 0,$$

and we claim that each $W_{\lambda_i}$ is $O(\tau)$-invariant. Recall from equation (4) that $\phi \in O(\tau)$ commutes with $\rho^{-1}\sigma$. We compute the eigenvalues of $\rho^{-1}\sigma$ in $W_{\lambda_i}$. Let $v \in W_{\lambda_i}$ with $v = v_1 + v_2$ where $\rho v_1 = v_1\lambda_i$, $\rho v_2 = -v_2\lambda_i$, and suppose that $\rho^{-1}\sigma v = v\mu$. Then

$$\sigma(v_1 + v_2) = (v_1\lambda_i - v_2\lambda_i)\mu = (v_1 - v_2)\lambda_i\mu.$$

Now $\sigma(v_1) = v_2' \in V_{-\lambda_i}$ and so

$$\sigma(v_1) = -v_2\lambda_i\mu \quad \text{and} \quad \sigma(v_2) = v_1\lambda_i\mu.$$

Hence

$$\sigma^2(v_1) = (\epsilon\iota - \rho^2)v_1 = v_1\cdot(\epsilon - \lambda_i^2) = -v_1\cdot(\lambda_i\mu)^2$$

and so $\epsilon - \lambda_i^2 = -(\lambda_i\mu)^2$ or $\mu = \pm\sqrt{1 - \epsilon/\lambda_i^2}$. Thus the eigenvalue $\mu$ is uniquely associated with the space $W_{\lambda_i}$ and so $W_{\lambda_i}$ is $O(\tau)$-invariant, as claimed. Thus it suffices to consider the action of $O(\tau)$ on $W_{\lambda_i}$ alone.

https://doi.org/10.4153/CMB-1985-032-7 Published online by Cambridge University Press

Let $e_1, \ldots, e_k$ be an orthonormal basis for $V_{\lambda_i}$. Then $\sigma e_1, \ldots, \sigma e_k$ is an orthogonal basis for $V_{-\lambda_i}$. We take a new orthonormal basis $f_1, \ldots, f'_1, \ldots, f'_k$ with

$$f_r = \frac{1}{\sqrt{2}}[e_r + \sigma(e_r)/\sqrt{\lambda_i^2 - \epsilon}]$$

and

$$f'_r = \frac{1}{\sqrt{2}}[e_r - \sigma(e_r)/\sqrt{\lambda_i^2 - \epsilon}].$$

We find immediately that

$$\rho(f_r) = \lambda_i f'_r, \qquad \rho(f'_r) = \lambda_i f_r$$
$$\sigma(f_r) = -\sqrt{\lambda_i^2 - \epsilon}\, f'_r, \qquad \sigma(f'_r) = +\sqrt{\lambda_i^2 - \epsilon}\, f_r.$$

In particular,

$$\rho^{-1}\sigma(f_r) = -\sqrt{1 - \epsilon/\lambda_i^2}\, f'_r \quad \text{and} \quad \rho^{-1}\sigma(f'_r) = +\sqrt{1 - \epsilon/\lambda_i^2}\, f'_r$$

so that the subspaces spanned by $f_1, \ldots, f_k$ and $f'_1, \ldots, f'_k$ are $O(\tau)$-invariant. Thus we have, for $\phi \in O(\tau)$ and $\rho$ restricted to $W_{\lambda_i}$

$$\phi = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad \rho = \begin{pmatrix} 0 & \lambda_i I_k \\ \lambda_i I_k & 0 \end{pmatrix}$$

with $\phi^*\rho\phi = \rho$. This yields $(B^*)^{-1} = A$. The condition $\phi^*\sigma\phi = \sigma$ yields no other restriction. Thus the group on $V_{\lambda_i}$ is $GL_k(F)$ as claimed, and the lemma is proved.

We are now in a position to state our main results:

THEOREM +. *Suppose that* $\epsilon > 0$. *Then*

$$O(\tau) \cong U_\psi \times GL_{n_1}(F) \times \ldots \times GL_{n_r}(F)$$

*where* $\psi$ *is a hermitian form of type* $(p, q)$ *where* $p$ *is the number of eigenvalues of* $\rho$ *equal to* $\epsilon$ *and* $q$ *is the number equal to* $-\epsilon$ *and* $n_1, \ldots, n_r$ *are the multiplicities of the positive eigenvalues different from* $\epsilon$ *of* $\rho$.

THEOREM −. *Suppose that* $\epsilon < 0$. *Then*

$$O(\tau) \cong \mathrm{Sp}(n) \times GL_{n_1}(F) \times \ldots \times GL_{n_r}(F)$$

*where* $n$ *is the dimension of* ker $\rho$ *and* $n_1, \ldots, n_r$ *are as above.*

THEOREM 0. *Suppose that* $\epsilon = 0$. *Then*

$$O(\tau) \cong GL_n(F) \times GL_{n_1}(F) \times \ldots \times GL_{n_r}(F)$$

*where* $n =$ dim ker $\rho$ *and* $n_1, \ldots, n_r$ *are as above.*

PROOF. Theorems + and − follow from Lemmas 1 and 2 together with the observation that, on $V_1^0$, $O(\tau)$ preserves a hermitian inner product while, on $V_0^1$, $O(\tau)$ preserves a skew-hermitian inner product. Finally, for Theorem 0, we have $V = V_0^0 \oplus V_1^1$ so that we can take

$$\phi = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & 0 \\ 0 & \rho_1 \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

with $\rho_1$ and $\sigma_1$ invertible. Then the conditions

$$\phi^*\rho\phi = \rho \quad \text{and} \quad \phi^*\sigma\phi = \sigma$$

yield

$$\gamma^*\rho_1\alpha = \rho_1 \quad \text{and} \quad \gamma^*\sigma_1\gamma = \sigma_1 \quad \text{and} \quad \beta = 0.$$

Thus $\alpha$ need only be invertible, while $\gamma \in O(\rho_1) \cap O(\sigma_1)$. Since $\rho_1$ and $\sigma_1$ are both invertible we can apply Lemma 2, and the proof is complete.

## References

1. C. Riehm, *The equivalence of bilinear forms*, J. Algebra **31** (1974). pp. 45–66.

University of Toronto
Toronto, Ontario
M5S 1A1