MAXIMA OF MOVING SUMS IN A POISSON RANDOM FIELD

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Abstract

In this paper we examine the extremal tail probabilities of moving sums in a marked Poisson random field. These sums are computed by adding up the weighted occurrences of events lying within a scanning set of fixed shape and size. We also provide an alternative representation of the constants of the asymptotic formulae in terms of the occupation measure of the conditional local random field at zero, and extend these representations to the constants of asymptotic tail probabilities of Gaussian random fields.

Keywords: Change of measure; Gaussian process; large deviations; marked Poisson process; moving sums; random field; scan statistics

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1. Introduction

The maxima of moving averages in Gaussian random fields in dimension $d > 1$ was studied in [19] and [21], with applications in imaging and signal detection. Two key techniques used are (i) the Karhunen–Loève expansion with the volume of tube formula and (ii) the Euler characteristic; see [1] for an overview, and [24] and [25] for more recent developments.

The maxima of moving sums in Poisson random fields, more commonly known as scan statistics in the statistical literature, is also used widely in molecular biology, epidemiology, geostatistics, and image analysis (cf. [4], [8], [9], and [11]), but the tail probability approximations are in comparison not as well developed for $d > 1$. While the tail probabilities of these sums have been studied in [3], [14], and [15], restrictions to rectangular scanning sets have been imposed for analytical convenience.

We set out here to study the tail probabilities of the maxima of moving sums for a wider choice of scanning sets. A theory parallel to the study of tail probabilities in Gaussian or Gaussian-like random fields in the classical framework of Pickands [16] and Qualls and Watanabe [18] (see also [7] and [17]) is first developed. Motivated by recent developments in molecular biology (see, for example, [8]), we consider a more general marked Poisson random field. This generalization entails careful consideration of overshoots in special cases of scanning sets that is not required in Poisson random fields. Albin [2] and Berman [5] studied tail probabilities of stationary processes, but their limiting results are of a different type and do not apply here. Kabluchko and Spodarev [13] considered scan statistics of Lévy noises and marked empirical processes, with the scan over all cubic subsets of $[0, 1]^d$.

The main results of this paper are Theorems 2.1 and 2.2. We illustrate their applications with examples in Section 3. Theorem 2.1 provides an expression of the asymptotic tail probability and the statement of the result requires a description of an induced local random field around

the boundary of the scanning set. The proof of Theorem 2.1 is given in Section 4. Lemma 4.1 is the basic building block of Theorem 2.1, providing the extremal tail probability over a local domain by using a change of measure approach. We then show how these building blocks can be combined together to provide the tail probability of the maxima of the sums over the whole domain, via an adaption of the Pickands–Qualls–Watanabe technique. Theorem 2.2 provides an alternative representation of the asymptotic constants in terms of the occupation measure of the local random field at zero. The proof is given in Section 5. We extend the representation to Gaussian random fields in Section 6. These representations provide simple upper and lower bounds of the asymptotic constants; see the examples in Sections 3 and 6. We consolidate the technical details in the appendices.

2. Main results

For vectors $t = (t_1, \ldots, t_d)$ and $u = (u_1, \ldots, u_d)$, and real number b, we shall let $t \geq u$ denote $t_j \geq u_j$ for all j and let $t \geq b$ denote $t_j \geq b$ for all j. Let $\sigma_k(\cdot)$, $k = d$ or $k = d - 1$, denote the Lebesgue or Hausdorff measure of a k-dimensional manifold in $\mathbb{R}^d$. For any $A \subset \mathbb{R}^d$, real number b, and $t \in \mathbb{R}^d$, let $\#A$ denote the number of elements in A and $t + bA = \{t + ba : a \in A\}$. Also, let $\|t\| = (t_1^2 + \cdots + t_d^2)^{1/2}$ and $\|t\|_\infty = \max_{1 \leq j \leq d} |t_j|$.

Let B be a bounded subset of $\mathbb{R}^d$ with boundary ∂B that can be expressed as a finite union of smooth $(d-1)$-dimensional submanifolds possibly with boundaries (see [22, p. 113] for the definition), satisfying $\sigma_{d-1}(\partial B) < \infty$. For example, if $B = \{t : \|t\|_\infty \leq 1\}$, a cube of length 2 centered at 0, then ∂B is a union of $2d$ faces, each a smooth $(d-1)$-dimensional submanifold with boundary.

Let $\{t_i : i \geq 1\}$ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\lambda > 0$, and let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables having distribution F and independent of the Poisson point process. Let $\mathcal{X} = \{(t_i, X_i) : i \geq 1\}$ denote the marked point process. Let $\mu = \mathrm{E}(X_1)$ and $M(\theta) = \mathrm{E}(\mathrm{e}^{\theta X_1})$. Assume that $\Theta = \{\theta : M(\theta) < \infty\}$ is an open neighborhood of 0. For any $A \subset \mathbb{R}^d$, define the sum $S(A) = \sum_{t_i \in A} X_i$. Let D be a Jordan measurable (bounded) subset of $\mathbb{R}^d$, and let $c > \mu \sigma_d(B)$. We analyze here the tail probability

$$p_\lambda = \mathrm{P}_\lambda \left(\sup_{v \in D} S(v + B) \geq \lambda c \right) \quad \text{as } \lambda \to \infty. \tag{2.1}$$

Through an appropriate transformation, we can also look at the limiting probability of p_λ as one involving fixed Poisson rate $\lambda_0 > 0$ and increasingly large scanning sets. Let $g_\lambda = (\lambda/\lambda_0)^{1/d}$ be the scaling constants. Then $p_\lambda = \mathrm{P}_{\lambda_0}(\sup_{v \in g_\lambda D} S(v + g_\lambda B) \geq \lambda c)$. For notational simplicity, the analysis here looks at p_λ in terms of (2.1), but in terms of practical use, the asymptotics in terms of increasing scanning sets is sometimes more appropriate.

We will now proceed with the description of a limiting local random field Y that is derived from both the distribution F and the geometry of the boundary ∂B. Let $\theta_c > 0$ and distribution F_c satisfy

$$M'(\theta_c) = \frac{c}{\sigma_d(B)} \quad \text{and} \quad F_c(\mathrm{d}x) = \frac{\mathrm{e}^{\theta_c x} F(\mathrm{d}x)}{M(\theta_c)}, \tag{2.2}$$

where a prime here denotes the first derivative. By (2.2), a random variable having distribution F_c has mean $M'(\theta_c)/M(\theta_c) = c/[\sigma_d(B)M(\theta_c)]$ and, hence, a marked Poisson process with intensity $\lambda M(\theta_c)$ and mark distribution F_c inside B will satisfy $\mathrm{E}_*[S(B)] = \lambda c$.

Consider the product space $\partial B \times \mathbb{R}$ with measure induced from the $(d-1)$-dimensional Hausdorff measure on ∂B and Lebesgue measure on $\mathbb{R}$. Let $\{v_i : i \geq 1\}$ be a nonhomogeneous

Poisson point process on $\partial B \times \mathbb{R}$ with intensity 1 on $\partial B \times [0, \infty)$ and intensity $M(\theta_c)$ on $\partial B \times (-\infty, 0)$. Let $Z_1, Z_2, \ldots$ be independent random variables (marks) conditioned on this nonhomogeneous point process, with $Z_i \sim F$ if $v_i \in \partial B \times [0, \infty)$ and $Z_i \sim F_c$ if $v_i \in \partial B \times (-\infty, 0)$. Let Q_λ be a probability measure under which $\mathcal{X} = \{(t_i, X_i) : i \geq 1\}$ is a nonhomogenous marked point process with intensity $\lambda M(\theta_c)$ and mark distribution F_c inside B, and intensity λ and mark distribution F outside B. The measure Q_λ describes the local behavior of $\mathcal{X}$ near ∂B under the original probability measure P_λ in which $\mathcal{X}$ is homogeneous, when conditioned on $\{S(B) \geq \lambda c\}$. Let $\partial B^{(\delta)} = \{u \in \mathbb{R}^d : \|u - t\| \leq \delta$ for some $t \in B\}$. For any $m > 0$ and large λ, we can apply a λ-scaling transformation such that $\mathcal{X}$ under Q_λ and restricted to $\partial B^{(m\lambda^{-1})}$ tends to $\mathcal{Z}$ restricted to $\partial B \times [-m, m]$ as $\lambda \to \infty$.

Let n_t be the unit normal vector of $t \in \partial B$ pointing away from B, and let a dot denote the scalar or inner product of two vectors. Let

$$Y^{(j)}(u) = \sum_{\{i : v_i^{(j)} \in A_u^{(j)}\}} Z_i \quad \text{for } j = 1, 2,$$

where

$$A_u^{(1)} = \bigcup_{\{t \in \partial B : n_t \cdot u > 0\}} t \times [0, n_t \cdot u) \quad \text{and} \quad A_u^{(2)} = \bigcup_{\{t \in \partial B : n_t \cdot u < 0\}} t \times [n_t \cdot u, 0). \tag{2.3}$$

We define the limiting local random field by

$$Y(u) = Y^{(1)}(u) - Y^{(2)}(u), \qquad u \in \mathbb{R}^d. \tag{2.4}$$

Let

$$I(= I_c) = \theta_c c - \sigma_d(B)[M(\theta_c) - 1]. \tag{2.5}$$

It follows from Theorem 2.1, below, that $I = -\lim_{\lambda \to \infty} \lambda^{-1} \log p_\lambda$ and, hence, it is the large deviation rate of the tail probability p_λ. If there exists $\eta > 0$ such that F is concentrated on $\pm\eta, \pm 2\eta, \ldots$, then we say that F is arithmetic. The largest η with this property is called the span of F; cf. Section 5.2 of [10]. If such η does not exist then we say that F is nonarithmetic. Let $\lfloor \cdot \rfloor$ denote the greatest integer function, and let a double prime denote the second derivative of a function.

Theorem 2.1. *Define $x_\lambda = \theta_c(\lambda c - \eta \lfloor \lambda c / \eta \rfloor)$ if F is arithmetic with span η and $x_\lambda = 0$ if F is nonarithmetic. Let*

$$K_m = \begin{cases} \eta\left[(1 - e^{-\eta\theta_c})^{-1} + \sum_{\ell \in \eta\mathbb{Z}^+} e^{\theta_c \ell} \, P\left(\sup_{0 \leq u \leq m} Y(u) \geq \ell\right)\right] & \text{if } F \text{ arithmetic with span } \eta, \\[2mm] \theta_c^{-1} + \int_0^\infty e^{\theta_c y} \, P\left(\sup_{0 \leq u \leq m} Y(u) \geq y\right) dy & \text{if } F \text{ nonarithmetic.} \end{cases} \tag{2.6}$$

Then

$$K = \lim_{m \to \infty} m^{-d} K_m \text{ is a well-defined positive and finite constant.} \tag{2.7}$$

Moreover, as $\lambda \to \infty$,

$$p_\lambda = P_\lambda\left(\sup_{v \in D} S(v + B) \geq \lambda c\right)$$
$$\sim [2\pi \sigma_d(B) M''(\theta_c)]^{-1/2} e^{-\lambda I + x_\lambda} \lambda^{d-1/2} \sigma_d(D) K. \tag{2.8}$$

Remark 2.1. By Jordan measurability of D, as $a \to 0$,

$$\#\left\{k \in (a\mathbb{Z})^d : \prod_{j=1}^{d}[k_j, k_j + a) \subset D\right\} \sim \#\left\{k \in (a\mathbb{Z})^d : \prod_{j=1}^{d}[k_j, k_j + a) \cap D \neq \varnothing\right\}. \quad (2.9)$$

Relation (2.8) still holds if D is replaced by domains D_λ that depend on λ, provided that (2.9) holds with D replaced by D_λ and a replaced by $m\lambda^{-1}$, with limit $\lambda \to \infty$ for all large m, and

$$\lim_{\lambda \to \infty} \lambda^{-1} \log[\sigma_d(D_\lambda)] = 0. \quad (2.10)$$

However, even without condition (2.10), the relation

$$P_\lambda\left(\sup_{v \in D_\lambda} S(v + B) \geq \lambda c\right)$$
$$\sim 1 - \exp\{-[2\pi \sigma_d(B) M''(\theta_c)]^{-1/2} e^{-\lambda I + x_\lambda} \lambda^{d-1/2} \sigma_d(D_\lambda) K\}$$

still holds.

We now provide an alternative expression of K which follows from a different proof of (2.8). This representation is less intuitive when compared to (2.6) and (2.7), but it has the advantage of being able to provide simple bounds for K. Let $\sigma_d^{-1}(\cdot)$ denote the reciprocal of $\sigma_d(\cdot)$.

Theorem 2.2. *The constant*

$$K = \chi_c^{-1} \operatorname{E}([(1 - \exp\{\theta_c \sup\{Y(u) : Y(u) < 0\}\}]\sigma_d^{-1}(\{u : Y(u) = 0\})), \quad (2.11)$$

where $\chi_c = \eta^{-1}(1 - e^{-\eta\theta_c})$ when F is arithmetic with span η and $\chi_c = \theta_c$ when F is nonarithmetic.

3. Examples

We first discuss two interesting cases of Theorem 2.1. In Example 3.1 we consider F to be degenerate at 1. Then (2.8) simplifies to (3.2), below, an extension of tail probability relations in Poisson random fields obtained earlier by other authors to nonrectangular scanning sets. In Example 3.2 we consider rectangular scanning sets on a marked Poisson random field. We show here that an overshoot constant derived from F plays an important role in the tail approximations. When F is degenerate at 1, that is, for Poisson random fields rather than marked Poisson random fields, the overshoot constant is equal to 1 and disappears from the resulting formula. We then show in Examples 3.3 and 3.4 how the alternative expression in Theorem 2.2 leads to simple upper and lower bounds of K.

Example 3.1. Let F be concentrated at 1. Then $S(v + B)$ counts the number of points t_i lying inside $v + B$. Since $M(\theta) = M'(\theta) = M''(\theta) = e^\theta$, it follows from (2.2) and (2.5) that

$$e^I = \left(\frac{c}{\sigma_d(B)}\right)^c e^{-c+\sigma_d(B)}. \quad (3.1)$$

Substituting (2.2) and (3.1) into (2.8) and noting that F is arithmetic with span 1, we obtain

$$p_\lambda \sim (2\pi)^{-1/2} \sigma_d^{\lfloor \lambda c \rfloor}(B) c^{-\lfloor \lambda c \rfloor - 1/2} e^{\lambda c - \lambda \sigma_d(B)} \lambda^{d-1/2} \sigma_d(D) K \quad \text{as } \lambda \to \infty. \quad (3.2)$$

Example 3.2. Let $B = \prod_{k=1}^{d}[0, b_k]$ with $b_k > 0$ for all k. Since ∂B is a union of $2d$ faces, with a pair of them orthogonal to each coordinate vector, by (2.3) and (2.4),

$$Y(u) = \sum_{k=1}^{d}[Y_k^{(1)}(u_k) - Y_k^{(2)}(u_k)],$$

where $Y_1^{(1)}, \ldots, Y_d^{(1)}, Y_1^{(2)}, \ldots, Y_d^{(2)}$ are independent one-dimensional compound Poisson processes. The process $Y_k^{(1)}$, $1 \leq k \leq d$, is constructed from a marked point process with intensity $\prod_{\ell \neq k} b_\ell$ (the surface area of the face of B orthogonal to the kth coordinate vector) and mark distribution F. The process $Y_k^{(2)}$, $1 \leq k \leq d$, is constructed from a marked point process with intensity $M(\theta_c) \prod_{\ell \neq k} b_\ell$ and mark distribution F_c. If X is a random variable with distribution F_c, we shall let $\bar{F}_c$ denote the distribution of $-X$. We define $\bar{F}$ in a similar manner. First consider F nonarithmetic, and let $Y_k = Y_k^{(1)} - Y_k^{(2)}$. Then, by (2.6) and as $\mathrm{P}(\sup_{0 \leq u \leq m} \sum_{k=1}^{d} Y_k(u_k) \geq y) = 1$ for $y \leq 0$,

$$
\begin{aligned}
K_m &= \int_{-\infty}^{\infty} \mathrm{e}^{\theta_c y}\, \mathrm{P}\left(\sup_{0 \leq u \leq m} \sum_{k=1}^{d} Y_k(u_k) \geq y \right) \mathrm{d}y \\
&= \theta_c^{-1}\, \mathrm{E}\left(\exp\left(\theta_c \sum_{k=1}^{d} \sup_{0 \leq u_k \leq m} Y_k(u_k) \right) \right) \\
&= \theta_c^{d-1} \prod_{k=1}^{d} \int_{-\infty}^{\infty} \mathrm{e}^{\theta_c y}\, \mathrm{P}\left(\sup_{0 \leq u_k \leq m} Y_k(u_k) \geq y \right) \mathrm{d}y.
\end{aligned}
\tag{3.3}
$$

Since $Y_k^{(1)}$ and $Y_k^{(2)}$ are independent compound Poisson processes, it follows that $Y_k(u_k) = \sum_{j=1}^{N_k(u_k)} U_{kj}$, where N_k is a Poisson process with rate parameter $(\prod_{\ell \neq k} b_\ell)[1 + M(\theta_c)]$ and $U_{k1}, U_{k2}, \ldots$ are i.i.d. random variables independent of N_k such that

$$\mathrm{P}(U_{k1} \in \mathrm{d}u) = \frac{M(\theta_c)\bar{F}_c(\mathrm{d}u) + F(\mathrm{d}u)}{1 + M(\theta_c)}.\tag{3.4}$$

Let P_* be a probability measure under which the distribution of N_k is unchanged, and let $U_{k1}, U_{k2}, \ldots$ be i.i.d. random variables independent of N_k satisfying

$$\mathrm{P}_*(U_{k1} \in \mathrm{d}u) = \frac{\bar{F}(\mathrm{d}u) + M(\theta_c)F_c(\mathrm{d}u)}{1 + M(\theta_c)}.\tag{3.5}$$

By (2.2), (3.4), and (3.5),

$$\frac{\mathrm{dP}_*}{\mathrm{dP}} U_{k1} = \exp\{\theta_c U_{k1}\}.\tag{3.6}$$

Suppressing the notation k, let $R_\ell = U_1 + \cdots + U_\ell$ and $\tau_y = \inf\{\ell \geq 1 : R_\ell \geq y\}$. Define the overshoot constant

$$\nu_c = \lim_{y \to \infty} \mathrm{E}_*(\exp\{-\theta_c(R_{\tau_y} - y)\}),\tag{3.7}$$

where E_* denotes expectation with respect to P_*. See [23], [26]–[28], and Chapter 8 of [20] for analytical expressions of ν_c and also of related constants for sample size calculations.

By (3.5)–(3.7),

$$\int_{-\infty}^{\infty} e^{\theta_c y} P\left(\sup_{0 \le u_k \le m} Y_k(u_k) \ge y\right) dy$$

$$= \theta_c^{-1} + \int_0^{\infty} E_*(\exp\{\theta_c(y - R_{\tau_y})\} \mathbf{1}_{\{\sup_{0 \le u_k \le m} Y_k(u_k) \ge y\}}) \, dy$$

$$\sim \nu_c E_*\left(\sup_{0 \le u_k \le m} Y_k(u_k)\right)$$

$$\sim \nu_c m [c \sigma_d^{-1}(B) - \mu] \prod_{\ell \ne k} b_\ell, \tag{3.8}$$

noting that, by (2.2), under F_c, $E_c(X_1) = M'(\theta_c)/M(\theta_c) = c/[\sigma_d(B)M(\theta_c)]$, and by definition, under F, $E(X_1) = \mu$. Substitute (3.8) into (3.3). Then, by (2.7) and (2.8),

$$p_\lambda \sim [2\pi \sigma_d(B) M''(\theta_c)]^{-1/2} e^{-\lambda I + x_\lambda} \lambda^{d-1/2} \sigma_d(D) \{\nu_c [c \sigma_d^{-1}(B) - \mu]\}^d$$

$$\times \left(\chi_c \prod_{k=1}^d b_k\right)^{d-1}, \tag{3.9}$$

where $\chi_c = \theta_c$ for F nonarithmetic. Using similar arguments, (3.9) can also be shown to hold for F arithmetic with span η, by defining ν_c in (3.7) with $y \to \infty$ over $\eta \mathbb{Z}$ and $\chi_c = \eta^{-1}(1 - e^{-\eta \theta_c})$.

Example 3.3. If F has point mass at 1 then $\sup\{Y(u) : Y(u) < 0\} = -1$ almost surely, and, by (2.11) and Jensen's inequality,

$$K = E(\sigma_d^{-1}(\{u : Y(u) = 0\})) \ge \{E(\sigma_d(\{u : Y(u) = 0\}))\}^{-1}. \tag{3.10}$$

Let $U_d = \{t : \|t\| \le 1\}$ be the d-dimensional unit ball, and let

$$C_d = \frac{\sigma_{d-1}(\partial U_d)}{\sigma_{d-1}^d(U_{d-1})} = \frac{d\pi^{d/2}/\Gamma(d/2+1)}{[\pi^{(d-1)/2}/\Gamma((d+1)/2)]^d}.$$

Then

$$E[\sigma_d(\{u : Y(u) = 0\})] = C_d \int_0^{\infty} r^{d-1} \sum_{k=0}^{\infty} e^{-r(1+c)} \frac{c^k r^{2k}}{(k!)^2} \, dr$$

$$= C_d \sum_{k=0}^{\infty} \frac{c^k \Gamma(2k+d)}{(k!)^2 (1+c)^{2k+d}}$$

$$= \frac{C_d}{(1+c)^d} f_d^{(d-1)} \frac{\sqrt{c}}{(1+c)},$$

where $f_d(x) = x^d/\sqrt{1-4x^2}$ and $f_d^{(d-1)}$ is the $(d-1)$th derivative of f_d. A positive lower bound of K follows from (3.10). Similar computations can also be carried out for kernels of other shapes.

Example 3.4. Let Ω denote the random set of all u for which the sums in (2.3) are sums over an empty set for both $j = 1$ and $j = 2$. For a more precise definition, let $\{(t_i, y_i) : i \ge 1\}$ be a Poisson point process on $\partial B \times \mathbb{R}$ with intensity 1 on $\partial B \times [0, \infty)$ and intensity $M(\theta_c)$ on $\partial B \times (-\infty, 0)$. Then

$$\Omega = \{u : y_i \ge n_{t_i} \cdot u > 0 \text{ or } y_i < n_{t_i} \cdot u < 0 \text{ for all } i\}.$$

By (2.4), $\Omega \subset \{u \colon Y(u) = 0\}$ and, hence, by (2.11),

$$K \le \begin{cases} |\eta| \, E(\sigma_d^{-1}(\Omega)) & \text{if } F \text{ has point mass at } \eta, \\ \chi_c^{-1} \, E(\sigma_d^{-1}(\Omega)) & \text{otherwise.} \end{cases}$$

The upper bound of K for F having point mass at η uses the property

$$\sup\{Y(u) \colon Y(u) < 0\} = -|\eta| \quad \text{almost surely.}$$

4. Proof of Theorem 2.1

A key idea here is a change of measure argument that allows us to obtain, in Lemma 4.1, below, the tail probability of the maxima over a local domain. To obtain the global probabilities in Theorem 2.1 from these local probabilities, we adapt the Pickands–Quall–Watanabe technique from the Gaussian random field literature. Hence, the characterization of the constant K in Theorem 2.1 bears a striking resemblance to constants seen in the earlier papers on Gaussian random fields though the distribution of $Y(u)$ here is compound Poisson rather than Gaussian. Hogan and Siegmund [12] provided a vital framework for this correspondence.

Lemma 4.1. *Let* $E_{t,m,\lambda} = \{\sup_{t \le v \le t+m\lambda^{-1}} S(v+B) \ge \lambda c\}$. *Define* x_λ *as in the statement of Theorem 2.1. Then, for all* $t \in D$,

$$P_\lambda(E_{t,m,\lambda}) = P_\lambda(E_{0,m,\lambda}) \sim [2\pi\lambda\sigma_d(B)M''(\theta_c)]^{-1/2} e^{-\lambda I + x_\lambda} K_m \quad \text{as } \lambda \to \infty. \qquad (4.1)$$

Proof. By stationarity, $P_\lambda(E_{t,m,\lambda}) = P_\lambda(E_{0,m,\lambda})$. Let us first consider the case in which F is arithmetic with span 1. Then

$$\begin{aligned} P_\lambda(E_{0,m,\lambda}) &= P_\lambda\left(\sup_{0 \le v \le m\lambda^{-1}} S(v+B) \ge \lambda c\right) \\ &= P_\lambda\{S(B) \ge \lfloor \lambda c \rfloor\} \\ &\quad + \sum_{\ell=1}^\infty P_\lambda\left(S(B) = \lfloor \lambda c \rfloor - \ell, \sup_{0 \le v \le m\lambda^{-1}} [S(v+B) - S(B)] \ge \ell\right). \qquad (4.2) \end{aligned}$$

Let Q_λ be a probability measure under which $\mathcal{X}$ is a nonhomogeneous marked point process with intensity $\lambda M(\theta_c)$ and mark distribution F_c inside B, and intensity λ and mark distribution F outside B. Then

$$\frac{dQ_\lambda}{dP_\lambda}(\mathcal{X}) = \exp\{\theta_c S(B) - \lambda\sigma_d(B)[M(\theta_c) - 1]\}. \qquad (4.3)$$

Let

$$B_a = \bigcap_{0 \le v \le a} (v + B).$$

Since $S(B \setminus B_{m\lambda^{-1}})$ and $\sup_{0 \le v \le m\lambda^{-1}}[S(v+B) - S(B)]$ are functions of the marked point

process outside $B_{m\lambda^{-1}}$ and, hence, independent of $S(B_{m\lambda^{-1}})$ under Q_λ, it follows from (4.3) that

$$
P_\lambda\left(S(B) = \lfloor \lambda c \rfloor - \ell, \sup_{0 \leq v \leq m\lambda^{-1}} [S(v+B) - S(B)] \geq \ell\right)
$$

$$
= \exp\{-\lambda I + x_\lambda + \theta_c \ell\} \, Q_\lambda\left(S(B) = \lfloor \lambda c \rfloor - \ell, \sup_{0 \leq v \leq m\lambda^{-1}} [S(v+B) - S(B)] \geq \ell\right)
$$

$$
= \exp\{-\lambda I + x_\lambda + \theta_c \ell\} \sum_{k=0}^{\infty} Q_\lambda(S(B_{m\lambda^{-1}}) = \lfloor \lambda c \rfloor - \ell - k)
$$

$$
\times Q_\lambda\left(S(B \setminus B_{m\lambda^{-1}}) = k, \sup_{0 \leq v \leq m\lambda^{-1}} [S(v+B) - S(B)] \geq \ell\right).
$$

$$(4.4)$$

It follows from the local central limit theorem that, for each $\ell \in \mathbb{Z}$,

$$
Q_\lambda(S(B_{m\lambda^{-1}}) = \lfloor \lambda c \rfloor - \ell - k) \leq [1 + o(1)] \, Q_\lambda(S(B) = \lfloor \lambda c \rfloor - \ell)
$$
$$
\sim [2\pi \lambda \sigma_d(B) M''(\theta_c)]^{-1/2} \quad \text{as } \lambda \to \infty, \qquad (4.5)
$$

uniformly over $k \geq 0$, with '$\leq$' replaced by '$=$' if we look at (4.5) with k fixed. Hence, by (4.4) and (4.5),

$$
P_\lambda\left(S(B) = \lfloor \lambda c \rfloor - \ell, \sup_{0 \leq v \leq m\lambda^{-1}} [S(v+B) - S(B)] \geq \ell\right)
$$
$$
\sim [2\pi \lambda \sigma_d(B) M''(\theta_c)]^{-1/2} \exp\{-\lambda I + x_\lambda + \theta_c \ell\}
$$
$$
\times Q_\lambda\left(\sup_{0 \leq v \leq m\lambda^{-1}} [S(v+B) - S(B)] \geq \ell\right). \qquad (4.6)
$$

By (4.6) and the weak convergence of $\{S(\lambda^{-1}u + B) - S(B) \colon 0 \leq u \leq m\}$ under Q_λ to $\{Y(u) \colon 0 \leq u \leq m\}$ as $\lambda \to \infty$ (see (2.3) and (2.4)),

$$
\sum_{\ell=1}^{\infty} P_\lambda\left(S(B) = \lfloor \lambda c \rfloor - \ell, \sup_{0 \leq v \leq m\lambda^{-1}} [S(v+B) - S(B)] \geq \ell\right)
$$
$$
\sim e^{-\lambda I + x_\lambda} [2\pi \lambda \sigma_d(B) M''(\theta_c)]^{-1/2} \sum_{\ell=1}^{\infty} e^{\theta_c \ell} \, P\left(\sup_{0 \leq u \leq m} Y(u) \geq \ell\right). \qquad (4.7)
$$

By a similar application of (4.3) and (4.5),

$$
P_\lambda(S(B) \geq \lfloor \lambda c \rfloor) \sim [2\pi \lambda \sigma_d(B) M''(\theta_c)]^{-1/2} e^{-\lambda I + x_\lambda} \sum_{\ell=-\infty}^{0} e^{\theta_c \ell}. \qquad (4.8)
$$

Substitution of (4.7) and (4.8) into (4.2) then proves Lemma 4.1 when F is arithmetic with span 1. For F arithmetic with span η, we prove Lemma 4.1 by replacing the sums in (4.2), (4.4), (4.7), and (4.8) by $\sum_{\ell \in \eta\mathbb{Z}^+}$, $\sum_{k \geq 0, k \in \eta\mathbb{Z}}$, or $\sum_{\ell \leq 0, \ell \in \eta\mathbb{Z}}$. For nonarithmetic F, the sums are replaced by corresponding integrals. The details are similar to the arguments provided above and shall be omitted.

We now proceed to the proof of Theorem 2.1. Let $J = [0, 1]^d$,

$$\underline{C}_a = \{k \in (a\mathbb{Z})^d : k + aJ \subset D\}, \quad \text{and} \quad \overline{C}_a = \{k \in (a\mathbb{Z})^d : (k + aJ) \cap D \neq \varnothing\}.$$

Then $\{k + aJ : k \in \underline{C}_a\}$ and $\{k + aJ : k \in \overline{C}_a\}$ are lower and upper coverings of D, respectively, by cubes of length a. We shall show via Lemmas 4.2 and 4.3, below, that

$$\lim_{m \to \infty} \limsup_{\lambda \to \infty} \left[\frac{\sum_{u \in \underline{C}_{m\lambda^{-1}}} P_\lambda(\bigcup_{\{w \in \underline{C}_{m\lambda^{-1}} : w \neq u\}}(E_{u,m,\lambda} \cap E_{w,m,\lambda}))}{\lambda^{d-1/2} e^{-\lambda I}} \right] = 0. \quad (4.9)$$

The proofs of Lemmas 4.2, 4.3, and (4.9) will be given in the appendices. Let $f(\lambda) = [2\pi \sigma_d(B) M''(\theta_c)]^{-1/2} \lambda^{d-1/2} e^{-\lambda I + x_\lambda}$. Then, by Lemma 4.1,

$$P_\lambda(E_{u,m,\lambda}) \sim \frac{K_m f(\lambda)}{\lambda^d} \quad \text{as } \lambda \to \infty. \quad (4.10)$$

Given $\varepsilon > 0$, let m_ε be large enough such that, for all $m \geq m_\varepsilon$, the expression in the square brackets on the left-hand side of (4.9) does not exceed ε for all large λ. Then, by (4.10), for all $m \geq m_\varepsilon$,

$$\liminf_{\lambda \to \infty} \left[\lambda^{-d}(\#\underline{C}_{m/\lambda}) K_m - \frac{\varepsilon \lambda^{d-1/2} e^{-\lambda I}}{f(\lambda)} \right] \leq \liminf_{\lambda \to \infty} \left[\frac{p_\lambda}{f(\lambda)} \right]$$

$$\leq \limsup_{\lambda \to \infty} \left[\frac{p_\lambda}{f(\lambda)} \right]$$

$$\leq \limsup_{\lambda \to \infty} [\lambda^{-d}(\#\overline{C}_{m/\lambda}) K_m]. \quad (4.11)$$

Since D is Jordan measurable,

$$\#\underline{C}_a \sim \#\overline{C}_a \sim a^{-d} \sigma_d(D) \quad \text{as } a \to 0. \quad (4.12)$$

Noting that $\liminf_{\lambda \to \infty}[p_\lambda / f(\lambda)]$ and $\limsup_{\lambda \to \infty}[p_\lambda / f(\lambda)]$ are fixed real numbers and x_λ is bounded, it follows from (4.11) and (4.12) that $m^{-d} K_m$ is Cauchy. Hence,

$$K = \lim_{m \to \infty} m^{-d} K_m$$

exists and (2.8) follows from (4.11).

We will now state Lemmas 4.2 and 4.3 before proving Theorem 2.1. To avoid repetitive arguments, we will state and prove all subsequent results in this section and in the appendices assuming that F is arithmetic with span 1. The modifications required to extend these results to arbitrary F are straightforward and will not be discussed.

Lemma 4.2. *We have*

$$\lim_{r \to \infty} \limsup_{\lambda \to \infty} \left[\frac{P_\lambda(S(B) < \lfloor \lambda c \rfloor - r, \sup_{0 \leq v \leq \lambda^{-1}} S(v + B) \geq \lfloor \lambda c \rfloor)}{\lambda^{-1/2} e^{-\lambda I}} \right] = 0.$$

Lemma 4.3. *Let $r \geq 0$ and $L > 0$ be given. Then*

$$\lim_{k \to \infty} \limsup_{\lambda \to \infty} \left[\frac{\sum_{\{v \in (\lambda^{-1}\mathbb{Z})^d : k\lambda^{-1} \leq \|v\|_\infty \leq L\}} P_\lambda(S(B) \geq \lfloor \lambda c \rfloor - r, S(v + B) \geq \lfloor \lambda c \rfloor - r)}{\lambda^{-1/2} e^{-\lambda I}} \right] = 0.$$

Proof of Theorem 2.1. By the arguments in (4.9)–(4.12), it remains for us to show that K is positive and finite. By Lemma 4.2, there exists r large enough such that

$$P_\lambda\Big(S(B) < \lfloor\lambda c\rfloor - r, \sup_{0\le v\le\lambda^{-1}} S(v+B) \ge \lfloor\lambda c\rfloor\Big) \le \lambda^{-1/2}e^{-\lambda I} \quad \text{for all large } \lambda. \quad (4.13)$$

Moreover, by a modification of (4.8) (with $\sum_{\ell=-\infty}^{r}$ in place of $\sum_{\ell=-\infty}^{0}$),

$$P_\lambda(S(B) \ge \lfloor\lambda c\rfloor - r) = O(\lambda^{-1/2}e^{-\lambda I}). \quad (4.14)$$

By adding up (4.13) and (4.14), and applying Lemma 4.1, we show that $K_1 < \infty$. Then, by (2.8), (4.11), and (4.12), $K \le K_1 < \infty$.

Next, select ε small enough so that $\delta = [2\pi\sigma_d(B)M''(\theta_c)]^{-1/2}\sum_{\ell=-\infty}^{0} e^{\theta_c\ell} - \varepsilon > 0$. By Lemma 4.3 with $r = 0$ and $L > \sup_{x,y\in D} \|x - y\|_\infty$, there exists k large enough such that

$$\sum_{\{v\in(k\lambda^{-1}\mathbb{Z})^d : 0<\|v\|_\infty\le L\}} P_\lambda(S(B) \ge \lfloor\lambda c\rfloor, S(v+B) \ge \lfloor\lambda c\rfloor) \le \varepsilon\lambda^{-1/2}e^{-\lambda I} \quad (4.15)$$

for all large λ. Then, by stationarity, (4.8), and (4.15), noting that $x_\lambda \ge 0$,

$$P_\lambda(E_{\mathbf{0},m,\lambda}) \ge P_\lambda(S(u+B) \ge \lfloor\lambda c\rfloor \text{ for some } u \in (k\lambda^{-1}\mathbb{Z})^d, 0 \le u \le m\lambda^{-1})$$

$$\ge \sum_{\{u\in(k\lambda^{-1}\mathbb{Z})^d : 0\le u\le m\lambda^{-1}\}} \Big(P_\lambda(S(u+B) \ge \lfloor\lambda c\rfloor)$$

$$- \sum_{\{w\in(k\lambda^{-1}\mathbb{Z})^d : 0<\|w-u\|_\infty\le L\}} P\{S(u+B) \ge \lfloor\lambda c\rfloor, S(w+B) \ge \lfloor\lambda c\rfloor\}\Big)$$

$$\ge \Big(\frac{m}{k}\Big)^d \delta\lambda^{-1/2}e^{-\lambda I} \quad \text{for all large } \lambda,$$

and, by letting $m \to \infty$ with k fixed, it follows from (2.7), (4.1), and x_λ bounded that $K > 0$.

5. Proof of Theorem 2.2

The key trick here is a telescoping sum of (5.3) or (5.5), below, which removes the dependence of the representation of K on the supremum of Y. To simplify the notation, we shall first consider F having point mass at 1, select λ such that $x_\lambda = 0$ (i.e. $\lambda c \in \mathbb{Z}$), and write $S(v)$ in place of $S(v+B)$. Let

$$p_c(t, k, \mathrm{d}w) = P_\lambda\Big(\sup_{v\in D} S(v) = \lambda c, S(t) = \lambda c - k, \sigma_d(\{v \in D: S(v) = \lambda c - k\}) \in \lambda^{-d} \,\mathrm{d}w\Big). \quad (5.1)$$

By the change of measure argument in the proof of Lemma 4.1, in particular (4.4) and (4.5), and Lemmas 4.2 and 4.3, for any integer $k \ge 0$, t lying in the interior of D and $\mathrm{d}w \in (0, \infty)$,

$$p_c(t, k, \mathrm{d}w) = e^{-\lambda I+\theta_c k} Q_\lambda(S(t) = \lambda c - k) Q_\lambda\Big(\sup_{v\in D}[S(v) - S(t)] = k,$$

$$\sigma_d(\{v \in D: [S(v) - S(t)] = 0\}) \in \lambda^{-d} \,\mathrm{d}w \mid S(t) = \lambda c - k\Big)$$

$$\sim [2\pi\lambda\sigma_d(B)M''(\theta_c)]^{-1/2}e^{-\lambda I+\theta_c k}$$

$$\times P\Big(\sup_{u\in\mathbb{R}^d} Y(u) = k, \sigma_d(\{u : Y(u) = 0\}) \in \mathrm{d}w\Big). \quad (5.2)$$

The second relation in (5.2) follows from the weak convergence of $\{S(\lambda^{-1}u + t) - S(t): 0 \leq u \leq m\}$, under Q_λ and with conditioning on $S(t) = \lambda c - k$, to $\{Y(u): 0 \leq u \leq m\}$. Multiplying $p_c(t, k, \mathrm{d}w)$ by $\lambda^d(\mathrm{e}^{-\theta_c k} - \mathrm{e}^{-\theta_c(k+1)})/w$ and integrating over $t \in D$ and $w > 0$, we obtain, from (5.1) and (5.2),

$$
(\mathrm{e}^{-\theta_c k} - \mathrm{e}^{-\theta_c(k+1)}) \, \mathrm{P}_\lambda\left(\sup_{v \in D} S(v) = \lambda c, \; S(t) = \lambda c - k \text{ for some } t \in D\right)
$$

$$
= \lambda^d(\mathrm{e}^{-\theta_c k} - \mathrm{e}^{-\theta_c(k+1)}) \int_D \int_0^\infty \frac{p_c(t, k, \mathrm{d}w)}{w} \, \mathrm{d}t
$$

$$
\sim [2\pi \sigma_d(B) M''(\theta_c)]^{-1/2} \mathrm{e}^{-\lambda I} \lambda^{d-1/2} \sigma_d(D)(1 - \mathrm{e}^{-\theta_c})
$$

$$
\times \mathrm{E}(\sigma_d^{-1}(\{u: Y(u) = 0\}) \mathbf{1}_{\{\sup_{u \in \mathbb{R}^d} Y(u) = k\}}). \tag{5.3}
$$

We then sum up over $0 \leq k \leq k_0$, let $\lambda \to \infty$ followed by $k_0 \to \infty$, and compare against the asymptotic relation

$$
\mathrm{P}_\lambda\left(\sup_{v \in D} S(v) = \lambda c\right) \sim [2\pi \sigma_d(B) M''(\theta_c)]^{-1/2}(1 - \mathrm{e}^{-\theta_c})\mathrm{e}^{-\lambda I} \lambda^{d-1/2} \sigma_d(D) K, \tag{5.4}
$$

which follows from a straightforward modification of Theorem 2.1, to obtain the expression $K = \mathrm{E}(\sigma_d^{-1}(\{u: Y(u) = 0\}))$.

Let us next consider more generally F arithmetic with span 1. Let $k_0 > 0$, and when the event $\{\sup_{v \in D} S(v) = \lambda c\}$ occurs, let $\{r_i: 1 \leq i \leq I\}$ be a monotone rearrangement of $\{\lambda c - S(t): t \in D, \; 0 \leq \lambda c - S(t) \leq k_0\}$. In other words, $0 = r_0 < r_1 < \cdots < r_I \leq k_0$. Define, for $0 \leq k < \ell \leq k_0$,

$$
p_c(t, k, \ell, \mathrm{d}w) = \mathrm{P}\Big(\sup_{v \in D} S(v) = \lambda c, \; \lambda c - S(t) = k = r_i \text{ and }
$$

$$
\ell = r_{i+1} \text{ for some } i, \; \sigma_d(\{v: S(v) = \lambda c - k\}) \in \lambda^{-d} \, \mathrm{d}w\Big).
$$

Then, by the arguments in (5.2) and (5.3),

$$
(\mathrm{e}^{-\theta_c k} - \mathrm{e}^{-\theta_c \ell}) \, \mathrm{P}_\lambda\left(\sup_{v \in D} S(v) = \lambda c, \; r_i = k, r_{i+1} = \ell \text{ for some } i\right)
$$

$$
= \lambda^d(\mathrm{e}^{-\theta_c k} - \mathrm{e}^{-\theta_c \ell}) \int_D \int_0^\infty \frac{p_c(t, k, \ell, \mathrm{d}w)}{w} \, \mathrm{d}t
$$

$$
\sim [2\pi \sigma_d(B) M''(\theta_c)]^{-1/2} \mathrm{e}^{-\lambda I} \lambda^{d-1/2} \sigma_d(D)(1 - \mathrm{e}^{-\theta_c(\ell-k)})
$$

$$
\times \mathrm{E}(\sigma_d^{-1}(\{u: Y(u) = 0\}) \mathbf{1}_{\{\sup_{u \in \mathbb{R}^d} Y(u) = k, \; \sup_{u \in \mathbb{R}^d: Y(u) < 0} Y(u) = k - \ell\}}). \tag{5.5}
$$

We then sum up (5.5) over all integers $0 \leq k < \ell \leq k_0$, let $\lambda \to \infty$ followed by $k_0 \to \infty$, and compare against (5.4) to obtain the more general expression (2.11). For F arithmetic with span η, the arguments are similar, while for nonarithmetic F, we need to partition the range of $S(t)$, $t \in D$, into intervals of length ε, then let $\varepsilon \to 0$. The details are omitted.

6. Extension of representation to Gaussian random fields

Let $X(t)$, $t \in \mathbb{R}^d$, be a stationary, mean zero, continuous Gaussian random field satisfying

$$
\mathrm{E}(X(t)X(t+s)) \sim 1 - a\|s\|^\alpha \quad \text{as } \|s\| \to 0
$$

for some $0 < \alpha \le 2$ and $a > 0$. Let $\Delta_c = (c^2 a)^{-1/\alpha}$. It was shown in [6], [16], and [18] that

$$P\left(\sup_{t \in D} X(t) \ge c\right) \sim (2\pi)^{-1/2} c^{-1} e^{-c^2/2} \Delta_c^{-d} \sigma_d(D) \widetilde{K} \quad \text{as } c \to \infty \tag{6.1}$$

for some $\widetilde{K} > 0$, which is expressed in terms of a continuous Gaussian random field $\widetilde{Y}$ satisfying

$$\mathrm{E}(\widetilde{Y}(u)) = -\|u\|^\alpha, \qquad \mathrm{cov}(\widetilde{Y}(u), \widetilde{Y}(v)) = \|u\|^\alpha + \|v\|^\alpha - \|u-v\|^\alpha. \tag{6.2}$$

In particular,

$$\widetilde{K} = \lim_{m \to \infty} m^{-d} \int_0^\infty e^y \, P\left(\sup_{0 \le u \le m} \widetilde{Y}(u) \ge y\right) \mathrm{d}y.$$

The key argument in the proof of (6.1) is the weak convergence

$$\{c[X(t + \Delta_c u) - X(t)]: 0 \le u \le m\} \Rightarrow \{\widetilde{Y}(u): 0 \le u \le m\} \quad \text{for any } m > 0, \tag{6.3}$$

conditioned on $X(t) = c - y/c$ for some $y > 0$. In Theorem 6.1, below, we apply the approach of Section 5 to obtain a different representation of $\widetilde{K}$, one that involves only the occupation measure of $\widetilde{Y}$ near 0.

Theorem 6.1. *We have* $\widetilde{K} = \lim_{\xi \to 0} \int_0^\xi \mathrm{E}(\sigma_d^{-1}(\{u: -b < \widetilde{Y}(u) \le \xi - b\})) \, \mathrm{d}b$.

Example 6.1. By Theorem 6.1 and Jensen's inequality,

$$\widetilde{K} \ge \limsup_{\xi \to 0} \int_0^\xi \{\mathrm{E}(\sigma_d(\{u: -b < \widetilde{Y}(u) \le \xi - b\}))\}^{-1} \, \mathrm{d}b = \left(\int_{\mathbb{R}^d} f_t(0) \, \mathrm{d}t\right)^{-1}, \tag{6.4}$$

where f_t denotes the normal density of $\widetilde{Y}(t)$, with means and variances given in (6.2). Let $U_d = \{t: \|t\| \le 1\}$. Then, by (6.4),

$$\widetilde{K} \ge \left(\sigma_{d-1}(\partial U_d) \int_0^\infty r^{d-1} (4\pi r^\alpha)^{-1/2} \exp\left\{-\frac{r^{2\alpha}}{4r^\alpha}\right\} \mathrm{d}r\right)^{-1}$$

$$= d^{-1} \pi^{(1-d)/2} 4^{1-d/\alpha} \alpha \Gamma\left(\frac{d}{2} + 1\right) \Big/ \Gamma\left(\frac{d}{\alpha} - \frac{1}{2}\right). \tag{6.5}$$

In the case $\alpha = 2$, $\widetilde{Y}$ has a simple characterization from which $\widetilde{K} = \pi^{-d/2}$ can be computed. For $d = 2$, the second line of (6.5) is $\pi^{-1}(= \widetilde{K})$ and, for $d = 3$, it is $(4\sqrt{\pi})^{-1}$.

It is quite straightforward to improve upon (6.5). Consider $0 < \alpha < 2$ and select $t \ne 0$. Let $f_{s|t}(\cdot \mid y)$ denote the density of $\widetilde{Y}(s)$ conditioned on $\widetilde{Y}(t) = y$. Then, by a more refined application of Jensen's inequality,

$$\widetilde{K} \ge \int_{-\infty}^\infty \left(\int_{\mathbb{R}^d} f_{s|t}(0 \mid y) \, \mathrm{d}s\right)^{-1} f_t(y) \, \mathrm{d}y.$$

In principle, we can get even sharper computable lower bounds by conditioning on two or more ts.

Proof of Theorem 6.1. Let $\xi > 0$, $0 \le v < \xi$, $X_{\sup} = \sup_{t \in D} X(t)$, and $Y_{\sup} = \sup_{u \in \mathbb{R}^d} \widetilde{Y}(u)$. Let k be a nonnegative integer, $I_k = (X_{\sup} + [v - (k+1)\xi]/c, X_{\sup} + (v - k\xi)/c]$, and $J_k = [k\xi - v, (k+1)\xi - v)$. Define

$$g_{c,\xi,v}(t, k, \mathrm{d}w) = P(X_{\sup} \ge c, \, X(t) \in I_k, \, \sigma_d(\{u \in D: X(u) \in I_k\}) \in \Delta_c^d \, \mathrm{d}w).$$

By (6.3), for any t lying in the interior of D and $dw \in (0, \infty)$,

$$g_{c,\xi,v}(t, k, dw)$$

$$\sim (2\pi)^{-1/2}c^{-1}e^{-c^2/2} \int_{-\infty}^{\infty} e^y \, P\Big(X_{\sup} \geq c, \; X(t) \in I_k,$$

$$\sigma_d(\{u \in D \colon X(u) \in I_k\}) \in \Delta_c^d \, dw \mid X(t) = c - \frac{y}{c} \Big) \, dy$$

$$\sim (2\pi)^{-1/2}c^{-1}e^{-c^2/2} \Big\{ \Big(\int_{-\infty}^{k\xi - v} e^y \, dy \Big) P(Y_{\sup} \in J_k, \; \sigma_d(\{u \colon Y_{\sup} - \widetilde{Y}(u) \in J_k\}) \in dw)$$

$$+ \int_{k\xi - v}^{(k+1)\xi - v} e^y \, P(y \leq Y_{\sup} < (k+1)\xi - v,$$

$$\sigma_d(\{u \colon Y_{\sup} - \widetilde{Y}(u) \in J_k\}) \in dw) \, dy \Big\}. \quad (6.6)$$

Multiply (6.6) by $(e^{-k\xi} - e^{-(k+1)\xi})/(\Delta_c^d w)$, integrate over $t \in D$ and $w > 0$, then add over $0 \leq k \leq k_0$ for some positive integer $k_0 > 0$. We obtain

$$\sum_{k=0}^{k_0}(e^{-k\xi} - e^{-(k+1)\xi}) \, P(X_{\sup} \geq c, \; X(t) \in I_k \text{ for some } t \in D)$$

$$= \Delta_c^{-d} \sum_{k=0}^{k_0}(e^{-k\xi} - e^{-(k+1)\xi}) \int_D \int_0^{\infty} \frac{g_{c,\xi,v}(t, k, dw)}{w} \, dt$$

$$\sim (2\pi)^{-1/2}c^{-1}e^{-c^2/2}\Delta_c^{-d}\sigma_d(D)$$

$$\times \Big\{ (e^{-v} - e^{-v-\xi}) \sum_{k=0}^{k_0} E(\mathbf{1}_{\{Y_{\sup} \in J_k\}} \, \sigma_d^{-1}(\{u \colon Y_{\sup} - \widetilde{Y}(u) \in J_k\})) + o_\xi(1) \Big\}, \quad (6.7)$$

where $o_\xi(1) \to 0$ as $\xi \to 0$. Since $Y_{\sup} \in J_k$ is equivalent to $k = \lfloor (Y_{\sup} + v)/\xi \rfloor$, by (6.1) and (6.7), and letting $k_0 \to \infty$,

$$\widetilde{K} = \lim_{\xi \to 0} \int_0^{\xi} E\Big(\sigma_d^{-1}\Big(\Big\{ u \colon Y_{\sup} - \Big(\xi \Big\lfloor \frac{Y_{\sup} + v}{\xi} \Big\rfloor - v - \xi \Big) < \widetilde{Y}(u)$$

$$\leq Y_{\sup} - \Big(\xi \Big\lfloor \frac{Y_{\sup} + v}{\xi} \Big\rfloor - v \Big) \Big\} \Big) \Big) \, dv,$$

and Theorem 6.1 is shown.

Appendix A. Proof of Lemma 4.2

By (4.6) with $m = 1$ and the weak convergence of $S(\lambda^{-1}u + B) - S(B)$ to $Y(u)$ under Q_λ, as $\lambda \to \infty$,

$$P_\lambda\Big(S(B) < \lfloor \lambda c \rfloor - r, \; \sup_{0 \leq v \leq \lambda^{-1}} S(v + B) \geq \lfloor \lambda c \rfloor \Big)$$

$$\sim [2\pi\lambda\sigma_d(B)M''(\theta_c)]^{-1/2}e^{-\lambda I + x_\lambda} \sum_{\ell=r+1}^{\infty} e^{\theta_c\ell} P\Big(\sup_{0 \leq u \leq 1} Y(u) \geq \ell \Big). \quad (A.1)$$

Let $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. It follows from (2.3) and (2.4) that

$$\sup_{0 \le u \le 1} Y(u) \le Z^* = \sum_{\{i : v_i^{(1)} \in A^{(1)}\}} [Z_i^{(1)}]^+ + \sum_{\{i : v_i^{(2)} \in A^{(2)}\}} [Z_i^{(2)}]^-, \tag{A.2}$$

where $A^{(1)} = \partial B \times [0, d^{1/2})$ and $A^{(2)} = \partial B \times [-d^{-1/2}, 0)$. We can also express $Z^* = \sum_{j=1}^N V_j$, where N is a Poisson random variable with mean $\kappa = 1 - F(0) + M(\theta_c)F_c(0)$ and $V_1, V_2, \ldots$ are i.i.d. random variables independent of N with $g = E(e^{\tilde{\theta} V_1}) < \infty$ for some $\tilde{\theta} > \theta_c$. Since $E(e^{\tilde{\theta} Z^*}) = e^{\kappa(g-1)}$, it follows from Markov's inequality that $P(Z^* \ge \ell) \le e^{\kappa(g-1)-\tilde{\theta}\ell}$. Hence, by (A.2),

$$\sum_{\ell=r+1}^{\infty} e^{\theta_c \ell} P_\lambda \left(\sup_{0 \le u \le 1} Y(u) \ge \ell \right) = O(\exp\{\kappa(g - 1) - (\tilde{\theta} - \theta_c)r\}). \tag{A.3}$$

Since x_λ is bounded, Lemma 4.2 follows from (A.1) and (A.3).

Appendix B. Proof of Lemma 4.3

Let $\widetilde{Q}_\lambda(= \widetilde{Q}_{\lambda, v})$ be the probability measure under which the marked point process $\mathcal{X}$ has intensity $\lambda M(\theta_c)$ on $B_1 = B \cap (v+B)$, intensity $\lambda M(\theta_c/2)$ on $B_2 = \{B \setminus (v+B)\} \cup \{(v+B) \setminus B\}$, and intensity λ elsewhere on $\mathbb{R}^d$. We also require that, under $\widetilde{Q}_\lambda$, the marks have distribution F_c on B_1, distribution $\widetilde{F}$ satisfying $\widetilde{F}(dx) = e^{\theta_c x/2} F(dx)/M(\theta_c/2)$ on B_2, and F elsewhere on $\mathbb{R}^d$. Then

$$\frac{d\widetilde{Q}_\lambda}{dP_\lambda}(\mathcal{X}) = \left(\prod_{\{i : t_i \in B_1\}} \exp\{\theta_c X_i\} \right) \exp\{-\lambda \sigma_d(B_1)[M(\theta_c) - 1]\}$$

$$\times \left(\prod_{\{i : t_i \in B_2\}} \exp\left\{ \frac{\theta_c X_i}{2} \right\} \right) \exp\left\{ -\lambda \sigma_d(B_2) \left[M\left(\frac{\theta_c}{2}\right) - 1 \right] \right\}. \tag{B.1}$$

Since M is convex and $M(0) = 1$, $\zeta = [M(\theta_c) - 1] - 2[M(\theta_c/2) - 1] > 0$. We can thus express (B.1) as

$$\frac{d\widetilde{Q}_\lambda}{dP_\lambda}(\mathcal{X}) = \exp\left\{ \frac{\theta_c}{2}[S(B) + S(v+B)] - \frac{\lambda[\sigma_d(B) + \sigma_d(v+B)][M(\theta_c) - 1]}{2} \right.$$

$$\left. + \frac{\lambda \zeta \sigma_d(B_2)}{2} \right\},$$

and it follows from (2.5) and an analogue of (4.5) that

$$P_\lambda(S(B) \ge \lfloor \lambda c \rfloor - r, \ S(v+B) \ge \lfloor \lambda c \rfloor - r)$$

$$\le P_\lambda(S(B) + S(v+B) \ge 2(\lfloor \lambda c \rfloor - r))$$

$$= E_{\widetilde{Q}_\lambda} \left(\frac{dP_\lambda}{d\widetilde{Q}_\lambda} \mathbf{1}_{\{S(B)+S(v+B) \ge 2(\lfloor \lambda c \rfloor - r)\}} \right)$$

$$= O(\exp\{-\lambda I - \lambda \zeta \sigma_d(B \setminus (v+B))\} \lambda^{-1/2}). \tag{B.2}$$

Let $\|e\| = 1$, and let $\Pi_e = \{b - (e \cdot b)e : b \in B\}$ be the projected surface of B on a $(d-1)$-dimensional hyperplane orthogonal to e. Then $\beta = \inf_{\|e\|=1} \sigma_{d-1}(\Pi_e) > 0$. Hence, there

exists $\varepsilon > 0$ such that

$$\sigma_d(B \setminus (v + B)) \geq \frac{\|v\|\beta}{2} \geq \frac{\|v\|_\infty \beta}{2} \quad \text{for all } \|v\|_\infty \leq \varepsilon. \tag{B.3}$$

By (B.2) and (B.3), it follows that

$$\sum_{\{v \in (\lambda^{-1}\mathbb{Z})^d : k\lambda^{-1} \leq \|v\|_\infty \leq \varepsilon\}} P_\lambda(S(B) \geq \lfloor \lambda c \rfloor - r, \; S(v + B) \geq \lfloor \lambda c \rfloor - r)$$

$$= O\left(\lambda^{-1/2} e^{-\lambda I} \sum_{\ell \geq k} \ell^{d-1} e^{-\zeta \ell \beta/2} \right)$$

$$= O(\lambda^{-1/2} e^{-\lambda I} k^{d-1} e^{-\zeta k \beta/2}). \tag{B.4}$$

Since $\alpha = \inf_{\|v\|_\infty > \varepsilon} \sigma_d(B \setminus (v + B)) > 0$, it also follows from (B.2) that

$$\sum_{\{v \in (\lambda^{-1}\mathbb{Z})^d : \varepsilon < \|v\|_\infty \leq L\}} P_\lambda(S(B) \geq \lfloor \lambda c \rfloor - r, \; S(v + B) \geq \lfloor \lambda c \rfloor - r) = O(\lambda^{d-1/2} e^{-\lambda I - \lambda \zeta \alpha}). \tag{B.5}$$

Lemma 4.3 then follows from combining (B.4) and (B.5).

Appendix C. Proof of (4.9)

Let $\varepsilon > 0$. By Lemma 4.2 and stationarity, we can select r large enough such that

$$\gamma_{u,\lambda} = P_\lambda\left(S(u + B) < \lfloor \lambda c \rfloor - r, \; \sup_{u \leq v \leq u+1} S(v + B) \geq \lfloor \lambda c \rfloor \right) \leq \varepsilon \lambda^{-1/2} e^{-\lambda I} \tag{C.1}$$

for all large λ. Let $k = \lfloor m^{1/2} \rfloor$, $\Gamma_m = \{t \in \mathbb{Z}^d : k \leq t \leq (m-k)\}$, and $\Omega_m = \{t \in \mathbb{Z}^d : 0 \leq t \leq m\} \setminus \Gamma_m$. Then

$$\sum_{u \in \underline{C}_{m\lambda^{-1}}} P_\lambda\left(\bigcup_{\{w \in \underline{C}_{m\lambda^{-1}} : w \neq u\}} (E_{u,m,\lambda} \cap E_{w,m,\lambda}) \right)$$

$$\leq \sum_{u \in \underline{C}_{\lambda^{-1}}} \gamma_{u,\lambda} + \sum_{\{u,w \in \underline{C}_{m\lambda^{-1}} : w \neq u\}} P_\lambda(G_{u,m,\lambda} \cap G_{w,m,\lambda}) + \sum_{u \in \underline{C}_{m\lambda^{-1}}} P_\lambda(H_{u,m,\lambda}), \tag{C.2}$$

where

$$G_{u,m,\lambda} = P_\lambda(S(v + B) \geq \lfloor \lambda c \rfloor - r \text{ for some } v \in u + \lambda^{-1}\Gamma_m), \tag{C.3}$$

$$H_{u,m,\lambda} = P_\lambda(S(v + B) \geq \lfloor \lambda c \rfloor - r \text{ for some } v \in u + \lambda^{-1}\Omega_m). \tag{C.4}$$

By (4.12) and (C.1),

$$\sum_{u \in \underline{C}_{\lambda^{-1}}} \gamma_{u,\lambda} \leq [\varepsilon + o(1)]\lambda^{d-1/2} e^{-\lambda I} \sigma_d(D). \tag{C.5}$$

Let $L > \sup_{x,y \in D} \|x - y\|_\infty$. Then, by (C.3), stationarity, and Lemma 4.3, there exists m large enough such that, for all $u \in \underline{C}_{m\lambda^{-1}}$ and large λ,

$$\sum_{\{w \in \underline{C}_{m\lambda^{-1}} : w \neq u\}} P_\lambda(G_{u,m,\lambda} \cap G_{w,m,\lambda})$$

$$\leq m^d \sum_{\{v \in (\lambda^{-1}\mathbb{Z})^d : k\lambda^{-1} \leq \|v\|_\infty \leq L\}} P_\lambda(S(B) \geq \lfloor \lambda c \rfloor - r, \ S(v + B) \geq \lfloor \lambda c \rfloor - r)$$

$$\leq [\varepsilon + o(1)]m^d \lambda^{-1/2} e^{-\lambda I}.$$

Hence, by (4.12),

$$\sum_{\{u,w \in \underline{C}_{m\lambda^{-1}} : w \neq u\}} P_\lambda(G_{u,m,\lambda} \cap G_{w,m,\lambda}) \leq \sigma_d(D)[\varepsilon + o(1)]\lambda^{d-1/2} e^{-\lambda I}. \qquad (C.6)$$

Since $k = \lfloor m^{1/2} \rfloor$, it follows that $\#\Omega_m = (m+1)^d - (m+1-2k)^d = O(m^{d-1/2})$. By (4.12), (C.4), and a modification of (4.8) (with $\sum_{\ell=-\infty}^{r}$ in place of $\sum_{\ell=-\infty}^{0}$),

$$\sum_{u \in \underline{C}_{m\lambda^{-1}}} P_\lambda(H_{u,m,\lambda}) = O(m^{d-1/2}(\lambda m^{-1})^d \sigma_d(D)\lambda^{-1/2} e^{-\lambda I}) \leq \varepsilon \lambda^{d-1/2} e^{-\lambda I} \sigma_d(D) \quad (C.7)$$

for all large m. We then obtain (4.9) from (C.2) and (C.5)–(C.7) by choosing ε arbitrarily small.

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