A NOTE ON COVERINGS OF E^n BY CONVEX SETS

J.H.H. Chalk

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1. Let $(x_1, x_2, \ldots, x_n)$ denote the coordinates of a point $\underset{\sim}{x}$ of Euclidean n-space E^n. Let $\underset{\sim}{a}_0, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ be a set of n+1 points of E^n with the property that

$$\underset{\sim}{b}_i = \underset{\sim}{a}_i - \underset{\sim}{a}_0 \quad (i=1, 2, \ldots, n)$$

form a linearly independent set and define a lattice Λ of points

$$u_1 \underset{\sim}{b}_1 + \ldots + u_n \underset{\sim}{b}_n,$$

by allowing $u_1, \ldots, u_n$ to take all integer values. Suppose that K is any closed, bounded, centrally symmetric convex set containing the points $\underset{\sim}{a}_0, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$. In a recent article [1], I showed that there is a constant $\lambda_n > 0$, depending only on n, such that the collection of convex sets formed by taking all Λ - translations of $\lambda_n K$ form a covering of E^n, i.e.,

$$(1) \qquad \bigcup_{\underset{\sim}{x} \in \Lambda} (\lambda_n K + \underset{\sim}{x}) \supset E^n.$$

This shows that if K provides a local covering for Λ (in the sense that it can be translated to cover a non-singular lattice simplex of Λ), then the vector sum of Λ and a uniformly bounded dilatation of K covers the whole of space. The method of proof gave an explicit value for λ_n; namely,

$$(2) \qquad \lambda_n = \begin{cases} \frac{1}{2}(n+1), & \text{if } n \text{ is odd,} \\ \frac{1}{2} n(1 + \frac{1}{n+1}), & \text{if } n \text{ is even.} \end{cases}$$

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That this is not far from the best possible may be judged by a consideration of the special case where $K = K_0$ is the generalised octahedron

$$K_0 : |x_1| + \ldots + |x_n| \leq 1,$$

$\underset{\sim}{a}_0 = \underset{\sim}{0}$, $\underset{\sim}{a}_i = \underset{\sim}{e}_i = (0, \ldots, 0, 1^{\text{th}}, 0, \ldots, 0)$, $\underset{\sim}{\varepsilon}_0 = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$ and Λ_0 is the lattice of points with integral coordinates. For then, with any ε satisfying $0 < \varepsilon < \frac{1}{2}n$, it is clear that $\underset{\sim}{\varepsilon}_0 \notin (\frac{1}{2}n - \varepsilon) K_0 + \underset{\sim}{x}$, for any $\underset{\sim}{x} \varepsilon \Lambda_0$, (c.f. [1], p. 238). Hence a necessary condition for (1) to hold is

$$\lambda_n \geq \frac{1}{2} n,$$

and, in [1], it was conjectured that this was also sufficient. In some unpublished work, I have since verified that this is correct when n is even, but false when n is odd; in fact, we now know that

$$(3) \qquad \lambda_n = \begin{cases} \frac{1}{2}(n+1), & \text{if } n \text{ is odd,} \\ \frac{1}{2} n, & \text{if } n \text{ is even.} \end{cases}$$

are the least possible values for λ_n in (1), for all $n \geq 2$.

So far as one could see, the method of [1] had reached the extent of its usefulness, offering little hope of improvement for the case of even n and giving no indication that (2) was best possible for odd n. However, the case $n=2$ was settled some 20 years ago [2] by a rather different approach and I have recently noticed that this argument can be adapted for the case of general n to give (3), (but leaving open the question of whether $\frac{1}{2}(n+1)$ is exact, when n is odd). Mr. D. L. Yates produced an example for $n=3$ to show that (1) implied $\lambda_3 \geq 2$ and subsequently, developed a more enlightened approach for the proof of (3), (Ph.D. thesis, University of Nottingham). It remains then, and this is the purpose of this note, to provide an example for all odd n to verify that (1) implies $\lambda_n \geq \frac{1}{2}(n+1)$.

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2. <u>An example for odd values of n.</u>

Let

$$\underset{\sim}{a}_0 = (-1, -1, \ldots, -1), \ \underset{\sim}{a}_1 = \underset{\sim}{e}_1, \ldots, \underset{\sim}{a}_n = \underset{\sim}{e}_n$$

and put $\underset{\sim}{\varepsilon} = \frac{1}{2}(n+1)\underset{\sim}{a}_0$. Define a set K by

(4) $$F(\underset{\sim}{x}) \leq 1,$$

where

(5) $$F(\underset{\sim}{x}) = \max|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n|$$

and the maximum is taken over all $\varepsilon_i = \pm 1$ with $\varepsilon_1 + \ldots + \varepsilon_n = +1$. Then K is a closed bounded convex set, symmetric in $\underset{\sim}{0}$, which clearly contains the points $\underset{\sim}{a}_0, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$. That it is closed, convex and symmetric in $\underset{\sim}{0}$ is obvious, since it is the intersection of finitely many closed half-spaces, together with their images in $\underset{\sim}{0}$. For the boundedness, observe that $F(\underset{\sim}{x}) \geq G(\underset{\sim}{x})$, where

$$G(\underset{\sim}{x}) = \max|\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n|$$

and the maximum is now taken over all $\varepsilon_i = \pm 1$ such that $(\varepsilon_1, \ldots, \varepsilon_n)$ is a cyclic permutation of

(6) $$(\underbrace{1, 1, \ldots, 1}_{\frac{1}{2}(n+1)}, \underbrace{-1, -1, \ldots, -1}_{\frac{1}{2}(n-1)})$$

These n vectors are linearly independent, since the circulant formed from (6) vanishes only when

$$1 + w + w^2 + \ldots + w^{\frac{1}{2}(n-1)} - w^{\frac{1}{2}(n+1)} - \ldots - w^{n-1} = 0$$

for some w with $w^n = 1$. But since n is odd and $1 + w + w^2 + \ldots + w^{n-1} = 0$, if $w \neq 1$, this is impossible.

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Hence the set defined by $G(\tilde{x}) \leq 1$ is a parallelopiped containing the polytope $F(\underset{\sim}{x}) \leq 1$ and so K is bounded.

Now, for any $\underset{\sim}{x} = \sum_{i=1}^{n} u_i \underset{\sim}{b}_i$ in Λ, we have

$$\underset{\sim}{x} - \underset{\sim}{\xi} = [\sum_{i=1}^{n} u_i (\underset{\sim}{a}_i - \underset{\sim}{a}_0)] - \frac{1}{2}(n+1) \underset{\sim}{a}_0$$

$$= \sum_{i=1}^{n} u_i \underset{\sim}{a}_i + \left(\sum_{i=1}^{n} u_i \sum_{i=1}^{n} \underset{\sim}{a}_i \right) + \frac{1}{2}(n+1) \sum_{i=1}^{n} \underset{\sim}{a}_i .$$

$$= \sum_{i=1}^{n} [\frac{1}{2}(n+1) + u_i + \sum_{j=1}^{n} u_j] \underset{\sim}{a}_i ,$$

from the definition of $\underset{\sim}{a}_0, \underset{\sim}{a}_1, \ldots, \underset{\sim}{a}_n$ and $\underset{\sim}{\xi}$. Hence

$$F(\underset{\sim}{x} - \underset{\sim}{\xi}) = \max_{\substack{\varepsilon_i = \pm 1 \\ \sum_{1 \leq i \leq n} \varepsilon_i = +1}} \left| \sum_{i=1}^{n} [\frac{1}{2}(n+1) + u_i + \sum_{j=1}^{n} u_j] \varepsilon_i \right|$$

$$= \max_{1 \leq r_1 < \ldots < r_m \leq n} \left| m + 2u_{r_1} + 2u_{r_2} + \ldots + 2u_{r_m} \right|$$

$$= \max_{1 \leq r_1 < \ldots < r_m \leq n} \left| v_{r_1} + v_{r_2} + \ldots + v_{r_m} \right|$$

where $m = \frac{1}{2}(n+1)$ and $v_1 = 2u_1 + 1, \ldots, v_n = 2u_n + 1$ are odd integers. But if $v_1, v_2, \ldots, v_n$ are any n non-zero integers (n odd), there are, (by Dirichlet's box principle!), at least $\frac{1}{2}(n+1)$ of them with the same sign. Hence

$$F(\underset{\sim}{x} - \underset{\sim}{\xi}) \geq \frac{1}{2}(n+1) \text{ for all } \underset{\sim}{x} \in \Lambda ,$$

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i.e., for any positive $\lambda < \frac{1}{2}(n+1)$,

$$\lambda K \cap (\underset{\sim}{x} - \underset{\sim}{\xi}) = \phi \quad \text{for all} \quad \underset{\sim}{x} \in \Lambda \ .$$

Thus the point $\underset{\sim}{\xi}$ does not belong to any of the set $\lambda K + \underset{\sim}{x}$, where $\underset{\sim}{x} \in \Lambda$, if $\lambda < \frac{1}{2}(n+1)$. We conclude that, although (1) is always satisfied with λ_n replaced by $\frac{1}{2}(n+1)$, in this particular case, it is not satisfied with λ_n replaced by any positive $\lambda < \frac{1}{2}(n+1)$.

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University of Toronto

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