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Extensions of Rings Having McCoy Condition

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Abstract. Let R be an associative ring with unity. Then R is said to be a *right McCoy ring* when the equation $f(x)g(x) = 0$ (over $R[x]$), where $0 \neq f(x), g(x) \in R[x]$, implies that there exists a nonzero element $c \in R$ such that $f(x)c = 0$. In this paper, we characterize some basic ring extensions of right McCoy rings and we prove that if R is a right McCoy ring, then $R[x]/(x^n)$ is a right McCoy ring for any positive integer $n \geq 2$.

1 Introduction

In [9, Theorem 2], McCoy proved that if R is a commutative ring then, whenever $g(x)$ is a zero divisor in $R[x]$, there exists a nonzero element $c \in R$ such that $cg(x) = 0$. Let $S[x]$ and $R[x]$ be the polynomial rings over rings S and R, respectively. Given a module M, let $M[x]$ be the set of all formal polynomials in indeterminate x with coefficients from M. Then $M[x]$ becomes an $(S[x], R[x])$-bimodule under usual addition and multiplication of polynomials. Assume that M is an R-module such that $ma = 0$ implies $mRa = 0$, for any $m \in M$ and $a \in R$. In [2, Corollary 2.8] it is proved that if $m'(x)$ is a torsion element in $M[x]$, then there exists a non zero element $c \in R$ such that $m'(x)c = 0$.

According to Nielsen [10], a ring R is said to be a *right McCoy ring* when the equation $f(x)g(x) = 0$ (over $R[x]$), where $f(x), g(x) \in R[x]/\{0\}$, implies that there exists a nonzero element $c \in R$ such that $f(x)c = 0$. The definition of a left McCoy ring is similar. If R is both a left and a right McCoy ring, then R is called a *McCoy ring*.

Recall that a ring R is called a *reduced ring* if it has no nonzero nilpotent elements. It is well known that if R is a reduced ring, then the following condition holds: $ab = 0$ implies $ba = 0$, for all $a, b \in R$. Cohn [3] called a ring R a *reversible ring* if it satisfies this condition. Clearly, reduced and commutative rings are reversible. By [10, Theorem 2], every reversible ring is a McCoy ring.

Another generalization of a reduced ring is an Armendariz ring. A ring R is said to be an *Armendariz ring* if, whenever polynomials $f(x) = a_0 + a_1x + \cdots + a_mx^m$, $g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j. It is easy to see that all Armendariz rings are McCoy rings.

Recall that a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$, for $a, b \in R$. Clearly, all reversible rings are semicommutative. In [4, Corollary 2.3], Hirano proved that if R is a semicommutative ring, then whenever $f(x)$ is a zero divisor

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in $R[x]$ there exists a nonzero element $c \in R$ such that $f(x)c = 0$. In this study, Hirano assumed that if R is a semicommutative ring then $R[x]$ is also a semicommutative ring. But, in [5, Example 3], the authors show that this assumption is false. Therefore, the question of whether semicommutativity implied the McCoy condition was left open.

In this note, we will discuss some basic ring extensions of right McCoy rings. The following will be proved:

Theorem 1.1 *Let R be a ring. Then R is a right McCoy ring if and only if $R[x]$ is a right McCoy ring.*

Given a ring R and a bimodule ${}_RM_R$, the trivial extension of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is the subring $\{\left(\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right) : a \in R, m \in M\}$ of the formal triangular ring $\left(\begin{smallmatrix} R & M \\ 0 & R \end{smallmatrix}\right)$. We also show the following.

Theorem 1.2 *Let R be a ring.*

(1) *The trivial extension $T(R, R)$ is a right McCoy ring if and only if R is a right McCoy ring.*
(2) *R is right McCoy ring if and only if the classical right quotient ring $Q(R)$ of R is right McCoy ring.*

Throughout this paper, we assume that R is an associative ring with unity.

2 Extensions of Right McCoy Rings

We start the trivial extension of a right McCoy ring.

Theorem 2.1 *Let R be a ring. Then R is a right McCoy ring if and only if the trivial extension $T(R, R)$ is a right McCoy ring.*

Proof Assume that the trivial extension $T(R, R)$ is a right McCoy ring. Let

$$0 \neq f(x) = \sum_{i=0}^{m} a_i x^i \quad \text{and} \quad 0 \neq g(x) = \sum_{j=0}^{n} b_j x^j$$

be two elements in $R[x]$ with $f(x)g(x) = 0$. So we may construct the following two elements in $T(R, R)$ such that $F(x) = \sum_{i=0}^{m} A_i x^i$ and $G(x) = \sum_{j=0}^{n} B_i x^j$, where $A_i = \left(\begin{smallmatrix} a_i & 0 \\ 0 & a_i \end{smallmatrix}\right)$ and $B_j = \left(\begin{smallmatrix} b_j & 0 \\ 0 & b_j \end{smallmatrix}\right)$. Since $0 \neq f(x)$ and $0 \neq g(x)$, we have $F(x) \neq 0$ and $G(x) \neq 0$. Note that $F(x)G(x) = 0$ in $T(R, R)[x]$, because $f(x)g(x) = 0$ in $R[x]$. Then there exists a nonzero element $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ in $T(R, R)$ such that $F(x)\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) = 0$. Since the element $\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right)$ is nonzero, we have $a \neq 0$ or $b \neq 0$. Then $\left(\begin{smallmatrix} a_ia & a_ib \\ 0 & a_ia \end{smallmatrix}\right) = 0$ for $0 \leq i \leq m$. This implies that $a_ia = 0$ and $a_ib = 0$, for $0 \leq i \leq m$. Hence we have $f(x)a = 0$ and $f(x)b = 0$. Therefore, the ring R is a right McCoy ring.

Assume that R is a right McCoy ring. Let $R' = T(R, R)$. Let

$$0 \neq F[x] = \begin{pmatrix} a_0 & b_0 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} x + \cdots + \begin{pmatrix} a_m & b_m \\ 0 & a_m \end{pmatrix} x^m$$

and

$$0 \neq G[x] = \begin{pmatrix} a_0' & b_0' \\ 0 & a_0' \end{pmatrix} + \begin{pmatrix} a_1' & b_1' \\ 0 & a_1' \end{pmatrix} x + \cdots + \begin{pmatrix} a_n' & b_n' \\ 0 & a_n' \end{pmatrix} x^n$$

be two elements in $R'[x]$ such that $F[x]G[x] = 0$. Let

$$f_1(x) = a_0 + a_1 x + \cdots + a_m x^m, \quad f_2(x) = b_0 + b_1 x + \cdots + b_m x^m$$

and

$$g_1(x) = a_0' + a_1' x + \cdots + a_n' x^n, \quad g_2(x) = b_0' + b_1' x + \cdots + b_n x^n.$$

Then $f_1(x), f_2(x), g_1(x), g_2(x) \in R[x]$, and it follows that

$$\begin{aligned} 0 &= \begin{pmatrix} f_1(x) & f_2(x) \\ 0 & f_1(x) \end{pmatrix} \begin{pmatrix} g_1(x) & g_2(x) \\ 0 & g_1(x) \end{pmatrix} \\ &= \begin{pmatrix} f_1(x)g_1(x) & f_1(x)g_2(x) + f_2(x)g_1(x) \\ 0 & f_1(x)g_1(x) \end{pmatrix}, \end{aligned}$$

since $F[x]G[x] = 0$. Thus we have $f_1(x)g_1(x) = 0$ and $f_1(x)g_2(x) + f_2(x)g_1(x) = 0$. Next we break the proof into nine cases.

Case 1 Let $f_1(x) \neq 0, f_2(x) \neq 0, g_1(x) \neq 0$, and $g_2(x) \neq 0$. So there exists a nonzero element c in R such that $f_1(x)c = 0$ since $f_1(x)g_1(x) = 0$ and R is right McCoy ring. Hence there exists a nonzero element $\left(\begin{smallmatrix} 0 & c \\ 0 & o \end{smallmatrix}\right)$ in R' such that $F(x)\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right) = 0$. Therefore, the ring R' is a right McCoy ring.

Case 2 Let $f_1(x) \neq 0, f_2(x) \neq 0, g_1(x) \neq 0$, and $g_2(x) = 0$. Then we may again choose $0 \neq \left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right) \in R'$.

Case 3 Let $f_1(x) \neq 0, f_2(x) \neq 0, g_1(x) = 0$, and $g_2(x) \neq 0$. Then we have $f_1(x)g_2(x) = 0$. Since R is a right McCoy ring, then there exists a nonzero element c in R such that $f_1(x)c = 0$. Hence there exists a nonzero element $\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right)$ in R' such that $F(x)\left(\begin{smallmatrix} 0 & c \\ 0 & 0 \end{smallmatrix}\right) = 0$. Therefore, the ring R' is a right McCoy ring.

Case 4 Let $f_1(x) \neq 0, f_2(x) = 0, g_1(x) \neq 0$, and $g_2(x) \neq 0$. Then we have $f_1(x)g_1(x) = 0$. So there exists a nonzero element c in R such that $f_1(x)c = 0$. Hence there exists a $0 \neq \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \in R'$ such that $F(x)\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) = 0$. Therefore, the ring R' is a right McCoy ring.

Case 5 Let $f_1(x) \neq 0, f_2(x) = 0, g_1(x) \neq 0$, and $g_2(x) = 0$. Then we have $f_1(x)g_1(x) = 0$. So there exists a nonzero element c in R such that $f_1(x)c = 0$. Hence there exists a $0 \neq \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) \in R'$ such that $F(x)\left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right) = 0$. Therefore, the ring R' is a right McCoy ring.

Case 6 Let $f_1(x) = 0$, $f_2(x) \neq 0$, $g_1(x) = 0$, and $g_2(x) \neq 0$. So we may choose the element $0 \neq \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in R'$.

The other possibilities are similar to cases (1)–(5). ∎

The following example shows that $T(R, R)$ is not an Armendariz ring, even if R is a right McCoy ring.

Example 2.2 Let T be a reduced ring. Then $R = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) : a, b \in T\}$ is an Armendariz ring by [6, Corollary 4]. Therefore, R is a right McCoy ring. By Theorem 2.1, the trivial extension $R' = \{\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right) : A, B \in R\}$ of R is a right McCoy ring. But R' is not an Armendariz ring by [6, Example 5].

The following example shows that $T(R, R)$ is not a reversible ring, even if R is a right McCoy ring.

Example 2.3 Let T be a reduced ring. By [10, Theorem 2], T is a right McCoy ring. We consider the ring

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in T \right\}.$$

By Theorem 2.1, the ring R is a right McCoy ring, but R is not a reversible ring by [7, Example 1.5].

Example 2.4 Let R be a right McCoy ring. The 6×6 upper triangular matrix ring $\mathbb{T}_6$ of R is not a right McCoy ring. Let

$$f(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $f(x)g(x) = 0$, but there is no nonzero element c in $\mathbb{T}_6$ such that $f(x)c = 0$. Hence $\mathbb{T}_6$ is not a right McCoy ring.

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By the same notation in [8], $B_4(R)$ stands for the ring

$$B_4(R) = \left\{ \begin{pmatrix} a_1 & a & b & c \\ 0 & a_1 & d & r \\ 0 & 0 & a_1 & s \\ 0 & 0 & 0 & a_1 \end{pmatrix} : a_1, a, b, c, d, r, s \in R \right\}.$$

Theorem 2.5 *Let R be a ring. Then R is a right McCoy ring if and only if $B_4(R)$ is a right McCoy ring.*

Proof Assume that $B_4(R)$ is a right McCoy ring. Let $0 \neq f(x) = \sum_{i=0}^{k} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{t} b_j x^j$ be two elements over $R[x]$ with $f(x)g(x) = 0$. Let $E_{i,j}$ be the usual matrix units (with 1 in the (i, j)-coordinate and zero elsewhere). Then $\left(\sum_{i=0}^{k} E_{i,j} x^i\right)\left(\sum_{j=0}^{t} E_{i,j} x^j\right) = 0$. Since $B_4(R)$ is a right Mccoy ring, there exists an element $0 \neq c \in B_4(R)$ such that $\left(\sum_{i=0}^{k} E_{i,j} x^i\right) c = 0$. This also implies that there exists an element $0 \neq r \in R$ such that $f(x)r = 0$.

For the converse, the proof is similar to the proof of Theorem 2.1. ■

Example 2.6 In Theorem 2.5, we proved that if R is a right McCoy ring, then $B_4(R)$ is a right McCoy ring but is not an Armendariz ring by [8, Example 1.1].

In [6, Theorem 16], Kim and Lee proved that a ring R is reduced if and only if the classical right quotient ring $Q(R)$ of R is reduced. The following theorem generalizes [5, Theorem 12] for Armendariz rings and [7, Theorem 2.6] for reversible rings to McCoy rings.

Theorem 2.7 *Suppose that the classical right quotient ring $Q(R)$ of R exists. Then R is a right McCoy ring if and only if $Q(R)$ is a right McCoy ring.*

Proof It is enough to show that if R is a right McCoy ring, then $Q(R)$ is a right McCoy ring. Let $0 \neq f(x) = \sum_{i=0}^{k} a_i x^i$ and $0 \neq g(x) = \sum_{j=0}^{t} b_j x^j$ be two elements in $Q(R)[x]$ with $f(x)g(x) = 0$. Then there exists $c_i, d_j, u, v \in R$ with u, v regular such that $a_i = c_i u^{-1}$ and $b_i = d_i v^{-1}$. Clearly $u^{-1} d_j \in Q(R)$ for each j, and so there exists $z_j, w \in R$ such that $u^{-1} d_j = z_j w^{-1}$. Let $f'(x) = \sum_{i=0}^{k} c_i x^i$ and $g'(x) = \sum_{j=0}^{t} z_j x^j$. It is easy to see that $0 \neq f'(x) \in R[x]$ and $0 \neq g'(x) \in R[x]$. Now from $f(x)g(x) = 0$, we have $f'(x)g'(x)(vw)^{-1} = 0$. Since R is a right McCoy ring, there exists $0 \neq c \in R$ such that $f'(x)c = 0$. Because $c \neq 0$ and $u^{-1}v^{-1} \in Q(R)$, there exists $0 \neq c', y \in R$ with y regular such that $u^{-1}c' = cy^{-1}$. Now from $f(x)c' = f'(x)cy^{-1} = 0$, the classical quotient $Q(R)$ is a right McCoy ring. ■

The following results are known:

(1) R is an Armendariz ring if and only if $R[x]$ is an Armendariz ring (see [1]).

(2) Let $n \geq 2$. Then R is reduced if and only if $R[x]/(x^n)$ is an Armendariz ring (see [8]).

Theorem 2.8 *Let R be a ring and $n \geq 2$ be a positive integer.*

(1) *R is a right McCoy ring if and only if $R[x]$ is a right McCoy ring.*

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(2) *If R is a right McCoy ring, then $R[x]/(x^n)$ is a right McCoy ring.*

Proof (1) Assume that R is a right McCoy ring. Let $0 \neq f(Y) = a_0 + a_1 Y + \cdots + a_m Y^m$ and $0 \neq g(Y) = b_0 + b_1 Y + \cdots + b_n Y^n$ be two elements in $R[x][Y]$ with $f(Y)g(Y) = 0$, where $a_i, b_j \in R[x]$. Let $t = \max(\deg(a_i)) + 1$. Clearly, $f(x^t) = a_0 + a_1 x + \cdots + a_m x^{mt}$ and $g(x^t) = b_0 + b_1 x + \cdots + b_n x^{nt}$. Note that the set of coefficients of the polynomial $f(x^t)$ is equal to the set of coefficients of $f(Y)$. This implies that $f(x^t)g(x^t) = 0$. Since R is a right McCoy ring, there exists an element $0 \neq c \in R$ (also $R[x]$) such that $f(x^t)c = 0$ and so $f(Y)c = 0$.

The converse is clear.

(2) Let $y = \overline{x} \in R[x]/(x^n) = S$. Then $S[y] = R + Ry + \cdots + Ry^{n-1}$ because $y^n = 0$. Let $0 \neq f = \sum_{i=1}^{k} f_i z^i$ and $g = \sum_{j=1}^{t} g_j z^j$ be two elements in $R[y][z]$ with $fg = 0$, where $f_i = \sum_{u=0}^{n-1} a_u^i y^u$ and $g_j = \sum_{v=0}^{n-1} b_v^j y^v$. If we repeat the proof of (1), we have two cases on $\sum_{i=0}^{k} a_s^i y^i$. Now, it is easy to see that S is a right McCoy ring ■

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