ODD RANK VECTOR BUNDLES IN ETA-PERIODIC MOTIVIC HOMOTOPY THEORY

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Abstract We observe that, in the eta-periodic motivic stable homotopy category, odd rank vector bundles behave to some extent as if they had a nowhere vanishing section. We discuss some consequences concerning SL^c-orientations of motivic ring spectra and the étale classifying spaces of certain algebraic groups. In particular, we compute the classifying spaces of diagonalisable groups in the eta-periodic motivic stable homotopy category.

Introduction

Around 40 years ago, Arason computed the Witt groups of projective spaces [Ara80]. This computation was later revisited by Gille [Gil01], Walter [Wal03] and Nenashev [Nen09]. It exhibited Witt groups as a somewhat exotic cohomology theory, whose value on projective spaces differs quite drastically from what is obtained in more classical cohomology theories such as Chow groups or K-theory. It is now understood that this behaviour reflects the lack of GL-orientation in Witt theory.

Ananyevskiy observed in [Ana16a] that the key property of Witt groups permitting to perform these computations turns out to be the fact that the Hopf map

$$\eta \colon \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1, \quad (x : y) \mapsto [x : y]$$

induces by pullback an isomorphism of Witt groups $\mathrm{W}(\mathbb{P}^1) \xrightarrow{\sim} \mathrm{W}(\mathbb{A}^2 \setminus \{0\})$. He thus extended in [Ana16a] the above-mentioned computations to arbitrary cohomology theories in which η induces an isomorphism.

Inverting the Hopf map η in the motivic stable homotopy category $\mathrm{SH}(S)$ over a base scheme S yields its η-periodic version $\mathrm{SH}(S)[\eta^{-1}]$, a category which has been studied in details by Bachmann–Hopkins [BH20]. In this paper, we lift Ananyevskiy's computations of the cohomology of projective spaces to the η-periodic stable homotopy category: We obtain for instance that a projective bundle of even relative dimension becomes an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$.

Another familiar feature of Witt groups is that twisting these groups by squares of line bundles has no effect, which may be viewed as a manifestation of the SL^c-orientability of Witt groups (see below). We show that that property of Witt groups in fact follows from their η-periodicity alone (see (4.3.2) for a more general statement).

Proposition. *Let $V \to X$ be a vector bundle and $L \to X$ a line bundle. Then we have an isomorphism of Thom spaces*

$$\mathrm{Th}_X(V) \simeq \mathrm{Th}_X(V \otimes L^{\otimes 2}) \in \mathrm{SH}(S)[\eta^{-1}].$$

Panin and Walter introduced [PW18, §3] the notion of SL^c-orientability for algebraic cohomology theories, which consists of the data of Thom classes for vector bundles equipped with a square root of their determinant and proved that Hermitian K-theory is SL^c-oriented. Ananyevskiy later showed [Ana20, Theorem 1.2] that a cohomology theory is SL^c-oriented as soon as it is a Zariski sheaf in bidegree $(0,0)$ and pointed out [Ana20, Theorem 1.1] the close relations between SL^c-orientations and SL-orientations (the latter consisting in the data of Thom classes for vector bundles with trivialised determinant). We show in this paper that the two notions actually coincide in the η-periodic context.

Theorem. *Every SL-orientation of an η-periodic motivic commutative ring spectrum is induced by a unique SL^c-orientation.*

These results are obtained as consequences of the following observation.

Proposition. *Let E be a vector bundle of odd rank over a smooth S-scheme X and E° the complement of the zero-section in E.*

  (i) *The projection $E^\circ \to X$ admits a section in $\mathrm{SH}(S)[\eta^{-1}]$.*
  (ii) *The diagram $E^\circ \times_X E^\circ \rightrightarrows E^\circ \to X$ becomes a split coequaliser diagram in $\mathrm{SH}(S)[\eta^{-1}]$.*

The first assertion may be viewed as a splitting principle, while the second permits performing a form of descent. To some extent, this proposition allows us to assume that odd rank vector bundles admit a nowhere-vanishing section (once η is inverted); in particular that line bundles are trivial.

Finally, we provide applications to the computation in $\mathrm{SH}(S)[\eta^{-1}]$ of the étale classifying spaces of certain algebraic groups.

Theorem. *For $r \in \mathbb{N} \setminus \{0\}$, there exist natural maps*

$$S \to \mathrm{B}\mathbb{G}_m \quad ; \quad S \to \mathrm{B}\mu_{2r+1} \quad ; \quad \mathbb{G}_m \to \mathrm{B}\mu_{2r}$$

which become isomorphisms in $\mathrm{SH}(S)[\eta^{-1}]$.

From this theorem, we deduce a computation in $\mathrm{SH}(S)[\eta^{-1}]$ of the classifying space of an arbitrary diagonalisable group. We obtain that all invariants of torsors under a diagonalisable group with values in an η-periodic cohomology theory arise from a single invariant of μ_2-torsors. In the appendix, we present an explicit construction of that invariant, exploiting the identification of the group μ_2 with the orthogonal group O_1.

Next, we obtain 'relative' computations in $\mathrm{SH}(S)[\eta^{-1}]$ of certain étale classifying spaces in terms of others.

Theorem. *For $n \in \mathbb{N} \setminus \{0\}$ and $r \in \mathbb{N}$, the natural morphisms*

$$\mathrm{BSL}_n \to \mathrm{BSL}_n^c \quad ; \quad \mathrm{BGL}_{2r} \to \mathrm{BGL}_{2r+1} \quad ; \quad \mathrm{BSL}_{2r+1} \to \mathrm{BGL}_{2r+1}$$

become isomorphisms in $\mathrm{SH}(S)[\eta^{-1}]$.

The first result can be viewed as a companion of the theorem on orientations stated above and cements the idea that the groups SL_n^c and SL_n are the same in the eyes of η-periodic stable homotopy theory. The second (resp. third) result expresses the fact that odd-dimensional vector bundles behave as if they had a nowhere-vanishing section (resp. trivial determinant) from the point of view of η-periodic stable homotopy theory.

The morphism $\mathrm{BSL}_{2r} \to \mathrm{BGL}_{2r}$ is not an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$, but we show that it admits a section, expressing the fact that every invariant (with values in an η-periodic cohomology theory) of even-dimensional vector bundles is determined by its value on those bundles having trivial determinant.

Finally, let us mention that the results of this paper serve as a starting point for the paper [Hau22] on Pontryagin classes.

1. Notation and basic facts

1.1.

Throughout the paper, we work over a noetherian base scheme S of finite dimension. The category of smooth separated S-schemes of finite type will be denoted by Sm_S. All schemes will be implicitly assumed to belong to Sm_S, and the notation $\mathbb{A}^n, \mathbb{P}^n, \mathbb{G}_m$ will refer to the corresponding S-schemes. We will denote by 1 the trivial line bundle over a given scheme in Sm_S.

1.2.

We will use the $\mathbb{A}^1$-homotopy theory introduced by Morel–Voevodsky [MV99]. We will denote by $\mathrm{Spc}(S)$ the category of motivic spaces (i.e., simplicial presheaves on Sm_S), by $\mathrm{Spc}_\bullet(S)$ its pointed version and by $\mathrm{Spt}(S)$ the category of T-spectra, where $T = \mathbb{A}^1/\mathbb{G}_m$. We endow these with the motivic equivalences, resp. stable motivic equivalences, and denote by $\mathrm{H}(S), \mathrm{H}_\bullet(S), \mathrm{SH}(S)$ the respective homotopy categories. We refer to, for example, [PPR09, Appendix A] for more details.

We have an infinite suspension functor $\Sigma^\infty \colon \mathrm{Spc}_\bullet(S) \to \mathrm{Spt}(S)$. Composing with the functor $\mathrm{Spc}(S) \to \mathrm{Spc}_\bullet(S)$ adding an external base point, we obtain a functor $\Sigma^\infty_+ \colon \mathrm{Spc}(S) \to \mathrm{Spt}(S)$.

The spheres are denoted as usual by $S^{p,q} \in \mathrm{Spc}_\bullet(S)$ for $p, q \in \mathbb{N}$ with $p \geq q$ (where $T \simeq S^{2,1}$). The motivic sphere spectrum $\Sigma_+^\infty S$ will be denoted by $\mathbf{1}_S \in \mathrm{Spt}(S)$. When A is a motivic spectrum, we denote its (p,q)-th suspension by $\Sigma^{p,q} A = S^{p,q} \wedge A$. This yields functors $\Sigma^{p,q} \colon \mathrm{SH}(S) \to \mathrm{SH}(S)$ for $p, q \in \mathbb{Z}$.

1.3.

When $E \to X$ is a vector bundle with $X \in \mathrm{Sm}_S$, we denote by $E^\circ = E \setminus X$ the complement of the zero section. The Thom space of E is the pointed motivic space $\mathrm{Th}_X(E) = E/E^\circ$. We will write $\mathrm{Th}_X(E) \in \mathrm{Spt}(S)$ instead of $\Sigma^\infty \mathrm{Th}_X(E)$, in order to lighten the notation. When $g \colon Y \to X$ is a morphism in Sm_S, we will usually write $\mathrm{Th}_Y(E)$ instead of $\mathrm{Th}_Y(g^*E)$. Since $E \to X$ is a weak equivalence, we have a cofiber sequence in $\mathrm{Spc}_\bullet(S)$, where $p \colon E^\circ \to X$ is the projection,

$$(E^\circ)_+ \xrightarrow{p_+} X_+ \to \mathrm{Th}_X(E). \tag{1.3.a}$$

If $F \to S$ is a vector bundle and $f \colon X \to S$ the structural morphism, we have by [MV99, Proposition 3.2.17 (1)] a natural identification in $\mathrm{Spc}_\bullet(S)$

$$\mathrm{Th}_X(E \oplus f^*F) = \mathrm{Th}_X(E) \wedge \mathrm{Th}_S(F). \tag{1.3.b}$$

When $V \to S$ is a vector bundle, we denote by $\Sigma^V \colon \mathrm{SH}(S) \to \mathrm{SH}(S)$ the derived functor induced by $A \mapsto A \wedge \mathrm{Th}_S(V)$. It is an equivalence of categories, with inverse denoted by Σ^{-V}.

1.4.

Let $i \colon Y \to X$ be a closed immersion in Sm_S, with normal bundle $N \to Y$, and open complement $u \colon U \to X$. The purity equivalence $X/U \simeq \mathrm{Th}_Y(N)$ (see, e.g., [MV99, Theorem 3.2.23]) yields a cofiber sequence in $\mathrm{Spc}_\bullet(S)$

$$U_+ \xrightarrow{u_+} X_+ \to \mathrm{Th}_Y(N). \tag{1.4.a}$$

More generally, if $V \to X$ is a vector bundle, we have a cofiber sequence in $\mathrm{Spc}_\bullet(S)$

$$\mathrm{Th}_U(V) \to \mathrm{Th}_X(V) \to \mathrm{Th}_Y(N \oplus i^*V).$$

This may be deduced from equation (1.4.a) by first reducing to the case $X = S$ using the functor $f_\sharp$ of (1.9) below and then applying the functor $- \wedge \mathrm{Th}_X(V)$ (both of which preserve homotopy colimits), in view of equation (1.3.b).

1.5.

Let $\varphi \colon E \xrightarrow{\sim} F$ be an isomorphism of vector bundles over $X \in \mathrm{Sm}_S$. Then φ induces a weak equivalence in $\mathrm{Spc}_\bullet(S)$ (and $\mathrm{Spt}(S)$)

$$\mathrm{Th}(\varphi) \colon \mathrm{Th}_X(E) \to \mathrm{Th}_X(F).$$

If $\psi \colon F \xrightarrow{\sim} G$ is an isomorphism of vector bundles over X, we have

$$\mathrm{Th}(\psi \circ \varphi) = \mathrm{Th}(\psi) \circ \mathrm{Th}(\varphi). \tag{1.5.a}$$

If $X = S$ and $f \in \mathrm{End}_{\mathrm{SH}(S)}(\mathbf{1}_S)$, then we have in $\mathrm{SH}(S)$

$$(\Sigma^F f) \circ \mathrm{Th}(\varphi) = \mathrm{Th}(\varphi) \circ (\Sigma^E f) \colon \mathrm{Th}_S(E) \to \mathrm{Th}_S(F). \tag{1.5.b}$$

(This follows from the fact that, as morphisms $\mathbf{1}_S \wedge \mathrm{Th}_S(E) \to \mathbf{1}_S \wedge \mathrm{Th}_S(F)$

$$(f \wedge \mathrm{id}_{\mathrm{Th}_S(F)}) \circ (\mathrm{id}_{\mathbf{1}_S} \wedge \mathrm{Th}(\varphi)) = f \wedge \mathrm{Th}(\varphi) = (\mathrm{id}_{\mathbf{1}_S} \wedge \mathrm{Th}(\varphi)) \circ (f \wedge \mathrm{id}_{\mathrm{Th}_S(E)}).)$$

1.6.

(See [Mor04, Lemma 6.3.4].) Let $X \in \mathrm{Sm}_S$ and $u \in H^0(X, \mathbb{G}_m)$. Consider the automorphism $u\,\mathrm{id}_1 \colon 1 \to 1$ of the trivial line bundle over X, and set

$$\langle u \rangle = \Sigma^{-2,-1} \mathrm{Th}(u\,\mathrm{id}_1) \in \mathrm{Aut}_{\mathrm{SH}(S)}(\Sigma_+^\infty X).$$

It follows from equation (1.5.a) that

$$\langle uv \rangle = \langle u \rangle \circ \langle v \rangle \quad \text{for } u, v \in H^0(X, \mathbb{G}_m). \tag{1.6.a}$$

When $X = S$ and $A \in \mathrm{SH}(S)$, we will denote again by $\langle u \rangle \in \mathrm{Aut}_{\mathrm{SH}(S)}(A)$ the morphism

$$A = \mathbf{1}_S \wedge A \xrightarrow{\langle u \rangle \wedge \mathrm{id}_A} \mathbf{1}_S \wedge A = A.$$

If $f \colon A \to B$ is a morphism in $\mathrm{SH}(S)$, then

$$f \circ \langle u \rangle = \langle u \rangle \circ f. \tag{1.6.b}$$

1.7.

We denote by $\eta \colon \mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^1$ in $\mathrm{Spc}_\bullet(S)$ the map $(x, y) \mapsto [x : y]$, where $\mathbb{A}^2 \setminus \{0\}$ is pointed by $(1, 1)$ and $\mathbb{P}^1$ by $[1 : 1]$.

1.8.

We will consider the categories $\mathrm{Spc}_\bullet(S)[\eta^{-1}]$ and $\mathrm{Spt}(S)[\eta^{-1}]$ obtained by monoidally inverting the map η of equation (1.7), which can be constructed as left Bousfield localisations, as discussed in [Bac18, §6]. Their respective homotopy categories will be denoted by $H_\bullet(S)[\eta^{-1}]$ and $\mathrm{SH}(S)[\eta^{-1}]$, and we will usually omit the mention of the localisation functors.

A spectrum $A \in \mathrm{Spt}(S)$ is called *η-periodic* if the map

$$A \wedge \Sigma^\infty(\mathbb{A}^2 \setminus \{0\}) \xrightarrow{\mathrm{id} \wedge \Sigma^\infty \eta} A \wedge \Sigma^\infty \mathbb{P}^1 \tag{1.8.a}$$

is an isomorphism in $\mathrm{SH}(S)$. The full subcategory of such objects in $\mathrm{Spt}(S)$ can be identified $\mathrm{Spt}(S)[\eta^{-1}]$.

1.9.

Let $X \in \mathrm{Sm}_S$ with structural morphism $f \colon X \to S$. Then there are Quillen adjunctions

$$f_\sharp \colon \mathrm{Spc}_\bullet(X) \leftrightarrows \mathrm{Spc}_\bullet(S) \colon f^* \quad ; \quad f_\sharp \colon \mathrm{Spt}(X) \leftrightarrows \mathrm{Spt}(S) \colon f^*.$$

The functor f^* is induced by base-change, while $f_\sharp$ arises from viewing a smooth X-scheme as a smooth S-scheme by composing with f. These induce Quillen adjunctions

$$f_\sharp \colon \mathrm{Spc}_\bullet(S)[\eta^{-1}] \leftrightarrows \mathrm{Spc}_\bullet(S)[\eta^{-1}] \colon f^* \quad ; \quad f_\sharp \colon \mathrm{Spt}(X)[\eta^{-1}] \leftrightarrows \mathrm{Spt}(S)[\eta^{-1}] \colon f^*.$$

We will also use the notation $f^*, f_\sharp$ for the derived functors on the respective homotopy categories.

1.10.

(See, e.g., [DHI04].) Let $V \to X$ be a vector bundle with $X \in \mathrm{Sm}_S$ and U_α an open covering of X. Then the map

$$\mathrm{hocolim}\left(\cdots \underset{\alpha,\beta}{\overset{\rightarrow}{\underset{\rightarrow}{\rightrightarrows}}} \coprod_{\alpha,\beta} \mathrm{Th}_{U_\alpha \cap U_\beta}(V|_{U_\alpha \cap U_\beta}) \rightrightarrows \coprod_\alpha \mathrm{Th}_{U_\alpha}(V|_{U_\alpha}) \right) \to \mathrm{Th}_X(V)$$

is a weak equivalence.

2. Splitting $\mathbb{G}_m$-torsors

2.1. Local splitting

In this section, we consider the schemes $\mathbb{A}^1, \mathbb{G}_m, \mathbb{P}^1, \mathbb{A}^2 \setminus \{0\}$ as pointed motivic spaces, respectively via $1, 1, [1:1], (1,1)$. We recall that $T = \mathbb{A}^1/\mathbb{G}_m$. We have a chain of weak equivalences

$$T = \mathbb{A}^1/\mathbb{G}_m \xrightarrow{\sim} \mathbb{P}^1/\mathbb{A}^1 \xleftarrow{\sim} \mathbb{P}^1, \tag{2.1.0.a}$$

where the first arrow is induced by the immersion $\mathbb{A}^1 \to \mathbb{P}^1, x \mapsto [x:1]$, and the quotient $\mathbb{P}^1/\mathbb{A}^1$ is taken with respect to the immersion $\mathbb{A}^1 \to \mathbb{P}^1, y \mapsto [1:y]$.

We first recall a well-known fact (see, e.g., [Ana20, Lemma 6.2] for a stable version):

2.1.1 Lemma. *Let $u \in H^0(S, \mathbb{G}_m)$. Then the morphism $\mathrm{Th}(u^2 \, \mathrm{id}_1) \colon T \to T$ (see (1.5)) coincides with the identity in $\mathrm{H}_\bullet(S)$.*

Proof. The endomorphism $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$ given by $[x:y] \mapsto [u^2 x:y] = [ux:u^{-1}y]$ is induced by the matrix

$$A = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}.$$

Since

$$A = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u-1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

is a product of transvections, the endomorphism φ induces the identity endomorphism of $\mathbb{P}^1_+$ in $\mathrm{H}_\bullet(S)$ (see, e.g., [Ana16a, Lemma 1]). The map φ stabilises the copies of $\mathbb{A}^1$ given by $x \mapsto [x:1]$ and $y \mapsto [1:y]$ and restricts to $u^2 \, \mathrm{id}_1$ on the former. Thus, the statement follows from the isomorphism (2.1.0.a). $\qquad\square$

2.1.2. Excision yields isomorphisms in $\mathrm{H}_\bullet(S)$

$$(\mathbb{A}^2 \setminus \{0\})/(\mathbb{G}_m \times \mathbb{A}^1) \xleftarrow{\sim} (\mathbb{A}^1 \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{\sim} (\mathbb{A}^1/\mathbb{G}_m) \wedge (\mathbb{G}_m)_+ = T \wedge (\mathbb{G}_m)_+.$$
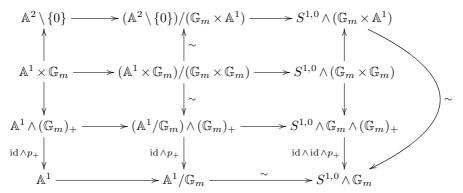
Composing with the quotient $\mathbb{A}^2 \setminus \{0\} \to (\mathbb{A}^2 \setminus \{0\})/(\mathbb{A}^1 \times \mathbb{G}_m)$, this yields a map

$$\mathbb{A}^2 \setminus \{0\} \to T \wedge (\mathbb{G}_m)_+. \tag{2.1.5.a}$$

Lemma 2.1.3. *The projection $p \colon \mathbb{G}_m \to S$ induces a cofiber sequence in $\mathrm{Spc}_\bullet(S)$*

$$\mathbb{A}^2 \setminus \{0\} \xrightarrow{(2.1.2.a)} T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{id} \wedge p_+} T.$$

Proof. This follows from the consideration of the following commutative diagram in $\mathrm{Spc}_\bullet(S)$, whose rows are cofiber sequences

and where the curved arrow is the weak equivalence induced by the projection $\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{G}_m$. $\qquad\square$

The next lemma is reminiscent of [Ana16b, Theorem 3.8]:

Lemma 2.1.4. *The morphism η of (1.7) factors in $\mathrm{H}_\bullet(S)$ as*

$$\mathbb{A}^2 \setminus \{0\} \xrightarrow{(2.1.2.a)} T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{Th}(t^{-1}\mathrm{id}_1)} T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{id} \wedge p_+} T \xrightarrow{(2.1.0.a)} \mathbb{P}^1,$$

where $t \in H^0(\mathbb{G}_m, \mathbb{G}_m)$ is the tautological section and $p \colon \mathbb{G}_m \to S$ the projection (and $T \wedge (\mathbb{G}_m)_+$ is identified with $\mathrm{Th}_{\mathbb{G}_m}(1)$).

Proof. Consider the commutative diagram in Sm_S

$$\begin{array}{ccc} \mathbb{A}^1 \times \mathbb{G}_m & \longrightarrow & \mathbb{A}^2 \setminus \{0\} \\ {\scriptstyle \mu} \downarrow & & \downarrow {\scriptstyle \eta} \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1, \end{array}$$

where the upper horizontal arrow is the natural open immersion, the lower horizontal arrow is given by $x \mapsto [x : 1]$ and μ is given by $(x,y) \mapsto xy^{-1}$. Excision yields the isomorphisms in the commutative diagram in $\mathrm{H}_\bullet(S)$

$$(\mathbb{A}^1 \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{\;\sim\;} (\mathbb{A}^2 \setminus \{0\})/(\mathbb{G}_m \times \mathbb{A}^1) \longleftarrow \mathbb{A}^2 \setminus \{0\}$$

$$\tilde{\mu} \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \eta$$

$$\mathbb{A}^1/\mathbb{G}_m \xrightarrow{\qquad\sim\qquad} \mathbb{P}^1/\mathbb{A}^1 \xleftarrow{\quad\sim\quad} \mathbb{P}^1,$$

where $\tilde{\mu}$ is induced by μ, and the lower horizontal arrows are the morphisms of equation (2.1.0.a). To conclude, observe that the morphism $\tilde{\mu}$ factors as the upper horizontal composite in the following commutative diagram in $H_\bullet(S)$:

$$(\mathbb{A}^1 \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{(x,y) \mapsto (xy^{-1}, y)} (\mathbb{A}^1 \times \mathbb{G}_m)/(\mathbb{G}_m \times \mathbb{G}_m) \xrightarrow{(x,y) \mapsto x} \mathbb{A}^1/\mathbb{G}_m$$

$$\downarrow \sim \qquad\qquad\qquad \downarrow \sim \qquad\qquad\qquad \| =$$

$$T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{Th}(t^{-1}\mathrm{id}_1)} T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{id} \wedge p_+} T.$$

$\square$

Proposition 2.1.5. *Let* $p \colon \mathbb{G}_m \to S$ *be the projection, and* $T = \mathbb{A}^1/\mathbb{G}_m = \mathrm{Th}_S(1)$. *Consider the composite (see (1.5))*

$$\pi \colon T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{Th}(t\,\mathrm{id}_1)} T \wedge (\mathbb{G}_m)_+ \xrightarrow{\mathrm{id} \wedge p_+} T.$$

Then the following square is homotopy co-Cartesian in $\mathrm{Spc}_\bullet(S)[\eta^{-1}]$

$$
\begin{array}{ccc}
T \wedge (\mathbb{G}_m)_+ & \xrightarrow{\;\pi\;} & T \\
{\scriptstyle \mathrm{id} \wedge p_+} \downarrow & & \downarrow \\
T & \longrightarrow & *.
\end{array}
$$

Proof. From (2.1.3), we deduce a commutative diagram

$$
\begin{array}{ccccc}
\mathbb{A}^2 \setminus \{0\} & \xrightarrow{(2.1.2.a)} & T \wedge (\mathbb{G}_m)_+ & \xrightarrow{\;\pi\;} & T \\
\downarrow & & {\scriptstyle \mathrm{id} \wedge p_+} \downarrow & & \downarrow \\
* & \longrightarrow & T & \longrightarrow & *,
\end{array}
\qquad (2.1.5.a)
$$

where the left inner square is homotopy co-Cartesian. Applying (2.1.1) over the base $\mathbb{G}_m$ and using the functor $p_\sharp$ of (1.9), we have in $H_\bullet(S)$

$$\mathrm{Th}(t\,\mathrm{id}_1) = \mathrm{Th}(t^{-1}\mathrm{id}_1) \colon T \wedge (\mathbb{G}_m)_+ \to T \wedge (\mathbb{G}_m)_+.$$

It thus follows from (2.1.4) that the upper composite in the diagram (2.1.5.a) is an isomorphism in $H_\bullet(S)[\eta^{-1}]$, hence the exterior square in equation (2.1.5.a) is homotopy co-Cartesian. We conclude that the right inner square is homotopy co-Cartesian (by [Hir03, Proposition 13.3.15, Remark 7.1.10]). $\square$

2.2. Global splitting

Definition 2.2.1. Let $L \to S$ be a line bundle. Denote by L° the complement of the zero-section in L, and by $p\colon L^\circ \to S$ the projection. Then the graph $L^\circ \to L^\circ \times_S L$ of the open immersion $L^\circ \to L$ may be viewed as a nowhere-vanishing section of the line bundle p^*L over L°, which induces the *tautological trivialisation* $\tau\colon 1 \xrightarrow{\sim} p^*L$. We define a morphism in $H_\bullet(S)$ (recall that $T = \mathrm{Th}_S(1)$, and see (1.5))

$$\pi_L\colon T \wedge (L^\circ)_+ = \mathrm{Th}_{L^\circ}(1) \xrightarrow{\mathrm{Th}(\tau)} \mathrm{Th}_{L^\circ}(p^*L) \xrightarrow{p} \mathrm{Th}_S(L).$$

Example 2.2.2. Assume that $L = 1$. Then the tautological trivialisation $\tau\colon 1 \to 1$ of the trivial line bundle over $L^\circ = \mathbb{G}_m$ is given by multiplication by the tautological section $t \in H^0(\mathbb{G}_m, \mathbb{G}_m)$, hence π_1 coincides with morphism π of (2.1.5).

2.2.3. Let $L \to S$ be a line bundle. If $f\colon R \to S$ is a scheme morphism, then the functor $f^*\colon \mathrm{Spc}_\bullet(S) \to \mathrm{Spc}_\bullet(R)$ maps π_L to π_{f^*L}.

2.2.4. If $\varphi\colon L \xrightarrow{\sim} M$ is an isomorphism of line bundles over S, then the following diagram commutes in $H_\bullet(S)$

$$
\begin{array}{ccc}
T \wedge (L^\circ)_+ & \xrightarrow{\ \pi_L\ } & \mathrm{Th}_S(L) \\
{\scriptstyle \mathrm{id} \wedge (\varphi^\circ)_+}\downarrow & & \downarrow{\scriptstyle \mathrm{Th}(\varphi)} \\
T \wedge (M^\circ)_+ & \xrightarrow{\ \pi_M\ } & \mathrm{Th}_S(M).
\end{array}
$$

Proposition 2.2.5. *Let $L \to S$ be a line bundle. Then the following square is homotopy co-Cartesian in $\mathrm{Spc}_\bullet(S)[\eta^{-1}]$*

$$
\begin{array}{ccc}
T \wedge (L^\circ)_+ & \xrightarrow{\ \pi_L\ } & \mathrm{Th}_S(L) \\
{\scriptstyle \mathrm{id} \wedge p_+}\downarrow & & \downarrow \\
T & \longrightarrow & *.
\end{array}
$$

Proof. Let F be the homotopy colimit of the diagram $\mathrm{Th}_S(L) \xleftarrow{\pi_L} T \wedge (L^\circ)_+ \xrightarrow{\mathrm{id} \wedge p_+} T$. By (1.10) and (1.9) (and in view of (2.2.3)), the fact that $F \simeq *$ may be verified Zariski locally on S. We may thus assume that L is trivial. By (2.2.4), we may further assume that $L = 1$ so that $L^\circ = \mathbb{G}_m$. Then, in view of (2.2.2) the result follows from (2.1.5). $\quad\square$

The next statement was initially inspired by [Lev19, Proof of Theorem 4.1].

Corollary 2.2.6. *Let $L \to S$ be a line bundle. Then in the notation of (2.2.1), we have an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$*

$$(\Sigma^\infty_+ p, \Sigma^{-2,-1}\Sigma^\infty \pi_L)\colon \Sigma^\infty_+ L^\circ \xrightarrow{\sim} \mathbf{1}_S \oplus \Sigma^{-2,-1}\mathrm{Th}_S(L).$$

Proof. The square induced in $\mathrm{Spt}(S)[\eta^{-1}]$ by the square of (2.2.5) is homotopy co-Cartesian, hence also homotopy Cartesian (see, e.g., [Hov99, Remark 7.1.12]). This yields an isomorphism

$$(\Sigma^\infty(\mathrm{id}\wedge p_+), \Sigma^\infty\pi_L)\colon \Sigma^\infty(T\wedge(L^\circ)_+) \xrightarrow{\sim} \Sigma^\infty T \oplus \mathrm{Th}_S(L),$$

from which the result follows by applying the functor $\Sigma^{-2,-1}$. □

Corollary 2.2.7. *Let $L \to S$ be a line bundle and $V \to S$ a vector bundle.*

(i) *The natural map $\mathrm{Th}_{L^\circ}(V) \to \mathrm{Th}_S(V)$ extends to a natural isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$*

$$\mathrm{Th}_{L^\circ}(V) \simeq \mathrm{Th}_S(V) \oplus \Sigma^{-2,-1}\mathrm{Th}_S(V\oplus L).$$

(ii) *Denote by $p\colon L^\circ \to S$ and $p_1, p_2\colon L^\circ \times_S L^\circ \to L^\circ$ the projections. Then*

$$\mathrm{Th}_{L^\circ\times_S L^\circ}(V) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Th}_{L^\circ}(V) \xrightarrow{p} \mathrm{Th}_S(V)$$

is a split coequaliser diagram in $\mathrm{SH}(S)[\eta^{-1}]$.

Proof. Statement (i) follows by applying the autoequivalence $\Sigma^V\colon \mathrm{SH}(S)[\eta^{-1}] \to \mathrm{SH}(S)[\eta^{-1}]$ to the decomposition of (2.2.6), in view of equation (1.3.b).

Certainly in the diagram of equation (ii), we have $p\circ p_1 = p\circ p_2$. The isomorphism (i) yields a section $s\colon \mathrm{Th}_S(V) \to \mathrm{Th}_{L^\circ}(V)$ of p in $\mathrm{SH}(S)[\eta^{-1}]$. Then, in $\mathrm{SH}(S)[\eta^{-1}]$, the composite

$$t\colon \mathrm{Th}_{L^\circ}(V) = (\Sigma_+^\infty L^\circ)\wedge\mathrm{Th}_S(V) \xrightarrow{\mathrm{id}\wedge s} (\Sigma_+^\infty L^\circ)\wedge\mathrm{Th}_{L^\circ}(V) = \mathrm{Th}_{L^\circ\times_S L^\circ}(V)$$

is a section of

$$p_1\colon \mathrm{Th}_{L^\circ\times_S L^\circ}(V) = (\Sigma_+^\infty L^\circ)\wedge\mathrm{Th}_{L^\circ}(V) \xrightarrow{\mathrm{id}\wedge p} (\Sigma_+^\infty L^\circ)\wedge\mathrm{Th}_S(V) = \mathrm{Th}_{L^\circ}(V).$$

On the other hand, in the commutative diagram in $\mathrm{SH}(S)[\eta^{-1}]$,

$$
\begin{array}{ccccccc}
\mathrm{Th}_{L^\circ}(V) & =\!=\!= & (\Sigma_+^\infty L^\circ)\wedge\mathrm{Th}_S(V) & \xrightarrow{\mathrm{id}\wedge s} & (\Sigma_+^\infty L^\circ)\wedge\mathrm{Th}_{L^\circ}(V) & =\!=\!= & \mathrm{Th}_{L^\circ\times_S L^\circ}(V) \\
{\scriptstyle p}\downarrow & & {\scriptstyle p\wedge\mathrm{id}}\downarrow & & {\scriptstyle p\wedge\mathrm{id}}\downarrow & & \downarrow{\scriptstyle p_2} \\
\mathrm{Th}_S(V) & =\!=\!= & \mathbf{1}_S\wedge\mathrm{Th}_S(V) & \xrightarrow{\mathrm{id}\wedge s} & \mathbf{1}_S\wedge\mathrm{Th}_{L^\circ}(V) & =\!=\!= & \mathrm{Th}_{L^\circ}(V)
\end{array}
$$

the upper composite is t, while the lower one is s. Therefore, $p_2\circ t = s\circ p$ as endomorphisms of $\mathrm{Th}_{L^\circ}(V)$ in $\mathrm{SH}(S)[\eta^{-1}]$, proving equation (ii). □

Corollary 2.2.8. *Let $L \to S$ be a line bundle, and denote by $p\colon L^\circ \to S$ the projection. Then the functor $p^*\colon \mathrm{SH}(S)[\eta^{-1}] \to \mathrm{SH}(L^\circ)[\eta^{-1}]$ is faithful and conservative.*

Proof. By the smooth projection formula and (2.2.6), the composite $p_\sharp\circ p^*\colon \mathrm{SH}(S)[\eta^{-1}] \to \mathrm{SH}(S)[\eta^{-1}]$ decomposes as

$$p_\sharp\circ p^* = \mathrm{id}\wedge(\Sigma_+^\infty L^\circ) = \mathrm{id}\wedge(\mathbf{1}_S \oplus \Sigma^{-2,-1}\mathrm{Th}_S(L)) = \mathrm{id}\oplus(\Sigma^{-2,-1}\circ\Sigma^L),$$

which is faithful, hence so is p^*. The above formula also shows that $p_\sharp \circ p^*$ reflects zero-objects, hence so does p^*. Since p^* is triangulated, it is conservative. $\qquad\square$

Remark 2.2.9. The results of this section on line bundles will be generalised to odd rank vector bundles in §4.2.

3. Applications to twisted cohomology

3.1. Cohomology theories represented by ring spectra

3.1.1. Let $A \in \mathrm{Spt}(S)$ be a motivic spectrum. For a pointed motivic space $\mathcal{X}$, we write

$$A^{p,q}(\mathcal{X}) = \mathrm{Hom}_{\mathrm{SH}(S)}(\Sigma^\infty \mathcal{X}, \Sigma^{p,q} A),$$

and $A^{*,*}(\mathcal{X}) = \bigoplus_{p,q \in \mathbb{Z}} A^{p,q}(\mathcal{X})$. When X is a smooth S-scheme, we will write $A^{*,*}(X)$ instead of $A^{*,*}(X_+)$. If $E \to X$ is a vector bundle of constant rank r, we write

$$A^{p,q}(X; E) = A^{p+2r,q+r}(\mathrm{Th}_X(E))$$

and extend this notation to arbitrary vector bundles in an obvious way. A morphism $f \colon \mathcal{Y} \to \mathcal{X}$ of pointed motivic spaces (resp. of smooth S-schemes) induces a pullback $f^* \colon A^{*,*}(\mathcal{X}) \to A^{*,*}(\mathcal{Y})$.

3.1.2. A commutative ring spectrum will mean a commutative monoid in $(\mathrm{SH}(S), \wedge, \mathbf{1}_S)$. When $A \in \mathrm{SH}(S)$ is a commutative ring spectrum and $X \in \mathrm{Sm}_S$, then $A^{*,*}(X)$ is naturally a ring, and $A^{*,*}(X; E)$ an $A^{*,*}(X)$-module. When $u \in H^0(X, \mathbb{G}_m)$, we will write $\langle u \rangle \in A^{0,0}(X)$ instead of $\langle u \rangle^*(1)$ (see (1.6)).

3.1.3. If A is an η-periodic motivic spectrum, for any pointed motivic space $\mathcal{X}$, we have natural isomorphisms for $p, q \in \mathbb{Z}$

$$A^{p,q}(\mathcal{X}) = \mathrm{Hom}_{\mathrm{SH}(S)[\eta^{-1}]}(\Sigma^\infty \mathcal{X}, \Sigma^{p,q} A).$$

Proposition 3.1.4. *Let A be an η-periodic motivic spectrum. Let $X \in \mathrm{Sm}_S$. Let $L \to X$ be a line bundle and $V \to X$ a vector bundle.*

(i) *Denoting by $p \colon L^\circ \to X$ the projection, we have a split short exact sequence*

$$0 \to A^{*,*}(X; V) \xrightarrow{p^*} A^{*,*}(L^\circ; V) \to A^{*,*}(X; V \oplus L) \to 0.$$

(ii) *Denoting by $p_1, p_2 \colon L^\circ \times_X L^\circ \to L^\circ$ the projections, we have an exact sequence*

$$0 \to A^{*,*}(X; V) \xrightarrow{p^*} A^{*,*}(L^\circ; V) \xrightarrow{p_1^* - p_2^*} A^{*,*}(L^\circ \times_X L^\circ; V).$$

Proof. This follows by applying (2.2.7) over the base X to the image of A under the pullback $\mathrm{Spt}(S) \to \mathrm{Spt}(X)$. □

3.2. SL- and SL^c-orientations

Definition 3.2.1. An SL-*oriented vector bundle* over a scheme X is a pair (E,δ), where $E \to X$ is a vector bundle and $\delta \colon 1 \xrightarrow{\sim} \det E$ is an isomorphism of line bundles. We will also say that δ is an SL-*orientation* of the vector bundle $E \to X$. An isomorphism of SL-oriented vector bundles $(E,\delta) \xrightarrow{\sim} (F,\epsilon)$ is an isomorphism of vector bundles $\varphi \colon E \xrightarrow{\sim} F$ such that $(\det \varphi) \circ \delta = \epsilon$.

Definition 3.2.2 See [PW18, §3]. An SL^c-*oriented vector bundle* over a scheme X is a triple (E,L,λ), where $E \to X$ is a vector bundle and $L \to X$ a line bundle, and $\lambda \colon L^{\otimes 2} \xrightarrow{\sim} \det E$ is an isomorphism. We will also say that (L,λ) is an SL^c-*orientation* of the vector bundle $E \to X$. An isomorphism of SL^c-oriented vector bundles $(E,L,\lambda) \xrightarrow{\sim} (F,M,\mu)$ is an isomorphism of vector bundles $\varphi \colon E \xrightarrow{\sim} F$ and an isomorphism of line bundles $\psi \colon L \xrightarrow{\sim} M$ such that $(\det \varphi) \circ \lambda = \mu \circ \psi^{\otimes 2}$.

3.2.3. Observe that each SL-orientation δ of a vector bundle E induces an SL^c-orientation (L,λ) of E, where $L = 1$ and λ is the composite $1^{\otimes 2} \simeq 1 \xrightarrow{\delta} \det E$.

3.2.4. Let (E,L,λ) be an SL^c-oriented vector bundle, and assume that the line bundle L is trivial. Then every trivialisation $\alpha \colon 1 \xrightarrow{\sim} L$ induces an SL-orientation of E given by

$$\delta_\alpha \colon 1 \simeq 1^{\otimes 2} \xrightarrow{\alpha^{\otimes 2}} L^{\otimes 2} \xrightarrow{\lambda} \det E.$$

Observe that the SL^c-oriented vector bundle induced (in the sense of (3.2.3)) by δ_α is isomorphic to (E,L,λ).

3.2.5. Consider a commutative ring spectrum $A \in \mathrm{SH}(S)$. By a SL-, resp. SL^c-, *orientation* of A, we will mean a normalised orientation in the sense of [Ana20, Definition 3.3]. Such data consists in Thom classes $\mathrm{th}_{(E,\delta)} \in A^{*,*}(X;E)$ for each SL-oriented vector bundle (E,δ) over $X \in \mathrm{Sm}_S$, resp. $\mathrm{th}_{(E,L,\lambda)} \in A^{*,*}(X;E)$ for each SL^c-oriented vector bundle (E,L,λ) over $X \in \mathrm{Sm}_S$, satisfying a series of axioms.

3.2.6. Let $A \in \mathrm{SH}(S)$ be a commutative ring spectrum. Then each SL^c-orientation of A induces an SL-orientation of A, by letting the Thom class of an SL-oriented vector bundle be the Thom class of the induced SL^c-oriented vector bundle, in the sense of (3.2.3).

Lemma 3.2.7. *Let $A \in \mathrm{SH}(S)$ be an* SL-*oriented commutative ring spectrum. Let (E,L,λ) be an* SL^c-*oriented vector bundle over $X \in \mathrm{Sm}_S$, and assume that the line bundle L is trivial. Then, in the notation of (3.2.4), the Thom class $\mathrm{th}_{(E,\delta_\alpha)} \in A^{*,*}(X;E)$ does not depend on the choice of the trivialisation α of L.*

Proof. If $\alpha\colon 1 \xrightarrow{\sim} L$ is a trivialisation, then every trivialisation is of the form $u\alpha$ for some $u \in H^0(X,\mathbb{G}_m)$. In the notation of (3.2.4), we then have $\delta_{u\alpha} = u^2\delta_\alpha$. By [Ana20, Lemma 7.3], we have

$$\mathrm{th}_{(E,\delta_{u\alpha})} = \mathrm{th}_{(E,u^2\delta_\alpha)} = \langle u^2\rangle\,\mathrm{th}_{(E,\delta_\alpha)} \in A^{*,*}(X;E).$$

Since $\langle u^2\rangle = 1 \in A^{0,0}(X)$ by (2.1.1) (or [Ana20, Lemma 6.2]), the statement follows. $\square$

Proposition 3.2.8. *Let $A \in \mathrm{SH}(S)$ be an η-periodic commutative ring spectrum. Then every SL-orientation of A is induced (in the sense of (3.2.6)) by a unique SL^c-orientation.*

Proof. We assume given an SL-orientation of A. Let (E,L,λ) be an SL^c-oriented vector bundle. Denoting by $p\colon L^\circ \to X$ the projection, the line bundle p^*L over L° admits a tautological trivialisation $\tau\colon 1 \xrightarrow{\sim} p^*L$. In view of (3.2.4), this yields an SL-orientation δ_τ of p^*E.

First, assume given an SL^c-orientation of A compatible with its SL-orientation. As observed in (3.2.4), the SL^c-oriented vector bundle $(p^*E,p^*L,p^*\lambda)$ is isomorphic to the one induced by the SL-oriented vector bundle (p^*E,δ_τ). Thus, we must have

$$p^*\mathrm{th}_{(E,L,\lambda)} = \mathrm{th}_{(p^*E,p^*L,p^*\lambda)} = \mathrm{th}_{(p^*E,\delta_\tau)} \in A^{*,*}(L^\circ;p^*E).$$

Since $p^*\colon A^{*,*}(X;E) \to A^{*,*}(L^\circ;p^*E)$ is injective by (3.1.4), we obtain the uniqueness part of the statement.

We now construct an SL^c-orientation of A from its SL-orientation. In the situation considered at the beginning of the proof, let $p_1,p_2\colon L^\circ \times_X L^\circ \to L^\circ$ be the projections, and set $q = p\circ p_1 = p\circ p_2$. The tautological trivialisation τ of p^*L yields two trivialisations $p_1^*\tau$ and $p_2^*\tau$ of q^*L, and thus two SL-orientations $\alpha_1 = \delta_{p_1^*\tau}$ and $\alpha_2 = \delta_{p_2^*\tau}$ of E. However, it follows from (3.2.7) that their Thom classes coincide so that (observe that $\alpha_i = p_i^*(\delta_\tau)$ for $i = 1,2$)

$$p_1^*\mathrm{th}_{(p^*E,\delta_\tau)} = \mathrm{th}_{(E,\alpha_1)} = \mathrm{th}_{(E,\alpha_2)} = p_2^*\mathrm{th}_{(p^*E,\delta_\tau)} \in A^{*,*}(L^\circ \times_X L^\circ;q^*E).$$

Therefore, it follows from (3.1.4.ii) that the element $\mathrm{th}_{(p^*E,\delta_\tau)} \in A^{*,*}(L^\circ;E)$ is the image of a unique element $\theta_{(E,L,\lambda)} \in A^{*,*}(X;E)$.

From the fact that $(E,\delta) \mapsto \mathrm{th}_{(E,\delta)}$ defines an SL-orientation of A, we deduce at once that $(E,L,\lambda) \mapsto \theta_{(E,L,\lambda)}$ defines an SL^c-orientation of A: Indeed, each axiom of [Ana20, Definition 3.3] can be verified after pulling back along $p\colon L^\circ \to X$ since $p^*\colon A^{*,*}(X;E) \to A^{*,*}(L^\circ;p^*E)$ is injective by (3.1.4).

To conclude the proof, it remains to show the SL^c-orientation $(E,L,\lambda) \mapsto \theta_{(E,L,\lambda)}$ induces the original SL-orientation of A. So let us assume that the SL^c-oriented vector bundle (E,L,λ) is induced by an SL-oriented vector bundle (E,δ), in the sense of (3.2.3). In particular, $L = 1$. Then the tautological trivialisation $\tau\colon 1 \xrightarrow{\sim} p^*L$ and the trivialisation $1 = p^*1 = p^*L$ yield two SL-orientations of p^*E. Their Thom classes in $A^{*,*}(L^\circ;p^*E)$ coincide by (3.2.7), and they are, respectively, $p^*\theta_{(E,L,\lambda)}$ and $p^*\mathrm{th}_{(E,\delta)}$. Since $p^*\colon A^{*,*}(X;E) \to A^{*,*}(L^\circ;p^*E)$ is injective by (3.1.4), we have $\theta_{(E,L,\lambda)} = \mathrm{th}_{(E,\delta)} \in A^{*,*}(X;E)$, as required. $\square$

Remark 3.2.9. Ananyevskiy constructed 'Thom isomorphisms' associated with SL^c-bundles in [Ana20, §4] when A is an arbitrary SL-oriented theory, but as explained in [Ana20, Remark 4.4] it is not clear whether this yields an SL^c-orientation, the problem being the multiplicativity axiom. When A is η-periodic, our construction leads to the same Thom isomorphisms for SL^c-bundles (in fact, the proof of (3.2.8) shows that there is at most one way to construct such functorial isomorphisms compatibly with the SL-orientation). Thus, the Thom isomorphisms constructed by Ananyevskiy do give rise to an SL^c-orientation when A is η-periodic.

3.3. Twisting by doubles and squares of line bundles

Proposition 3.3.1. *Let L be a line bundle over S.*

(i) *There exist an isomorphism $\Sigma^{4,2}\mathbf{1}_S \simeq \mathrm{Th}_S(L^{\oplus 2})$ in $\mathrm{SH}(S)[\eta^{-1}]$.*

(ii) *If $s_1, s_2 \in \mathbb{Z}$ are of the same parity, there exist an isomorphism*

$$\mathrm{Th}_S(L^{\otimes s_1}) \simeq \mathrm{Th}_S(L^{\otimes s_2}) \ in \ \mathrm{SH}(S)[\eta^{-1}].$$

Proof. Let us first assume that the line bundle $L \to S$ admits a trivialisation $\alpha\colon 1 \xrightarrow{\sim} L$. Then we have an isomorphism in $\mathrm{SH}(S)$ (see (1.5))

$$\mathrm{Th}(\alpha^{\oplus 2})\colon \mathrm{Th}_S(1^{\oplus 2}) \xrightarrow{\sim} \mathrm{Th}_S(L^{\oplus 2}). \qquad (3.3.1.\mathrm{a})$$

Every trivialisation of L is of the form $u\alpha$ with $u \in H^0(S, \mathbb{G}_m)$. As automorphisms of $\mathrm{Th}_S(1^{\oplus 2})$ in $\mathrm{SH}(S)$ we have,

$$\mathrm{Th}((u\,\mathrm{id}_1)^{\oplus 2}) = \mathrm{Th}((u^2\,\mathrm{id}_1) \oplus \mathrm{id}_1) \circ \mathrm{Th}((u^{-1}\,\mathrm{id}_1) \oplus (u\,\mathrm{id}_1)) = \mathrm{Th}((u^2\,\mathrm{id}_1) \oplus \mathrm{id}_1)$$

because $\mathrm{Th}((u^{-1}\,\mathrm{id}_1) \oplus (u\,\mathrm{id}_1))$ is the identity of $\mathrm{Th}_S(1^{\oplus 2})$, being given by a product of transvections (see the proof of (2.1.1)). Now, by (2.1.1), under the identification $\mathrm{Th}_S(1^{\oplus 2}) = \mathrm{Th}_S(1) \wedge \mathrm{Th}_S(1)$ (see equation (1.3.b)), we have

$$\mathrm{Th}((u^2\,\mathrm{id}_1) \oplus \mathrm{id}_1) = \mathrm{Th}(u^2\,\mathrm{id}_1) \wedge \mathrm{id}_{\mathrm{Th}_S(1)} = \mathrm{id}_{\mathrm{Th}_S(1)} \wedge \mathrm{id}_{\mathrm{Th}_S(1)} = \mathrm{id}_{\mathrm{Th}_S(1^{\oplus 2})}.$$

Therefore, $\mathrm{Th}((u\,\mathrm{id}_1)^{\oplus 2})$ is the identity of $\mathrm{Th}_S(1^{\oplus 2})$ in $\mathrm{SH}(S)$, hence

$$\mathrm{Th}((u\alpha)^{\oplus 2}) = \mathrm{Th}(\alpha^{\oplus 2}) \circ \mathrm{Th}((u\,\mathrm{id}_1)^{\oplus 2}) = \mathrm{Th}(\alpha^{\oplus 2})$$

so that the isomorphism $\mathrm{Th}(\alpha^{\oplus 2})$ in $\mathrm{SH}(S)$ considered in equation (3.3.1.a) is independent of the choice of the trivialisation α.

Next, let us consider the case (ii). If $\alpha\colon 1 \xrightarrow{\sim} L$ is a trivialisation, we have an isomorphism in $\mathrm{SH}(S)$

$$\mathrm{Th}(\mathrm{id}_{L^{\otimes s_1}} \otimes \alpha^{\otimes s_2 - s_1})\colon \mathrm{Th}_S(L^{\otimes s_1}) \xrightarrow{\sim} \mathrm{Th}_S(L^{\otimes s_2}). \qquad (3.3.1.\mathrm{b})$$

(Here and below, for $r \in \mathbb{N}$, the notation $\alpha^{\otimes -r}$ refers to the morphism $((\alpha^\vee)^{-1})^{\otimes r}$.) Now for $u \in H^0(S, \mathbb{G}_m)$, the composite in $\mathrm{SH}(S)$

$$\mathrm{Th}_S(1) \xrightarrow{\mathrm{Th}(\alpha^{\otimes s_1})} \mathrm{Th}_S(L^{\otimes s_1}) \xrightarrow{\mathrm{Th}(\mathrm{id}_{L^{\otimes s_1}} \otimes (u\alpha)^{\otimes s_2 - s_1})} \mathrm{Th}_S(L^{\otimes s_2}) \xrightarrow{\mathrm{Th}(\alpha^{\otimes s_2})^{-1}} \mathrm{Th}_S(1)$$

coincides with $\mathrm{Th}(u^{s_2-s_1}\,\mathrm{id}_1)$, which is the identity by (2.1.1) (recall that $s_2 - s_1$ is even), and in particular does not depend on $u \in H^0(S,\mathbb{G}_m)$. Since the left and right arrows in the above composite are isomorphisms, we deduce that the middle arrow does not depend on $u \in H^0(S,\mathbb{G}_m)$, which shows as above that the isomorphism (3.3.1.b) is independent of the choice of the trivialisation α.

Let us come back to the general case, where $L \to S$ is a possibly nontrivial line bundle. Let $p\colon L^\circ \to S$ be the projection, and consider the tautological trivialisation τ of the line bundle p^*L over L°. Let us consider the isomorphism $\varphi\colon \mathrm{Th}_{L^\circ}(B) \xrightarrow{\sim} \mathrm{Th}_{L^\circ}(C)$ in $\mathrm{SH}(S)[\eta^{-1}]$, where

- $B = 1^{\oplus 2}, C = L^{\oplus 2}, \varphi = \mathrm{Th}(\tau^{\oplus 2})$ in case (i).
- $B = L^{\otimes s_1}, C = L^{\otimes s_2}, \varphi = \mathrm{Th}(\mathrm{id}_{L^{\otimes s_1}} \otimes \alpha^{\otimes s_2 - s_1})$ in case (ii).

Let $p_1, p_2\colon L^\circ \times_S L^\circ \to L^\circ$ be the projections, and set $q = p \circ p_1 = p \circ p_2$. For $i \in \{1,2\}$, the isomorphism $p_i^*\varphi\colon \mathrm{Th}_{L^\circ \times_S L^\circ}(B) \to \mathrm{Th}_{L^\circ \times_S L^\circ}(C)$ in $\mathrm{SH}(S)[\eta^{-1}]$ is induced by the trivialisation $p_i^*\tau$ of the line bundle q^*L over $L^\circ \times_S L^\circ$, hence does not depend on i, by the special case considered at the beginning of the proof. Thus, by (2.2.7.ii) there exists a unique morphism f fitting into the commutative diagram in $\mathrm{SH}(S)[\eta^{-1}]$

$$
\begin{array}{ccc}
\mathrm{Th}_{L^\circ \times_S L^\circ}(B) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Th}_{L^\circ}(B) \xrightarrow{p} \mathrm{Th}_S(B) \\
\Big\downarrow{\scriptstyle p_1^*\varphi = p_2^*\varphi} \qquad\quad \Big\downarrow{\scriptstyle \varphi} \qquad\quad \Big\downarrow{\scriptstyle f} \\
\mathrm{Th}_{L^\circ \times_S L^\circ}(C) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Th}_{L^\circ}(C) \xrightarrow{p} \mathrm{Th}_S(C)
\end{array}
$$

as well as a unique morphism g into the commutative diagram in $\mathrm{SH}(S)[\eta^{-1}]$

$$
\begin{array}{ccc}
\mathrm{Th}_{L^\circ \times_S L^\circ}(C) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Th}_{L^\circ}(C) \xrightarrow{p} \mathrm{Th}_S(C) \\
\Big\downarrow{\scriptstyle p_1^*\varphi^{-1} = p_2^*\varphi^{-1}} \qquad\quad \Big\downarrow{\scriptstyle \varphi^{-1}} \qquad\quad \Big\downarrow{\scriptstyle g} \\
\mathrm{Th}_{L^\circ \times_S L^\circ}(B) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Th}_{L^\circ}(B) \xrightarrow{p} \mathrm{Th}_S(B).
\end{array}
$$

As $p\colon \mathrm{Th}_{L^\circ}(B) \to \mathrm{Th}_S(B)$ and $p\colon \mathrm{Th}_{L^\circ}(C) \to \mathrm{Th}_S(C)$ are epimorphisms in $\mathrm{SH}(S)[\eta^{-1}]$ (see (2.2.7)), it follows that f and g are mutually inverse isomorphisms in $\mathrm{SH}(S)[\eta^{-1}]$. $\qquad\square$

Remark 3.3.2. Proposition (3.3.1) will be improved in (4.3.2).

4. Nowhere-vanishing sections of odd rank bundles

4.1. Projective bundles

The results of this section are slight generalisations of those of [Ana16a, §4].

4.1.1. Let us consider the linear embeddings $i_k\colon \mathbb{P}^k \to \mathbb{P}^{k+1}$ given by the vanishing of the last coordinate. Denote by $\iota_k\colon S \to \mathbb{P}^k$ the S-point given by the composite $S = \mathbb{P}^0 \xrightarrow{i_0} \mathbb{P}^1 \xrightarrow{i_1} \cdots \xrightarrow{i_{k-1}} \mathbb{P}^k$.

4.1.2. Assume given a collection $\mathcal{D} = (d_1, \ldots, d_r) \in \mathbb{Z}^r$ for some $r \in \mathbb{N}$. We will denote by $\mathcal{O}(\mathcal{D})$ the vector bundle $\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r)$ over $\mathbb{P}^k$, for each $k \in \mathbb{N}$. When $k = 0$, we have a canonical isomorphism $\mathcal{O}(\mathcal{D}) \simeq 1^{\oplus r}$ over $\mathbb{P}^0 = S$. This yields, for any $k \in \mathbb{N}$, a canonical map in $\mathrm{Spt}(S)$

$$\Sigma^{2r,r} \mathbf{1}_S = \mathrm{Th}_S(1^{\oplus r}) \simeq \mathrm{Th}_{\mathbb{P}^0}(\mathcal{O}(\mathcal{D})) \xrightarrow{\iota_k} \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})). \qquad (4.1.2.\mathrm{a})$$

Proposition 4.1.3. *Let $k, r \in \mathbb{N}$, and $d_1, \ldots, d_r \in \mathbb{Z}$. Set $\mathcal{D} = (d_1, \ldots, d_r)$, and let $d = d_1 + \cdots + d_r$. We use the notation $\mathcal{O}(\mathcal{D})$ described in (4.1.2).*

(i) *If k and d are odd, then $\mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) = 0$ in $\mathrm{SH}(S)[\eta^{-1}]$.*

(ii) *If k and d are even, then equation (4.1.2.a) induces an isomorphism $\Sigma^{2r,r} \mathbf{1}_S \simeq \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D}))$ in $\mathrm{SH}(S)[\eta^{-1}]$.*

(iii) *If k is even and d is odd, then $\mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) \simeq \Sigma^{2(k+r),k+r} \mathbf{1}_S$.*

(iv) *If k is odd and d is even, then $\mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) \simeq \Sigma^{2(k+r),k+r} \mathbf{1}_S \oplus \Sigma^{2r,r} \mathbf{1}_S$.*

Proof. Let us first prove (i). Assume that k and d are odd. Consider a linear embedding $\mathbb{P}^1 \to \mathbb{P}^k$. Its normal bundle is $\mathcal{O}(1)^{\oplus k-1}$, and its open complement is a vector bundle over $\mathbb{P}^{k-2}$. The corresponding zero-section $\mathbb{P}^{k-2} \to \mathbb{P}^k \setminus \mathbb{P}^1$ induces an isomorphism in $\mathrm{SH}(S)$ and is the restriction of a linear embedding $\mathbb{P}^{k-2} \to \mathbb{P}^k$. Thus, (1.4) yields a distinguished triangle in $\mathrm{SH}(S)$

$$\mathrm{Th}_{\mathbb{P}^{k-2}}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(\mathcal{D}) \oplus \mathcal{O}(1)^{\oplus k-1}) \to \Sigma^{1,0} \mathrm{Th}_{\mathbb{P}^{k-2}}(\mathcal{O}(\mathcal{D})).$$

Using induction on the odd integer k, we are reduced to assuming that $k = 1$. Now, by (3.3.1.ii) we have in $\mathrm{SH}(S)[\eta^{-1}]$

$$\mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(\mathcal{D})) \simeq \Sigma^{2s,s}(\mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(-1))^{\wedge r-s}),$$

where s is the number of indices $i \in \{1, \ldots, r\}$ such that d_i is even. Since d is odd, so is $r - s$ and using (3.3.1.i), we deduce that

$$\mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(\mathcal{D})) \simeq \Sigma^{2(r-1),r-1} \mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(-1)) \quad \text{in } \mathrm{SH}(S)[\eta^{-1}].$$

But $\mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(-1))$ vanishes in $\mathrm{SH}(S)[\eta^{-1}]$ because of the distinguished triangle (see (1.3.a))

$$\Sigma^\infty_+ \mathcal{O}(-1)^\circ \to \Sigma^\infty_+ \mathbb{P}^1 \to \mathrm{Th}_{\mathbb{P}^1}(\mathcal{O}(-1)) \to \Sigma^{1,0} \Sigma^\infty_+ \mathcal{O}(-1)^\circ$$

and the definition of the map η (recall that $\mathcal{O}(-1)^\circ = \mathbb{A}^2 \setminus \{0\}$). We have proved equation (i).

Let us come back to the situation when k and d are arbitrary. Consider a linear embedding $\mathbb{P}^{k-1} \to \mathbb{P}^k$ avoiding the S-point $\iota_k \colon S \to \mathbb{P}^k$ (we write $\mathbb{P}^{-1} = \varnothing$). It is a closed immersion defined by the vanishing of a regular section of $\mathcal{O}(1)$. Its open complement is isomorphic to $\mathbb{A}^k$, and the morphism $j_k \colon S \to \mathbb{A}^k$ induced by ι_k induces an isomorphism in $\mathrm{SH}(S)$. The canonical trivialisation of $\mathcal{O}(\mathcal{D})$ over $\mathbb{P}^0 = S$ is the restriction along j_k of a trivialisation of $\mathcal{O}(\mathcal{D})|_{\mathbb{A}^k}$ (induced by the trivialisation of $\mathcal{O}(1)|_{\mathbb{A}^k}$ corresponding to the regular section of $\mathcal{O}(1)$ mentioned above). It follows that the map

$\mathrm{Th}_{\mathbb{P}^0}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{A}^k}(\mathcal{O}(\mathcal{D}))$ induced by j_k induces an isomorphism in $\mathrm{SH}(S)$. Thus, (1.4) yields a distinguished triangle in $\mathrm{SH}(S)$

$$\Sigma^{2r,r}\mathbf{1}_S \xrightarrow{(4.1.2.a)} \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^{k-1}}(\mathcal{O}(\mathcal{D}) \oplus \mathcal{O}(1)) \to \Sigma^{2r+1,r}\mathbf{1}_S$$

so that equation (ii) follows from equation (i).

Consider now a linear embedding $s \colon S = \mathbb{P}^0 \to \mathbb{P}^k$ avoiding $i_{k-1}(\mathbb{P}^{k-1})$. Its open complement U is a line bundle over $\mathbb{P}^{k-1}$. The corresponding zero-section $\mathbb{P}^{k-1} \to U$ induces an isomorphism in $\mathrm{SH}(S)$ and is the restriction of the linear embedding $i_{k-1} \colon \mathbb{P}^{k-1} \to \mathbb{P}^k$. Since the vector bundle $s^*\mathcal{O}(\mathcal{D})$ and the normal bundle $s^*\mathcal{O}(1)^{\oplus k}$ to s are both trivial, we have by (1.4) a distinguished triangle in $\mathrm{SH}(S)$

$$\mathrm{Th}_{\mathbb{P}^{k-1}}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) \to \Sigma^{2(k+r),k+r}\mathbf{1}_S \to \Sigma^{1,0}\mathrm{Th}_{\mathbb{P}^{k-1}}(\mathcal{O}(\mathcal{D})). \qquad (4.1.3.\mathrm{a})$$

Therefore, equation (iii) follows from equation (i).

Finally, assume that k is odd and d is even. It follows from equation (ii) that the composite $\mathrm{Th}_{\mathbb{P}^{k-1}}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^{k+1}}(\mathcal{O}(\mathcal{D}))$ is an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$, hence $\mathrm{Th}_{\mathbb{P}^{k-1}}(\mathcal{O}(\mathcal{D})) \to \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(\mathcal{D}))$ admits a retraction, giving a splitting of the triangle (4.1.3.a). In view of equation (ii), this proves equation (iv). $\qquad \square$

Corollary 4.1.4. *If $k \in \mathbb{N}$ is even, the structural morphism $\mathbb{P}^k \to S$ induces an isomorphism $\Sigma_+^\infty \mathbb{P}^k \xrightarrow{\sim} \mathbf{1}_S$ in $\mathrm{SH}(S)[\eta^{-1}]$.*

Proof. The structural morphism is retraction of ι_k, so the corollary follows from (4.1.3.ii) applied with $r = 0$. $\qquad \square$

Proposition 4.1.5. *Let $E, V_1, \ldots, V_n$ be vector bundles of constant rank over S, and $d_1, \ldots, d_n \in \mathbb{Z}$. Assume that $\mathrm{rank}\, E$ is even and that $d_1 \mathrm{rank}\, V_1 + \cdots + d_n \mathrm{rank}\, V_n$ is odd. Then*

$$\mathrm{Th}_{\mathbb{P}(E)}((\mathcal{O}(d_1) \otimes q^* V_1) \oplus \cdots \oplus (\mathcal{O}(d_n) \otimes q^* V_n)) = 0 \in \mathrm{SH}(S)[\eta^{-1}],$$

where $q \colon \mathbb{P}(E) \to S$ is the projective bundle.

Proof. By (1.10) and (1.9), this may be verified Zariski locally on S, so we may assume that $E, V_1, \ldots, V_n$ are all trivial. Then the statement follows from (4.1.3.i). $\qquad \square$

Proposition 4.1.6. *Let E, V be vector bundles over S. Assume that E has constant odd rank. Then $\mathrm{Th}_{\mathbb{P}(E)}(V) \to \mathrm{Th}_S(V)$ is an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$. In particular, $\Sigma_+^\infty \mathbb{P}(E) \xrightarrow{\sim} \mathbf{1}_S$ in $\mathrm{SH}(S)[\eta^{-1}]$.*

Proof. By (1.10) and (1.9), this may be verified Zariski locally on S, so we may assume that E and V are both trivial. Then the statement follows after suspending (4.1.4). $\qquad \square$

4.2. Odd rank vector bundles

4.2.1. Let $E \to S$ be a vector bundle. The composite $\mathcal{O}(-1) \subset E \times_S \mathbb{P}(E) \to E$ restricts to an isomorphism $\mathcal{O}(-1)^\circ \xrightarrow{\sim} E^\circ$, which is $\mathbb{G}_m$-equivariant. We thus obtain a commutative diagram in $\mathrm{Spt}(S)$

$$\begin{array}{ccc}
\Sigma_+^\infty \mathcal{O}(-1)^\circ & \longrightarrow & \Sigma_+^\infty \mathbb{P}(E) \\
\downarrow {\scriptstyle \simeq} & & \downarrow \\
\Sigma_+^\infty E^\circ & \longrightarrow & \mathbf{1}_S
\end{array} \qquad (4.2.1.\text{a})$$

which induces a morphism of the homotopy cofibers of the horizontal morphisms:

$$\mathrm{Th}_{\mathbb{P}(E)}(\mathcal{O}(-1)) \to \mathrm{Th}_S(E) \in \mathrm{SH}(S). \qquad (4.2.1.\text{b})$$

Lemma 4.2.2. *Let $E \to S$ be a vector bundle of constant odd rank. Then equation (4.2.1.b) is an isomorphism $\mathrm{Th}_{\mathbb{P}(E)}(\mathcal{O}(-1)) \xrightarrow{\sim} \mathrm{Th}_S(E)$ in $\mathrm{SH}(S)[\eta^{-1}]$.*

Proof. It follows from (4.1.6) that both vertical arrows in the diagram (4.2.1.a) are isomorphisms in $\mathrm{SH}(S)[\eta^{-1}]$, hence so the induced map on homotopy cofibers. $\square$

Proposition 4.2.3. *Let $E \to S$ be a vector bundle of constant odd rank. Then the projection $E^\circ \to S$ admits a section in $\mathrm{SH}(S)[\eta^{-1}]$, inducing a decomposition in $\mathrm{SH}(S)[\eta^{-1}]$*

$$\Sigma_+^\infty E^\circ \simeq \mathbf{1}_S \oplus \Sigma^{-2,-1} \mathrm{Th}_S(E).$$

Proof. By (2.2.6), we have a splitting in $\mathrm{SH}(S)[\eta^{-1}]$

$$\Sigma_+^\infty \mathcal{O}(-1)^\circ \simeq \Sigma_+^\infty \mathbb{P}(E) \oplus \Sigma^{-2,-1} \mathrm{Th}_{\mathbb{P}(E)}(\mathcal{O}(-1))$$

and the statement follows from (4.2.1), (4.1.6) and (4.2.2). $\square$

Remark 4.2.4. One may deduce that (2.2.7) and (2.2.8) remain valid when L is an odd rank vector bundle instead of a line bundle, using exactly the same arguments but substituting (4.2.3) for (2.2.6).

We deduce the following splitting principle.

Corollary 4.2.5. *Let $X \in \mathrm{Sm}_S$, and $E \to X$ be a vector bundle of constant odd rank. Then there exists a morphism $f \colon Y \to X$ in Sm_S whose image in $\mathrm{SH}(S)[\eta^{-1}]$ admits a section, and a vector bundle $F \to Y$ such that $f^*E \simeq F \oplus 1$.*

Proof. Applying the functor $\mathrm{SH}(X)[\eta^{-1}] \to \mathrm{SH}(S)[\eta^{-1}]$ of (1.9) we may assume that $X = S$. Let us denote by $p \colon (E^\vee)^\circ \to S$ the projection. Then $p^*E^\vee$ admits a nowhere vanishing section s. Its dual $s^\vee \colon p^*E \to 1$ is surjective. Letting $Q = \ker s^\vee$, we have an exact sequence of vector bundles over $(E^\vee)^\circ$

$$0 \to Q \to p^*E \to 1 \to 0. \qquad (4.2.5.\text{a})$$

Then we may find an affine bundle $g \colon Y \to (E^\vee)^\circ$ along which the pullback of the sequence (4.2.5.a) splits (we may take for Y the scheme parametrising the sections of $p^*E^\vee \to Q^\vee$; see, e.g., [Rio10, p.243]). Then $\Sigma_+^\infty g \colon \Sigma_+^\infty Y \to \Sigma_+^\infty (E^\vee)^\circ$ is an isomorphism in $\mathrm{SH}(S)$, and $\Sigma_+^\infty p \colon \Sigma_+^\infty (E^\vee)^\circ \to \mathbf{1}_S$ admits a section in $\mathrm{SH}(S)[\eta^{-1}]$ by (4.2.3). So we may set $f = p \circ g$. $\square$

4.3. Thom spaces of tensor products by line bundles

We are now in position to slightly improve the result obtained in §3.3.

Lemma 4.3.1. *Let $E \to S$ be a vector bundle and $u \in H^0(S, \mathbb{G}_m)$. Assume that E has constant rank r. Then in the notation of (1.5) and (1.6), we have in $\mathrm{SH}(S)[\eta^{-1}]$*

$$\mathrm{Th}(u \, \mathrm{id}_E) = \langle u^r \rangle \colon \mathrm{Th}_S(E) \to \mathrm{Th}_S(E).$$

Proof. Since, under the identification $\mathrm{Th}_S(E \oplus 1) = \mathrm{Th}_S(E) \wedge \mathrm{Th}_S(1)$ (see equation (1.3.b)) we have

$$\mathrm{Th}(u \, \mathrm{id}_{E \oplus 1}) = \mathrm{Th}(u \, \mathrm{id}_E) \wedge \mathrm{Th}(u \, \mathrm{id}_1) = \mathrm{Th}(u \, \mathrm{id}_E) \wedge \langle u \rangle,$$

we may replace E with $E \oplus 1$ if necessary and thus assume that r is odd. The $\mathbb{G}_m$-equivariant isomorphism $E^\circ \simeq \mathcal{O}(-1)^\circ$ (see (4.2.1)) yields a commutative square in $\mathrm{SH}(S)$

$$
\begin{array}{ccc}
\mathrm{Th}_{\mathbb{P}(E)}(\mathcal{O}(-1)) & \xrightarrow{\mathrm{Th}(u \, \mathrm{id}_{\mathcal{O}(-1)})} & \mathrm{Th}_{\mathbb{P}(E)}(\mathcal{O}(-1)) \\
\downarrow & & \downarrow \\
\mathrm{Th}_S(E) & \xrightarrow{\mathrm{Th}(u \, \mathrm{id}_E)} & \mathrm{Th}_S(E)
\end{array}
$$

where the vertical arrows coincide and are isomorphisms in $\mathrm{SH}(S)[\eta^{-1}]$ by (4.2.2). In view of equation (1.6.b), we may replace $E \to S$ with $\mathcal{O}(-1) \to \mathbb{P}(E)$, and thus assume that E is a line bundle. By (2.2.8), we may replace S with E°, and thus assume that the line bundle $E \to S$ admits a trivialisation $\alpha \colon 1 \xrightarrow{\sim} E$. Then we have a commutative square of isomorphisms in $\mathrm{SH}(S)$

$$
\begin{array}{ccc}
\mathrm{Th}_S(E) & \xrightarrow{\mathrm{Th}(u \, \mathrm{id}_E)} & \mathrm{Th}_S(E) \\
\uparrow{\scriptstyle \mathrm{Th}(\alpha)} & & \uparrow{\scriptstyle \mathrm{Th}(\alpha)} \\
\mathrm{Th}_S(1) & \xrightarrow{\mathrm{Th}(u \, \mathrm{id}_1)} & \mathrm{Th}_S(1).
\end{array}
$$

By definition $\mathrm{Th}(u \, \mathrm{id}_1) = \langle u \rangle$, and we deduce using equation (1.6.b) that $\mathrm{Th}(u \, \mathrm{id}_E) = \langle u \rangle \in \mathrm{Aut}_{\mathrm{SH}(S)}(\mathrm{Th}_S(E))$. Since $\langle u^2 \rangle = \mathrm{id}$ by (2.1.1) and r is odd, it follows that $\langle u \rangle = \langle u^r \rangle$, concluding the proof. $\qquad \square$

Proposition 4.3.2. *Let $L \to S$ be a line bundle, and $V \to S$ a vector bundle of constant rank r. If $s \in \mathbb{Z}$ is such that rs is even, then there exists an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$*

$$\mathrm{Th}_S(V) \simeq \mathrm{Th}_S(V \otimes L^{\otimes s}).$$

Proof. Upon replacing V with $V \otimes L^{\otimes s}$, we may assume that $s \geq 0$. When $\alpha \colon 1 \xrightarrow{\sim} L$ is a trivialisation of the line bundle L over S we have an isomorphism in $\mathrm{SH}(S)$

$$\mathrm{Th}(\mathrm{id}_V \otimes \alpha^{\otimes s}) \colon \mathrm{Th}_S(V) \to \mathrm{Th}_S(V \otimes L^{\otimes s}). \tag{4.3.2.a}$$

Any trivialisation of L is of the form $u\alpha$ for some $u \in H^0(S, \mathbb{G}_m)$, and we have in $\mathrm{SH}(S)[\eta^{-1}]$, by (4.3.1) and (2.1.1)

$$\mathrm{Th}(\mathrm{id}_V \otimes (u\alpha)^{\otimes s}) = \mathrm{Th}(\mathrm{id}_V \otimes \alpha^{\otimes s}) \circ \mathrm{Th}(u^s \, \mathrm{id}_V) = \mathrm{Th}(\mathrm{id}_V \otimes \alpha^{\otimes s}) \circ \langle u^{rs} \rangle = \mathrm{Th}(\mathrm{id}_V \otimes \alpha^{\otimes s}).$$

It follows that the image of the isomorphism (4.3.2.a) in $\mathrm{SH}(S)[\eta^{-1}]$ is independent of the choice of the trivialisation α, and we conclude as in the proof of (3.3.1). $\qquad\square$

5. Classifying spaces and characters

5.1. Models for étale classifying spaces

Here, we recall some facts concerning the geometric models of the étale classifying space of a linear algebraic group given in [MV99, §4.2].

5.1.1. Let G be a linear algebraic group over S. Let (V_m, U_m, f_m), for $m \in \mathbb{N} \setminus \{0\}$, be an admissible gadget with a nice (right) G-action, in the sense of [MV99, Definition 4.2.1]. Here, $V_m \to S$ are G-equivariant vector bundles, and $U_m \subset V_m$ are G-invariant open subschemes where the G-action is free. Set $E_m G = U_m$ and $B_m G = (E_m G)/G$. Let us define $BG \in \mathrm{Spc}(S)$ as the colimit of the motivic spaces $B_m G$ as m runs over $\mathbb{N} \setminus \{0\}$. It is proved in [MV99, Proposition 4.2.6] that the weak-equivalence class of BG does not depend on the choice of (V_m, U_m, f_m). More precisely if $(V_m, U_m, f_m), (V'_m, U'_m, f'_m)$ are admissible gadgets with a nice G-action, and $U_m \to U'_m$ are G-equivariant morphisms commuting with the morphisms f_m, f'_m, then the induced morphism of motivic spaces $\mathrm{colim}_m(U_m/G) \to \mathrm{colim}_m(U'_m/G)$ is a weak equivalence. In the sequel, we will refer to a system (V_m, U_m, f_m) as above as a model for BG and use the notation $E_m G, B_m G$.

5.1.2. In the situation of (5.1.1), since $B_1 G$ is cofibrant and each $B_m G \to B_{m+1} G$ is a cofibration (for the model structure of [MV99]), it follows that the colimit BG is canonically weakly equivalent to the homotopy colimit of the motivic spaces $B_m G$ in $\mathrm{Spc}(S)$ (see, e.g., [Hir03, Theorem 19.9.1]).

5.1.3. Let G be a linear algebraic group over S, and choose a model for BG. Since the map $\mathrm{colim}_m E_m G \to S$ is a weak equivalence of motivic spaces [MV99, Proposition 4.2.3], we obtain a canonical morphism $S \to BG$ in $\mathrm{H}(S)$. We say that the model is *pointed* if we are given an S-point of $E_1 G$. This yields map $S \to BG$ in $\mathrm{Spc}(S)$, whose image in $\mathrm{H}(S)$ is the canonical morphism described just above.

5.1.4. (See also [MV99, p.133].) Let us fix an integer $n \in \mathbb{N}$ and describe an explicit model for BGL_n. Fix an integer $p \geq n$ (we will typically take $p = n$). For $s \in \mathbb{N}$, we denote by $\mathrm{Gr}(n, s)$ the Grassmannian of rank n subbundles $U \subset 1^{\oplus s}$ over S (for us a subbundle is locally split, so $1^{\oplus s}/U$ is a vector bundle). For each $m \in \mathbb{N} \setminus \{0\}$, consider the S-scheme $V_{m,p}$ parametrising the vector bundles maps $1^{\oplus n} \to 1^{\oplus pm}$; then $V_{m,p} \to S$ is a vector bundle. Let $U_{m,p}$ the open subscheme of $V_{m,p}$ parametrising those vector bundle maps admitting Zariski locally a retraction (i.e., making $1^{\oplus n}$ a subbundle of $1^{\oplus pm}$). Then the natural left GL_n-action on $1^{\oplus n}$ induces a right GL_n-action on $U_{m,p}$, which is free, and the

quotient $U_{m,p}/\mathrm{GL}_n$ can be identified with the Grassmannian $\mathrm{Gr}(n,pm)$. The inclusion $1^{\oplus m} \subset 1^{\oplus m+1}$ given by the vanishing of the last coordinate induces an inclusion

$$1^{\oplus pm} = (1^{\oplus m})^{\oplus p} \subset (1^{\oplus m+1})^{\oplus p} = 1^{\oplus p(m+1)},$$

which yields a GL_n-equivariant morphism $f_{m,p} \colon U_{m,p} \to U_{m+1,p}$.

Then the family $(V_{m,p}, U_{m,p}, f_{m,p})$ is an admissible gadget with a nice GL_n-action. Indeed, the first condition of [MV99, Definition 4.2.1] is satisfied because $U_{1,p}$ possesses an S-point, and the second condition is satisfied with $j = 2i$. The fact that the group GL_n is special implies the validity of condition (3) of [MV99, Definition 4.2.4]. We thus obtained a model for BGL_n. We have just seen that this model is pointed (in the sense of (5.1.3)); a canonical pointing when $p = n$ is induced by the identity of $1^{\oplus n}$.

5.1.5. Let $H \subset G$ be an inclusion of linear algebraic groups over S. Then any admissible gadget with a nice G-action is also one with a nice H-action (where the H-action is given by restricting the G-action). Indeed, the only nonimmediate point is condition (3) of [MV99, Definition 4.2.4]. So let F be a smooth S-scheme with a free right H-action. Consider the quotient $E = (F \times G)/H$, where the right H-action on G is given by letting $h \in H$ act via $g \mapsto h^{-1}g$. Right multiplication in G induces a free right G-action on E. For any $U \in \mathrm{Sm}_S$ with a right G-action, we have isomorphisms

$$(E \times U)/G \simeq ((F \times G)/H \times U)/G \simeq (F \times (G \times U)/G)/H \simeq (F \times U)/H,$$

which are functorial in U and thus permit to identify the morphisms $(E \times U)/G \to E/G$ and $(F \times U)/H \to F/H$. Since the former is an epimorphism in the Nisnevich topology (as the group G is nice), so is the latter.

Thus, given a model for $\mathrm{B}G$, we obtain a model for $\mathrm{B}H$, where $E_m H = E_m G$ with the induced H-action. This yields morphisms

$$B_m H = (E_m H)/H = (E_m G)/H \to (E_m G)/G = B_m G$$

which are compatible with the transition maps as m varies and thus a map $\mathrm{B}H \to \mathrm{B}G$.

5.1.6. (See also [MV99, Remark 4.2.7].) Assume that G is a linear algebraic group over S, and fix an embedding $G \subset \mathrm{GL}_n$ as a closed subgroup. By (5.1.5), every (pointed) model for BGL_n induces a (pointed) model for $\mathrm{B}G$. Since BGL_n admits a pointed model by (5.1.4), so does $\mathrm{B}G$.

5.2. Products

5.2.1. Let G, G' be linear algebraic groups over S. Choose admissible gadgets (V_m, U_m, f_m) with a nice G-action, and (V'_m, U'_m, f'_m) with a nice G'-action (in the sense of [MV99, Definition 4.2.1], recall from (5.1.6) that such exist). Then the family $(V_m \times V'_m, U_m \times U'_m, f_m \times f'_m)$ constitutes an admissible gadget with a nice $G \times G'$-action. Indeed, to check the last condition of [MV99, Definition 4.2.4], let $T \to X$ be a $G \times G'$-torsor in Sm_S. Then the projection $(T \times U \times U')/(G \times G') \to T/(G \times G')$ factors as

$$(T_1 \times U')/G' \to T_1/G'$$

where $T_1 = (T \times U)/G$ followed by

$$(T_2 \times U)/G \to T_2/G,$$

where $T_2 = T/G'$. Each morphism is an epimorphism in the Nisnevich topology by assumption, hence so is their composite.

Under this choice of a model for BG, we have

$$B_m(G \times G') = B_m G \times B_m G'. \tag{5.2.1.a}$$

Lemma 5.2.2. *If G, G' are linear algebraic groups over S, we have an isomorphism*

$$\mathrm{B}(G \times G') \simeq \mathrm{B}G \times \mathrm{B}G' \in \mathrm{H}(S).$$

Proof. Since the product with a given motivic space commutes with homotopy colimits, we have isomorphisms in $\mathrm{H}(S)$

$$
\begin{aligned}
\mathrm{B}(G \times G') &\simeq \operatorname{hocolim}_m(B_m G \times B_m G') && \text{by equation (5.2.1.a) and (5.1.2)} \\
&\simeq \operatorname{hocolim}_m \operatorname{hocolim}_d(B_m G \times B_d G') && \text{by a cofinality argument} \\
&\simeq \operatorname{hocolim}_m(B_m G \times \operatorname{hocolim}_d B_d G') && \\
&\simeq (\operatorname{hocolim}_m B_m G) \times (\operatorname{hocolim}_d B_d G') && \\
&\simeq \mathrm{B}G \times \mathrm{B}G' && \text{by (5.1.2).} \qquad \square
\end{aligned}
$$

5.3. Characters

In this section, we discuss general facts relating the classifying space of a linear algebraic group G to that of the kernel H of a character of G, which will be applied to explicit situations in §6.

5.3.1. Let G be a linear algebraic group over S, and fix a model for BG (see (5.1.1)). Assume given a character of G, that is a morphism of algebraic groups $\chi \colon G \to \mathbb{G}_m$. Considering the right G-action on $\mathbb{A}^1$ given by letting $g \in G$ act via $\lambda \mapsto \chi(g)^{-1}\lambda$, we define for each $m \in \mathbb{N} \setminus \{0\}$ a line bundle over $B_m G$:

$$C_m(\chi) = (E_m G \times \mathbb{A}^1)/G. \tag{5.3.1.a}$$

The assignment $\chi \mapsto C_m(\chi)$ satisfies

$$C_m(\chi\chi') = C_m(\chi) \otimes C_m(\chi'), \tag{5.3.1.b}$$

yielding group morphisms

$$\operatorname{Hom}_{\text{alg. groups}}(G, \mathbb{G}_m) \to \operatorname{Pic}(B_m G) \quad ; \quad \chi \mapsto C_m(\chi).$$

Let now $H \subset G$ be a closed subgroup, and consider the morphisms $B_m H \to B_m G$ defined in (5.1.5). If $\chi|_H$ denotes the restriction of the character χ to H, then

$$C_m(\chi) \times_{B_m G} B_m H \simeq C_m(\chi|_H). \tag{5.3.1.c}$$

If G' is a linear algebraic group over S, letting $\tilde{\chi}\colon G \times G' \to G \xrightarrow{\chi} \mathbb{G}_m$ be the induced character of $G \times G'$, we have

$$C_m(\tilde{\chi}) = C_m(\chi) \times B_m G'. \tag{5.3.1.d}$$

5.3.2. Let G be a linear algebraic group over S and χ a surjective character of G. Letting $H = \ker \chi$, we thus have an exact sequence of algebraic groups over S

$$1 \to H \to G \xrightarrow{\chi} \mathbb{G}_m \to 1. \tag{5.3.2.a}$$

Let us fix a model for BG. As explained in (5.1.5), this yields a model for BH, and morphisms $p_m\colon B_m H \to B_m G$ for $m \in \mathbb{N} \setminus \{0\}$. By (5.3.1), we also have a line bundle $C_m(\chi)$ over $B_m G$ such that

$$C_m(\chi)^\circ = ((E_m G \times \mathbb{A}^1)/G)^\circ = (E_m G \times \mathbb{G}_m)/G = ((E_m G)/H \times \mathbb{G}_m)/\mathbb{G}_m = (E_m G)/H = B_m H.$$

In view of (1.3.a), this yields a cofiber sequence in $\mathrm{Spc}_\bullet(S)$, for each $m \in \mathbb{N} \setminus \{0\}$

$$(B_m H)_+ \xrightarrow{p_{m+}} (B_m G)_+ \to \mathrm{Th}_{B_m G}(C_m(\chi)). \tag{5.3.2.b}$$

More generally (as in (1.4)), if $V \to B_m G$ is a vector bundle, we have a cofiber sequence in $\mathrm{Spc}_\bullet(S)$,

$$\mathrm{Th}_{B_m H}(V) \to \mathrm{Th}_{B_m G}(V) \to \mathrm{Th}_{B_m G}(C_m(\chi) \oplus V). \tag{5.3.2.c}$$

5.3.3. In the situation of (5.3.2), let us define

$$\mathrm{Th}_{BG}(C(\chi)) = \mathrm{colim}_m \mathrm{Th}_{B_m G}(C_m(\chi)) \in \mathrm{Spc}_\bullet(S).$$

As in (5.1.2), this coincides with the homotopy colimit (the transition morphisms are again monomorphisms, being directed colimits of such). We will also write $\mathrm{Th}_{BG}(C(\chi)) \in \mathrm{Spt}(S)$ instead of $\Sigma^\infty \mathrm{Th}_{BG}(C(\chi))$. Taking the (homotopy) colimit of equation (5.3.2.b) yields a cofiber sequence in $\mathrm{Spc}_\bullet(S)$

$$(BH)_+ \to (BG)_+ \to \mathrm{Th}_{BG}(C(\chi)). \tag{5.3.3.a}$$

5.3.4. In the situation of (5.3.2), assume that the model for BG is pointed. Then we have a commutative diagram of S-schemes with Cartesian squares

$$\begin{array}{ccccc}
G & \xrightarrow{\chi} & \mathbb{G}_m & \longrightarrow & S \\
\downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle i_1} \\
E_1 G & \longrightarrow & B_1 H & \longrightarrow & B_1 G,
\end{array} \tag{5.3.4.a}$$

where e_1 is induced by the S-point and the G-action on $E_1 G$, and j_1, resp. i_1, is obtained by taking the H-quotient, resp. G-quotient of e_1. Composing i_1 and j_1 with the natural maps $B_1 G \to BG$ and $B_1 H \to BH$ respectively, we obtain maps $i\colon S \to BG$ and $j\colon \mathbb{G}_m \to BH$ in $\mathrm{Spc}(S)$. Note that i is the map described in (5.1.3) and that the left-hand Cartesian square in equation (5.3.4.a) shows that the map $j\colon \mathbb{G}_m \to BH$ classifies the H-torsor $\chi\colon G \to \mathbb{G}_m$.

The right-hand Cartesian square in equation (5.3.4.a) shows that the $\mathbb{G}_m$-torsor $B_1H \to B_1G$ pulls back to the trivial torsor along i_1, which yields a trivialisation of the line bundle $i_1^* C_1(\chi)$ over S and thus a morphism in $\mathrm{Spc}_\bullet(S)$

$$t \colon T = \mathrm{Th}_S(1) \to \mathrm{Th}_{B_1G}(C_1(\chi)) \to \mathrm{Th}_{\mathrm{B}G}(C(\chi)).$$

We thus obtain a commutative diagram in $\mathrm{Spc}_\bullet(S)$, whose rows are cofiber sequences

$$
\begin{array}{ccccc}
(\mathrm{B}H)_+ & \longrightarrow & (\mathrm{B}G)_+ & \longrightarrow & \mathrm{Th}_{\mathrm{B}G}(C(\chi)) \\
{\scriptstyle j_+}\big\uparrow & & {\scriptstyle i_+}\big\uparrow & & {\scriptstyle t}\big\uparrow \\
(\mathbb{G}_m)_+ & \longrightarrow & S_+ & \longrightarrow & T.
\end{array}
\qquad (5.3.4.\mathrm{b})
$$

5.3.5. In the situation of (5.3.2), using (2.2.5) for the line bundle $C_m(\chi) \to B_mG$, and applying the functor $\mathrm{Spc}_\bullet(B_mG)[\eta^{-1}] \to \mathrm{Spc}_\bullet(S)[\eta^{-1}]$ of (1.9), we have homotopy cocartesian squares in $\mathrm{Spc}_\bullet(S)[\eta^{-1}]$

$$
\begin{array}{ccc}
T \wedge (B_mH)_+ & \longrightarrow & \mathrm{Th}_{B_mG}(C(\chi_m)) \\
\big\downarrow & & \big\downarrow \\
T \wedge (B_mG)_+ & \longrightarrow & *,
\end{array}
$$

which are compatible with the transition maps as m varies by (2.2.3). Taking the homotopy colimit, and proceeding as in the proof of (2.2.6), we obtain an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$

$$\Sigma_+^\infty \mathrm{B}H \simeq \Sigma_+^\infty \mathrm{B}G \oplus \Sigma^{-2,-1} \mathrm{Th}_{\mathrm{B}G}(C(\chi)) \in \mathrm{SH}(S)[\eta^{-1}]. \qquad (5.3.5.\mathrm{a})$$

6. Computations of classifying spaces

6.1. Diagonalisable groups

Using the embeddings $\mathbb{P}^k \subset \mathbb{P}^{k+1}$ of (4.1.1) for $k \in \mathbb{N}$, we define, for $n \in \mathbb{Z}$

$$\mathbb{P}^\infty = \mathrm{hocolim}_k \mathbb{P}^k \in \mathrm{Spc}(S) \quad \text{and} \quad \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n)) = \mathrm{hocolim}_k \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(n)) \in \mathrm{Spc}_\bullet(S),$$

and as usual write $\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n)) \in \mathrm{Spt}(S)$ instead of $\Sigma^\infty \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n))$. We have a natural map $\iota_\infty \colon S = \mathbb{P}^0 \to \mathbb{P}^\infty$ in $\mathrm{Spc}(S)$. For each $n \in \mathbb{Z}$, the line bundle $\mathcal{O}(n)$ over $\mathbb{P}^0$ admits a canonical trivialisation so that equation (4.1.2.a) yields a canonical map in $\mathrm{Spt}(S)$

$$\Sigma^{2,1} \mathbf{1}_S \to \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n)). \qquad (6.1.0.\mathrm{a})$$

Proposition 6.1.1. *Let $n \in \mathbb{Z}$. The following hold in $\mathrm{SH}(S)[\eta^{-1}]$:*

(i) *The morphism ι_∞ induces an isomorphism $\mathbf{1}_S \simeq \Sigma_+^\infty \mathbb{P}^\infty$.*

(ii) *If n is odd, then $\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n)) = 0$.*

(iii) *If n is even, then equation (6.1.0.a) induces an isomorphism $\Sigma^{2,1} \mathbf{1}_S \simeq \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n))$.*

Proof. We apply (4.1.3) with $\mathcal{D} = (n)$, and so $\mathcal{O}(\mathcal{D}) = \mathcal{O}(n)$. We obtain that $\Sigma^{2,1} \mathbf{1}_S \to \mathrm{Th}_{\mathbb{P}^k}(\mathcal{O}(n))$ is a weak equivalence in $\mathrm{Spt}(S)[\eta^{-1}]$ when n, k are even. Taking the homotopy

colimit over k yields a weak equivalence $\Sigma^{2,1}\mathbf{1}_S \to \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(n))$ in $\mathrm{Spt}(S)[\eta^{-1}]$ when n is even. This proves equation (iii). The other statements are deduced in a similar way from (4.1.3). $\qquad\square$

6.1.2. Consider the model for $\mathrm{B}\mathbb{G}_m$ described in (5.1.4) with $p = n = 1$, under the identification $\mathbb{G}_m = \mathrm{GL}_1$. Then $B_m\mathbb{G}_m = \mathbb{P}^{m-1}$, and thus $\mathrm{B}\mathbb{G}_m = \mathbb{P}^\infty$. Furthermore, the line bundle $C_m(\mathrm{id}_{\mathbb{G}_m})$ over B_mG defined in equation (5.3.1.a) may be identified with the tautological bundle $\mathcal{O}(-1)$ over $\mathbb{P}^{m-1}$.

Theorem 6.1.3. *Let $n \in \mathbb{N} \setminus \{0\}$. The following hold in $\mathrm{SH}(S)[\eta^{-1}]$:*

   (i) *The natural morphism $\mathbf{1}_S \to \Sigma_+^\infty \mathrm{B}\mathbb{G}_m$ is an isomorphism.*

   (ii) *If n is odd, the natural morphism $\mathbf{1}_S \to \Sigma_+^\infty \mathrm{B}\mu_n$ is an isomorphism.*

   (iii) *If n is even, the morphism $\mathbb{G}_m \to \mathrm{B}\mu_n$ classifying the μ_n-torsor $\mathbb{G}_m \to \mathbb{G}_m$ given by taking n-th powers induces an isomorphism $\Sigma_+^\infty \mathbb{G}_m \simeq \Sigma_+^\infty \mathrm{B}\mu_n$.*

Proof. Let us consider the model for $\mathrm{B}\mathbb{G}_m$ described in (6.1.2), where $B_m\mathbb{G}_m = \mathbb{P}^{m-1}$ and $\mathrm{B}\mathbb{G}_m = \mathbb{P}^\infty$. Then the first statement follows from (6.1.1.i).

Next, consider the character $n\colon \mathbb{G}_m \to \mathbb{G}_m$ given by taking n-th powers. Its kernel is μ_n, and the line bundle $C_m(n)$ over $B_m\mathbb{G}_m$ (defined in equation (5.3.1.a)) corresponds to the line bundle $\mathcal{O}(-n)$ over $\mathbb{P}^{m-1}$ (this may be seen for instance by combining (6.1.2) with equation (5.3.1.b)). So we are in the situation of (5.3.2) with $G = \mathbb{G}_m, \chi = n, H = \mu_n$. Thus, equation (5.3.3.a) yields a distinguished triangle in $\mathrm{SH}(S)$

$$\Sigma_+^\infty \mathrm{B}\mu_n \to \Sigma_+^\infty \mathrm{B}\mathbb{G}_m \to \mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(-n)) \to \Sigma^{1,0}\Sigma_+^\infty \mathrm{B}\mu_n.$$

If n is odd, then $\mathrm{Th}_{\mathbb{P}^\infty}(\mathcal{O}(-n)) = 0$ in $\mathrm{SH}(S)[\eta^{-1}]$ by (6.1.1.ii), and the above distinguished triangle shows that the morphism $\Sigma_+^\infty \mathrm{B}\mu_n \to \Sigma_+^\infty \mathrm{B}\mathbb{G}_m$ is an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$. Thus, the second statement follows from the first.

Assume that n is even. Then in the diagram (5.3.4.b) the maps i_+ and t become isomorphisms in $\mathrm{SH}(S)[\eta^{-1}]$ by (6.1.1.i) and (6.1.1.iii), hence $\Sigma_+^\infty j\colon \Sigma_+^\infty \mathbb{G}_m \to \Sigma_+^\infty \mathrm{B}\mu_n$ is also one. As observed in (5.3.2), the latter is induced by the μ_n-torsor $n\colon \mathbb{G}_m \to \mathbb{G}_m$. $\qquad\square$

Remark 6.1.4. Combining (6.1.3) with (5.2.2), we have thus obtained a 'computation' in $\mathrm{SH}(S)[\eta^{-1}]$ of the classifying space $\mathrm{B}G$ of every finitely generated diagonalisable group G.

6.2. SL versus SL^c

Let $n \in \mathbb{N} \setminus \{0\}$. Consider the character

$$\nu_n\colon \mathrm{GL}_n \times \mathbb{G}_m \to \mathbb{G}_m \quad ; \quad (M,t) \mapsto t^{-2}\det M.$$

By definition (see [PW18, §3]), we have $\mathrm{SL}_n^c = \ker \nu_n$. We view SL_n as a subgroup of SL_n^c via the mapping $M \mapsto (M,1)$.

Proposition 6.2.1. *For $n \in \mathbb{N} \setminus \{0\}$, the inclusion $\mathrm{SL}_n \subset \mathrm{SL}_n^c$ induces an isomorphism*

$$\Sigma_+^\infty \mathrm{BSL}_n \xrightarrow{\sim} \Sigma_+^\infty \mathrm{BSL}_n^c \ \ in \ \mathrm{SH}(S)[\eta^{-1}].$$

Proof. The character

$$\delta_n \colon \mathrm{SL}_n^c \to \mathbb{G}_m \quad ; \quad (M,t) \mapsto t$$

is surjective (recall that $n \geq 1$), and satisfies $\mathrm{SL}_n = \ker \delta_n$.

Set $P_n = \mathrm{GL}_n \times \mathbb{G}_m$, and denote by $q_n \colon P_n \to \mathbb{G}_m$ the second projection. Let us fix an arbitrary model for BGL_n, but choose the model for $\mathrm{B}\mathbb{G}_m$ described in (6.1.2) so that $B_m\mathbb{G}_m = \mathbb{P}^{m-1}$. Recall from (5.2.1.a) that this yields a model for BP_n such that

$$B_m P_n = (B_m \mathrm{GL}_n) \times (B_m \mathbb{G}_m) = (B_m \mathrm{GL}_n) \times \mathbb{P}^{m-1}.$$

Letting $g_n \colon P_n \to \mathrm{GL}_n$ be the first projection, we have, as characters $P_n \to \mathbb{G}_m$

$$\nu_n = g_n^*(\det_n) \cdot q_n^*(\mathrm{id}_{\mathbb{G}_m})^{-2},$$

where $\det_n \colon \mathrm{GL}_n \to \mathbb{G}_m$ denotes the determinant morphism. It follows from equations (5.3.1.b) and (5.3.1.d) that, as line bundles over $B_m P_n$, we have in the notation of (5.3.1.a)

$$C_m(\nu_n) \simeq C_m(\det_n) \boxtimes C_m(\mathrm{id}_{\mathbb{G}_m})^{\otimes -2} \quad ; \quad C_m(q_n) \simeq 1 \boxtimes C_m(\mathrm{id}_{\mathbb{G}_m}).$$

Recall from (6.1.2) that the line bundle $C_m(\mathrm{id}_{\mathbb{G}_m}) \to B_m\mathbb{G}_m$ corresponds to $\mathcal{O}(-1) \to \mathbb{P}^{m-1}$. Applying (4.1.5) to the projective bundle $B_m P_n \to B_m \mathrm{GL}_n$, for m even we have

$$\mathrm{Th}_{B_m P_n}(C_m(\nu_n) \oplus C_m(q_n)) = 0 = \mathrm{Th}_{B_m P_n}(C_m(q_n)) \in \mathrm{SH}(S)[\eta^{-1}]. \tag{6.2.1.a}$$

Since the character δ_n is the restriction of $q_n \colon P_n \to \mathbb{G}_m$, it follows from equation (5.3.1.c) that the line bundle $C_m(\delta_n)$ over $B_m \mathrm{SL}_n^c$ is the pullback of $C_m(q_n)$ over $B_m P_n$. By (5.3.2.c), we have a distinguished triangle in $\mathrm{SH}(S)$

$$\mathrm{Th}_{B_m \mathrm{SL}_n^c}(C_m(\delta_n)) \to \mathrm{Th}_{B_m P_n}(C_m(q_n)) \to \mathrm{Th}_{B_m P_n}(C_m(\nu_n) \oplus C_m(q_n))$$
$$\to \Sigma^{1,0} \mathrm{Th}_{B_m \mathrm{SL}_n^c}(C_m(\delta_n))$$

so that, in view of equation (6.2.1.a)

$$\mathrm{Th}_{B_m \mathrm{SL}_n^c}(C_m(\delta_n)) = 0 \in \mathrm{SH}(S)[\eta^{-1}] \quad \text{for } m \text{ even}.$$

Now, the distinguished triangle in $\mathrm{SH}(S)$ (see (5.3.2.b))

$$\Sigma_+^\infty B_m \mathrm{SL}_n \to \Sigma_+^\infty B_m \mathrm{SL}_n^c \to \mathrm{Th}_{B_m \mathrm{SL}_n^c}(C_m(\delta_n)) \to \Sigma^{1,0} \Sigma_+^\infty B_m \mathrm{SL}_n$$

implies that, for m even, the natural map induces an isomorphism

$$\Sigma_+^\infty B_m \mathrm{SL}_n \xrightarrow{\sim} \Sigma_+^\infty B_m \mathrm{SL}_n^c \quad \text{in } \mathrm{SH}(S)[\eta^{-1}].$$

The statement follows by taking the homotopy colimit. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.2.2. Let $A \in \mathrm{SH}(S)$ be an η-periodic commutative ring spectrum, and consider the corresponding cohomology theory $A^{*,*}(-)$ (see (3.1.1)). Then by (6.2.1), we have a natural isomorphism

$$A^{*,*}(\mathrm{BSL}_n^c) \simeq A^{*,*}(\mathrm{BSL}_n).$$

If A is SL-oriented (see (3.2.5)) and $S = \operatorname{Spec} k$ with k a field of characteristic not two, Ananyevskiy computed in [Ana15, Theorem 10] that

$$A^{*,*}(\mathrm{BSL}_n) = \begin{cases} A^{*,*}(S)[[p_1,\ldots,p_{r-1},e]]_h & \text{if } n = 2r \text{ with } r \in \mathbb{N} \setminus \{0\} \\ A^{*,*}(S)[[p_1,\ldots,p_r]]_h & \text{if } n = 2r+1 \text{ with } r \in \mathbb{N}, \end{cases}$$

where p_i has degree $(4i,2i)$ and e has degree $(2r,r)$ (here, the notation $R[[x_1,\ldots,x_m]]_h$ refers to the homogeneous power series ring in m variables over the graded ring R; see [Ana15, Definition 27]). This computation remains valid (with exactly the same arguments) when S is an arbitrary noetherian scheme of finite dimension under the assumption that 2 is invertible in S. Removing that last assumption seems to require a modification of the arguments of [Ana15], the problem being with [Ana15, Lemma 6] (which is used to prove [Ana15, Theorem 9]).

6.3. GL and SL

In this section, we compare the classifying spaces BGL_{2r}, BGL_{2r+1}, BSL_{2r}, BSL_{2r+1}.

6.3.1. Recall that under the model for BGL_n described in (5.1.4) for $p = n$, the scheme $B_m \mathrm{GL}_n$ is identified with the Grassmannian $\mathrm{Gr}(n,nm)$. The closed immersion $\mathrm{Gr}(n,nm) \to \mathrm{Gr}(n+1,(n+1)m)$ mapping a subbundle $E \subset (1^{\oplus m})^{\oplus n}$ to $E \oplus 1 \subset (1^{\oplus m})^{\oplus n} \oplus 1^{\oplus m} = (1^{\oplus m})^{\oplus n+1}$, where the inclusion $1 \subset 1^{\oplus m}$ is given by the vanishing of the last $m-1$ coordinates, induces a morphism $f_m \colon B_m \mathrm{GL}_n \to B_m \mathrm{GL}_{n+1}$ which is compatible with the transition maps as m varies. This yields a morphism in $\mathrm{Spc}(S)$

$$\mathrm{BGL}_n \to \mathrm{BGL}_{n+1}. \tag{6.3.1.a}$$

6.3.2. For integers $u,v,w \in \mathbb{N}$, we denote by $\mathrm{Gr}(u \subset v,w)$ the flag variety of subbundles $P \subset Q \subset 1^{\oplus w}$ with $\operatorname{rank} P = u$ and $\operatorname{rank} Q = v$. Let $r,s \in \mathbb{N}$, and consider the morphisms
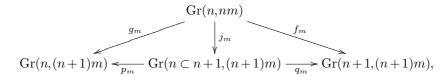
$$\mathrm{Gr}(2r,s) \xleftarrow{p} \mathrm{Gr}(2r \subset 2r+1,s) \xrightarrow{q} \mathrm{Gr}(2r+1,s)$$

given by mapping a flag $P \subset Q$ to P, resp. Q.

For $n \in \{2r, 2r+1\}$, let us denote by $\mathcal{U}_n \subset 1^{\oplus s}$ the tautological rank n subbundle over $\mathrm{Gr}(n,s)$, and write $\mathcal{Q}_n = 1^{\oplus s}/\mathcal{U}_n$. Then the morphism p may be identified with the projective bundle $\mathbb{P}(\mathcal{Q}_{2r})$, and the morphism q is the projective bundle $\mathbb{P}(\mathcal{U}_{2r+1}^\vee)$.

Proposition 6.3.3. *The map* $\mathrm{BGL}_{2r} \to \mathrm{BGL}_{2r+1}$ *of equation (6.3.1.a) becomes an isomorphism in* $\mathrm{SH}(S)[\eta^{-1}]$.

Proof. Let $n = 2r$. For $m \in \mathbb{N} \setminus \{0\}$, consider the commutative diagram in Sm_S

$$\mathrm{Gr}(n,nm)$$

$$\mathrm{Gr}(n,(n+1)m) \xleftarrow{\quad p_m \quad} \mathrm{Gr}(n \subset n+1,(n+1)m) \xrightarrow{\quad q_m \quad} \mathrm{Gr}(n+1,(n+1)m),$$

with arrows g_m, j_m, f_m.

where the morphism j_m is given by mapping $E \subset (1^{\oplus m})^{\oplus n}$ to

$$E \subset E \oplus 1 \subset (1^{\oplus m})^{\oplus n} \oplus 1^{\oplus m} = (1^{\oplus m})^{\oplus n+1},$$

with the inclusion $1 \subset 1^{\oplus m}$ given by the vanishing of the $m-1$ last coordinates. Here, the morphisms p_m, q_m are the morphisms p, q described in (6.3.2) when $s = (n+1)m$. The morphism f_m is the one described in (6.3.1), and the morphism g_m is induced by the inclusion

$$(1^{\oplus m})^{\oplus n} = (1^{\oplus m})^{\oplus n} \oplus 0 \subset (1^{\oplus m})^{\oplus n} \oplus 1^{\oplus m} = (1^{\oplus m})^{\oplus n+1}. \tag{6.3.3.a}$$

The morphisms of this diagram are compatible with the transition maps as m varies, induced by the inclusions $1^{\oplus m} \subset 1^{\oplus m+1}$ given by the vanishing of the last coordinate.

The morphism q_m is a $\mathbb{P}^n$-bundle, hence is an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$ by (4.1.6) (recall that $n = 2r$ is even). The morphism p_m is a $\mathbb{P}^{(n+1)m-n-1}$-bundle, hence is also an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$ when m is odd by (4.1.6).

In the notation of (5.1.4), the morphism g_m is the GL_n-quotient of the morphism $U_{m,n} \to U_{m,n+1}$ induced by equation (6.3.3.a). Therefore, it follows from (5.1.1) that the map $\mathrm{colim}_m g_m$ is a weak equivalence of motivic spaces.

Applying the functor $\Sigma_+^\infty \colon \mathrm{Sm}_S \to \mathrm{Spt}(S)[\eta^{-1}]$ to the above diagram and taking the homotopy colimit over m, we thus obtain a commutative diagram in $\mathrm{Spt}(S)[\eta^{-1}]$, where all maps are weak equivalences. Since the map (6.3.1.a) is obtained as $\mathrm{colim}_m f_m$, the proposition follows. $\qquad\square$

6.3.4. Let $n \in \mathbb{N} \setminus \{0\}$. The group SL_n is the kernel of the determinant morphism $\det_n \colon \mathrm{GL}_n \to \mathbb{G}_m$, which is surjective (as $n \geq 1$). We are thus in the situation of (5.3.2) so that we have by equation (5.3.5.a) a splitting

$$\Sigma_+^\infty \mathrm{BSL}_n = \Sigma_+^\infty \mathrm{BGL}_n \oplus \Sigma^{-2,-1} \mathrm{Th}_{\mathrm{BGL}_n}(C(\det_n)) \in \mathrm{SH}(S)[\eta^{-1}]. \tag{6.3.4.a}$$

6.3.5. Using the model for BGL_n described in (5.1.4) with $p = n$, the variety $B_m \mathrm{GL}_n$ coincides with the Grassmannian $\mathrm{Gr}(n, nm)$. Observe that the tautological bundle $\mathcal{U}_n$ over this variety is isomorphic to the quotient $(E_m \mathrm{GL}_n \times \mathbb{A}^n)/\mathrm{GL}_n$, where the right GL_n-action on $\mathbb{A}^n$ is given by letting $\varphi \in \mathrm{GL}_n$ act via $v \mapsto \varphi^{-1}(v)$. The GL_n-equivariant isomorphism $\det(E_m \mathrm{GL}_n \times \mathbb{A}^n) \simeq E_m \mathrm{GL}_n \times \mathbb{A}^1$, where the right GL_n-action on $\mathbb{A}^1$ is given by letting $\varphi \in \mathrm{GL}_n$ act via $\lambda \mapsto \det_n(\varphi^{-1})\lambda$, yields an isomorphism of line bundles over $B_m \mathrm{GL}_n$

$$\det \mathcal{U}_n \simeq C_m(\det_n). \tag{6.3.5.a}$$

Proposition 6.3.6. *Let $r \in \mathbb{N}$. Then in $\mathrm{SH}(S)[\eta^{-1}]$, the natural morphism $\mathrm{BSL}_{2r} \to \mathrm{BGL}_{2r}$ acquires a section, and the natural morphism $\mathrm{BSL}_{2r+1} \to \mathrm{BGL}_{2r+1}$ becomes an isomorphism.*

Proof. The first statement follows from (6.3.4.a). Let us prove the second. We use the model for BGL_{2r+1} described in (5.1.4) with $n = p = 2r+1$ so that $B_m\,\mathrm{GL}_{2r+1} = \mathrm{Gr}(2r+1,s)$ where $s = (2r+1)m$. We consider the situation of (6.3.2) and use the notation thereof. We have an exact sequence of vector bundles over $Y_m = \mathrm{Gr}(2r \subset 2r+1,s) = \mathbb{P}(\mathcal{Q}_{2r})$

$$0 \to p^*\mathcal{U}_{2r} \to q^*\mathcal{U}_{2r+1} \to \mathcal{O}_{\mathbb{P}(\mathcal{Q}_{2r})}(-1) \to 0.$$

Taking determinants and using equation (6.3.5.a), we obtain an isomorphism of line bundles

$$p^*(\det\mathcal{U}_{2r}) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{Q}_{2r})}(-1) \simeq q^*(\det\mathcal{U}_{2r+1}) \simeq q^*C_m(\det_{2r+1}). \tag{6.3.6.a}$$

When m is even, the vector bundle $\mathcal{Q}_{2r} = 1^{\oplus s}/\mathcal{U}_{2r}$ has even rank, hence it follows from (4.1.5) and equation (6.3.6.a) that $\mathrm{Th}_{Y_m}(q^*C_m(\det_{2r+1})) = 0$ in $\mathrm{SH}(S)[\eta^{-1}]$. Since $q \colon Y_m \to \mathrm{Gr}(2r+1,s)$ is a $\mathbb{P}^{2r}$-bundle, it then follows from (4.1.6) (applied with $V = C_m(\det_{2r+1})$) that $\mathrm{Th}_{\mathrm{Gr}(2r+1,s)}(C_m(\det_{2r+1})) = 0$ in $\mathrm{SH}(S)[\eta^{-1}]$ for m even. Taking the homotopy colimit over m, we deduce that $\mathrm{Th}_{\mathrm{BGL}_{2r+1}}(C(\det_{2r+1})) = 0$ in $\mathrm{SH}(S)[\eta^{-1}]$, and the second statement of the proposition follows from equation (6.3.4.a). $\qquad\square$

Remark 6.3.7. Let $A \in \mathrm{SH}(S)$ be an η-periodic SL-oriented commutative ring spectrum (see (3.2.5)), and consider the corresponding cohomology theory $A^{*,*}(-)$ (see (3.1.1)). Assume that 2 is invertible in S. Combining (6.3.6) and (6.3.3) with Ananyevskiy's computation of $A^{*,*}(\mathrm{BSL}_{2r+1})$ (see (6.2.2)), we recover Levine's computation [Lev19, Theorem 4.1]

$$A^{*,*}(\mathrm{BGL}_{2r}) = A^{*,*}(\mathrm{BGL}_{2r+1}) = A^{*,*}(S)[[p_1, \ldots, p_r]]_h.$$

(Note that this permits to remove some of the technical assumptions present in the statement of [Lev19, Theorem 4.1].)

Appendix A. An invariant of μ_2-torsors

We have proved in (6.1.3) that in $\mathrm{SH}(S)[\eta^{-1}]$, for $n \in \mathbb{N}$

$$\Sigma_+^\infty \mathrm{B}\mu_n = \begin{cases} \mathbf{1}_S & \text{if } n = 0 \,(\text{i.e., } \mu_n = \mathbb{G}_m) \text{ or } n \text{ is odd,} \\ \Sigma_+^\infty \mathbb{G}_m = \mathbf{1}_S \oplus \mathbf{1}_S & \text{if } n > 0 \text{ is even.} \end{cases}$$

Thus, when $n > 0$ is even, there is essentially one nontrivial invariant of μ_n-torsors over S in $\mathrm{SH}(S)[\eta^{-1}]$, in the form of an element of $\mathrm{End}_{\mathrm{SH}(S)[\eta^{-1}]}(\mathbf{1}_S)$. Moreover, it also follows from (6.1.3) that the morphism $\mathrm{B}\mu_2 \to \mathrm{B}\mu_{2r}$ becomes an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$ for $r > 0$. Therefore, the above-mentioned invariant of μ_n-torsors is induced by an invariant of μ_2-torsors, which is, however, not really explicit from this description. In this section, we provide an explicit construction of this invariant (the connection with the above discussion is made in (A.9) below).

A.1.

Let $L \to S$ be a line bundle. As observed by Ananyevskiy [Ana20, Lemma 4.1], the isomorphism of S-schemes $L^\circ \xrightarrow{\sim} (L^\vee)^\circ$, given locally by $l \mapsto l^\vee$, where $l^\vee(l) = 1$, induces an isomorphism in $H_\bullet(S)$

$$\sigma_L \colon \mathrm{Th}_S(L) \xrightarrow{\sim} \mathrm{Th}_S(L^\vee).$$

A.2.

If $\varphi \colon L \to M$ is an isomorphism of line bundles over X, we have (see (1.5))

$$\sigma_L = \mathrm{Th}(\varphi^\vee) \circ \sigma_M \circ \mathrm{Th}(\varphi). \tag{A.2.a}$$

Definition A.3. It will be convenient to think of a μ_2-torsor over S as a pair (L, λ), where $L \to S$ is a line bundle, and $\lambda \colon L \xrightarrow{\sim} L^\vee$ is an isomorphism of line bundles over S. Isomorphisms $(L, \lambda) \to (L', \lambda')$ are given by isomorphisms of line bundles $\varphi \colon L \xrightarrow{\sim} L'$ such that $\lambda = \varphi^\vee \circ \lambda' \circ \varphi$. The set of isomorphism classes of μ_2-torsors is denoted $H^1_{et}(S, \mu_2)$; it is endowed with a group structure induced by the tensor product of line bundles.

Definition A.4. Consider a μ_2-torsor, given by a line bundle $L \to S$ and an isomorphism $\lambda \colon L \xrightarrow{\sim} L^\vee$. Let us consider the composite isomorphism in $\mathrm{SH}(S)$ (see (A.1) for the definition of σ_L, and (1.5) for that of $\mathrm{Th}(\lambda)$)

$$\mathrm{Th}_S(L) \xrightarrow{\mathrm{Th}(\lambda)} \mathrm{Th}_S(L^\vee) \xrightarrow{\sigma_L^{-1}} \mathrm{Th}_S(L).$$

This yields an element $a_{(L, \lambda)} = \Sigma^{-L}(\sigma_L^{-1} \circ \mathrm{Th}(\lambda)) \in \mathrm{Aut}_{\mathrm{SH}(S)}(\mathbf{1}_S)$. We define

$$\alpha(L, \lambda) = (a_{(1, \mathrm{can})})^{-1} \circ a_{(L, \lambda)} \in \mathrm{Aut}_{\mathrm{SH}(S)}(\mathbf{1}_S),$$

where can$\colon 1 \to 1^\vee$ is the canonical isomorphism of line bundles over S. (The element $a_{(1, \mathrm{can})}$ corresponds to the element ϵ of [Mor04, §6.1].)

A.5.

This construction is compatible with pullbacks, in the sense that if $f \colon R \to S$ is a morphism of noetherian schemes of finite dimension and (L, λ) a μ_2-torsor, then the composite in $\mathrm{SH}(R)$

$$\mathbf{1}_R = f^* \mathbf{1}_S \xrightarrow{f^* \alpha(L, \lambda)} f^* \mathbf{1}_S = \mathbf{1}_R$$

is $\alpha(f^* L, f^* \lambda)$.

Example A.6. Consider a μ_2-torsor (L, λ), where $L = 1$ is the trivial line bundle. Then $\lambda = u\,\mathrm{can}$ for some $u \in H^0(S, \mathbb{G}_m)$, hence $\mathrm{Th}(\lambda) = \mathrm{Th}(\mathrm{can}) \circ \mathrm{Th}(u\,\mathrm{id}_1)$. Therefore, (see (1.6))

$$\begin{aligned}
\alpha(1, \lambda) &= \Sigma^{-2, -1}((\sigma_1^{-1} \circ \mathrm{Th}(\mathrm{can}))^{-1} \circ (\sigma_1^{-1} \circ \mathrm{Th}(\lambda))) \\
&= \Sigma^{-2, -1}(\mathrm{Th}(\mathrm{can})^{-1} \circ \mathrm{Th}(\lambda)) \\
&= \Sigma^{-2, -1}\,\mathrm{Th}(u\,\mathrm{id}_1) \\
&= \langle u \rangle.
\end{aligned}$$

Proposition A.7. *The assignment* $(L,\lambda) \mapsto \alpha(L,\lambda)$ *induces a morphism of pointed sets*

$$\alpha\colon H^1_{et}(S,\mu_2) \to \mathrm{Aut}_{\mathrm{SH}(S)}(\mathbf{1}_S).$$

Proof. By construction, we have $\alpha(1,\mathrm{can}) = \mathrm{id}$. Consider an isomorphism of μ_2-torsors $(L,\lambda) \xrightarrow{\sim} (M,\mu)$ (see (A.3)), given by an isomorphism $\varphi\colon L \xrightarrow{\sim} M$. Let us set $a_L = a_{(L,\lambda)}$ and $a_M = a_{(M,\mu)}$ (see (A.4)). Then, in $\mathrm{Aut}_{\mathrm{SH}(S)}(\mathrm{Th}_S(L))$,

$$
\begin{aligned}
\Sigma^L a_L = \sigma_L^{-1} \circ \mathrm{Th}(\lambda) &= \sigma_L^{-1} \circ \mathrm{Th}(\varphi^\vee) \circ \mathrm{Th}(\mu) \circ \mathrm{Th}(\varphi) && (\text{as } \lambda = \varphi^\vee \circ \mu \circ \varphi) \\
&= \mathrm{Th}(\varphi)^{-1} \circ \sigma_M^{-1} \circ \mathrm{Th}(\mu) \circ \mathrm{Th}(\varphi) && \text{by equation (A.2.a)} \\
&= \mathrm{Th}(\varphi)^{-1} \circ (\Sigma^M a_M) \circ \mathrm{Th}(\varphi) \\
&= \mathrm{Th}(\varphi)^{-1} \circ \mathrm{Th}(\varphi) \circ (\Sigma^L a_M) && (\text{by equation (1.5.b)}) \\
&= \Sigma^L a_M,
\end{aligned}
$$

whence $a_L = a_M$, and $\alpha(L,\lambda) = \alpha(M,\mu)$. $\qquad\square$

Proposition A.8. *The assignment* $(L,\lambda) \mapsto \alpha(L,\lambda)$ *induces a group morphism*

$$\alpha\colon H^1_{et}(S,\mu_2) \to \mathrm{Aut}_{\mathrm{SH}(S)[\eta^{-1}]}(\mathbf{1}_S).$$

Proof. Consider μ_2-torsors given by line bundles L,M over S and isomorphisms $L \xrightarrow{\sim} L^\vee$ and $M \xrightarrow{\sim} M^\vee$. As the functor $f^*\colon \mathrm{SH}(S)[\eta^{-1}] \to \mathrm{SH}(L^\circ)[\eta^{-1}]$ is faithful (2.2.8), by functoriality (A.5) we may assume that the line bundle L is trivial. Similarly, we may assume that M is also trivial. By (A.7), we may assume that $L = M = 1$. Then, in view of (A.6), the statement follows from equation (1.6.a). $\qquad\square$

Remark A.9. Let $\rho\colon \mathbb{G}_m \to \mathrm{B}\mu_2$ be the map classifying the μ_2-torsor $(1,t\,\mathrm{id})$ over $\mathbb{G}_m$, where $t \in H^0(\mathbb{G}_m,\mathbb{G}_m)$ is the tautological section. Recall from (6.1.3.iii) that $\Sigma^\infty_+ \rho\colon \Sigma^\infty_+ \mathbb{G}_m \to \Sigma^\infty_+ \mathrm{B}\mu_2$ is an isomorphism in $\mathrm{SH}(S)[\eta^{-1}]$. If a μ_2-torsor (L,λ) is classified by the map $f\colon S \to \mathrm{B}\mu_2$, we claim that $\alpha(L,\lambda)$ is the composite in $\mathrm{SH}(S)[\eta^{-1}]$ (the map π_1 was defined in (2.2.1))

$$\mathbf{1}_S \xrightarrow{\Sigma^\infty_+ f} \Sigma^\infty_+ \mathrm{B}\mu_2 \xrightarrow{(\Sigma^\infty_+ \rho)^{-1}} \Sigma^\infty_+ \mathbb{G}_m \xrightarrow{\Sigma^{-2,-1}\Sigma^\infty_+ \pi_1} \mathbf{1}_S. \qquad (\text{A.9.a})$$

Indeed, applying the functor $\mathrm{SH}(S)[\eta^{-1}] \to \mathrm{SH}(L^\circ)[\eta^{-1}]$ which is faithful by (2.2.8) and using (A.7), we may assume that $L = 1$. Then λ is given by multiplication by an element $u \in H^0(S,\mathbb{G}_m)$, and the map f factors as $S \xrightarrow{u} \mathbb{G}_m \xrightarrow{\rho} \mathrm{B}\mu_2$. Denoting by $p\colon \mathbb{G}_m \to S$ is the projection, the composite (A.9.a) is given by

$$\mathbf{1}_S \xrightarrow{\Sigma^\infty_+ u} \Sigma^\infty_+ \mathbb{G}_m \xrightarrow{\langle t \rangle} \Sigma^\infty_+ \mathbb{G}_m \xrightarrow{\Sigma^\infty_+ p} \mathbf{1}_S,$$

which coincides with $\langle u \rangle \in \mathrm{Aut}_{\mathrm{SH}(S)[\eta^{-1}]}(S)$. Thus, the claim follows from (A.6).

This work was supported by the DFG research grant HA 7702/5-1 and Heisenberg grant HA 7702/4-1.

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