A NOTE ON A FUNCTIONAL EQUATION

CHUNG-MING AN

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The object of this note is to give an aspect to the problem of the functional equation of the generalized gamma function and Dirichlet series which are defined in [1].[1] In general, we cannot answer the problem yet. But it is worthy to attack this problem for some special cases.

Throughout this note $\mathbb{R}$ and Z denote the reals and the integers respectively. Let $F(X)$ and $E(X)$ be two homogeneous polynomials in $\mathbb{R}[X]$, $X = (X_1, \cdots, X_n)$, of degree $d > 0$. Asume that $F(x) \neq 0$, $E(x) \neq 0$ for non-zero $x \in \mathbb{R}^n$. We put

$$\zeta(F, s) = \sum_{\gamma \in Z^n - \{0\}} F(\gamma)^{-s},$$

$$\Gamma(E, s) = \int_{\mathbb{R}^n} e^{-|x|^2} E(x)^{s-(n/d)} dx,$$

$$\xi(E, F, s) = \pi^{-(ds)/2} \Gamma(E, s) \zeta(F, s),$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$ and $dx = dx_1 \cdots dx_n$. It is proved in [1] that $Z(F, s)$ and $\Gamma(E, s)$ are meromorphic functions of s.

If $n = 1$, we may assume $F(X) = aX^d$, $a > 0$, and put $F^{-1}(X) = (1/a)X^d$. We denote the number a by $|F|$. We shall use the same notations for $E(X)$.

If $n \geqq 2$, we only consider the case of the quadratic forms, i.e., $d = 2$. Let

$$E(X) = \sum_{i,j=1}^{n} e_{ij} X_i X_j, \quad F(X) = \sum_{i,j=1}^{n} f_{ij} X_i X_j$$

be two positive-definite quadratic forms. We may assume that $E = (e_{ij})$ and $F = (f_{ij})$ are two $n \times n$ positive-definite matrices with real entries. So $|E| = \det(E) \neq 0$, $|F| = \det(F) \neq 0$. Let E^{-1} and F^{-1} be the inverse matrices of E and F, respectively.

[1] The result in [1] appeared in the Bulletin of the American Mathematical Society, May 1969.

The main result of this paper is

THEOREM. *Under the above assumptions, $\xi(E,F,s)$ satisfies the following functional equation*

$$\xi\left(E^{-1},F^{-1},\frac{n}{d}-s\right)=|E|^{\frac{1}{2}},|F|^{\frac{1}{2}},\xi(E,F,s).$$

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For $n=1$, we observe

$$\zeta(F,s) = a^{-s}(1+K^{-s})\,\zeta(ds)$$

$$\Gamma(E,s) = a^{s-n/d}(1+K^{s-n/d})\,\Gamma\left(\frac{ds}{2}\right)$$

where $K=(-1)^d$, $\xi(s)$ is the Riemann-zeta function and $\Gamma(s)$ is the gamma function. Then, the theorem for $n=1$ follows immediately from the functional equation of the Riemann zeta function.

For $n \geqq 2$ and $d=2$, we shall use the polar coordinates in n-dimension ([1]). Thus, we have

$$\Gamma(E,s)=\tfrac{1}{2}\Gamma(s)\int_{S^{n-1}}E(\omega)^{s-n/2}d\omega.$$

We want to prove the following lemma:

LEMMA. *If $n \geqq 2$ and $d=2$, then*

$$|E|^{\frac{1}{2}}\int_{S^{n-1}}E(\omega)^{s-n/2}d\omega=\int_{S^{n-1}}E^{-1}(\omega)^{-s}d\omega.$$

PROOF. It is well known that there is an orthogonal $n\times n$ matrix U, i.e., $U^{-1}={}^tU$, such that

$${}^tUEU=A=\begin{bmatrix} a_1 & & & \bigcirc \\ & \cdot & & \\ & & \cdot & \\ \bigcirc & & & \cdot \\ & & & & a_n \end{bmatrix},\qquad a_i>0.$$

Viewing X as a $n\times 1$ matrix, $E(X)={}^tXEX$ and changing variables by $X=UY$, we shall have $dx=dy$, $E(X)=A(Y)$ and $|x|^2=|y|^2$. Hence, $\Gamma(E,s)=\Gamma(A,s)$, i.e.,

$$\int_{S^{n-1}}A(\omega)^{s-n/2}d\omega=\int_{S^{n-1}}E(\omega)^{s-n/2}d\omega.$$

Thus, it is enough to prove the lemma for the matrix A.

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Let

$$T = \begin{bmatrix} (a_1)^{\frac{1}{2}} & & & \\ & \cdot & & O \\ & & \cdot & \\ & O & & \cdot \\ & & & (a_n)^{\frac{1}{2}} \end{bmatrix}, \qquad \text{i.e., } A = T^2$$

and $z = Ty$. We shall have $dy = |T|^{-1}dz$, $A(y) = |z|^2$ and $A^{-1}(z) = |y|^2$.

So

$$\Gamma(A, s) = |A|^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-A^{-1}(z)} |z|^{2s-1} dz.$$

By changing variables into the polar coordinates, i.e., $z = r\omega$,

we get

$$\Gamma(A, s) = |A|^{-\frac{1}{2}} \int_{S^{n-1}} \int_0^\infty e^{-r^2 A^{-1}(w)} r^{2s-1} dr d\omega.$$

Put

$$H(\omega) = \int_0^\infty e^{-r^2 A^{-1}(w)} r^{2s-1} dr.$$

Since $A^{-1}(\omega) > 0$, for all $\omega \in S^{n-1}$, we may put $t = (A^{-1}(\omega))^{\frac{1}{2}} r$.
Then

$$H(\omega) = \int_0^\infty e^{-t^2} t^{2s-1} (A^{-1}(\omega))^{-s} dt = \tfrac{1}{2}(A^{-1}(\omega))^{-s}\Gamma(s).$$

So, we shall have

$$\Gamma(A, s) = \tfrac{1}{2}|A|^{-\frac{1}{2}}\Gamma(s) \int_{S^{n-1}} (A^{-1}(\omega))^{-s} d\omega.$$

But, on the other hand

$$\Gamma(A, s) = \tfrac{1}{2}\Gamma(s) \int_{S^{n-1}} (A(\omega))^{s-\frac{1}{2}n} d\omega.$$

From above two forms, it is clear to see the lemma. q.e.d.

If we apply the functional equation for Epstein zeta function, i.e., ([2]),

$$|F|^{\frac{1}{2}} \pi^{-s} \Gamma(s) \zeta(F, s) = \pi^{-(\frac{1}{2}n - s)} \Gamma(\tfrac{1}{2}n - s) \zeta(F^{-1}, \tfrac{1}{2}n - s)$$

and the lemma to $\zeta(E, F, s)$, we shall obtain

$$\zeta(E^{-1}, F^{-1}, \tfrac{1}{2}n - s) = |E|^{\frac{1}{2}} |F|^{\frac{1}{2}} \zeta(E, F, s),$$

which proves the theorem.

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For general cases, I think that all the difficulties in solving this problem of the functional equation come from the integral over n-sphere

$$\int_{S^{n-1}} E(\omega)^{s-n/d} d\omega$$

in the form of the generalized gamma function and from the lack of theta-formula for the polynomials of higher degree. For example, we may define the function

$$\theta(\tau, x) = \sum_{\gamma \varepsilon Z^n} \exp(2\pi i \tau F(\gamma + x))$$

for a positive-definite homogeneous polynomial $F(X) \in \mathbb{R}[X]$. But, the information for non-quadratic forms is inadequate. Some work in this direction has been done by Ekkehard Krätzel [3].

References

[1] Chung-ming An, *On a generalization of gamma function and its application to certain Dirichlet series, Dissertation*, (University of Pennsylvania, 1969).

[2] C. L. Siegel, *Lecture on advanced analytic number theory*, (Tata Institute of Fundamental Research, Bombay, India, 1961).

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Department of Mathematics
The Johns Hopkins University
Baltimore, Maryland

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