Can. J. Math. Vol. **49** (5), 1997 pp. 1066–1088

MULTIPARAMETER VARIATIONAL EIGENVALUE PROBLEMS WITH INDEFINITE NONLINEARITY

TETSUTARO SHIBATA

ABSTRACT. We consider the multiparameter nonlinear Sturm-Liouville problem

$$u''(x) - \sum_{k=1}^{m} \mu_k u(x)^{p_k} + \sum_{k=m+1}^{n} \mu_k u(x)^{p_k} = \lambda u(x)^q, \quad x \in I := (-1, 1),$$
$$u(x) > 0, \quad x \in I,$$
$$u(-1) = u(1) = 0,$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_m, \mu_{m+1}, \ldots \mu_n) \in \bar{R}_+^m \times R_+^{n-m} \Big(R_+ := (0, \infty)\Big)$ and $\lambda \in R$ are parameters. We assume that

$$1 \le q \le p_1 < p_2 < \cdots < p_n < 2q + 3.$$

We shall establish an asymptotic formula of variational eigenvalue $\lambda = \lambda(\mu, \alpha)$ obtained by using Ljusternik-Schnirelman theory on general level set $N_{\mu,\alpha}(\alpha > 0$: parameter of level set). Furthermore, we shall give the optimal condition of $\{(\mu, \alpha)\}$, under which $\mu_i(m + 1 \le i \le n$: fixed) dominates the asymptotic behavior of $\lambda(\mu, \alpha)$.

1. **Introduction.** This paper is concerned with the following nonlinear multiparameter problem

$$(1.1)\qquad \begin{gathered} u''(x) - \sum_{k=1}^{m} \mu_k u(x)^{p_k} + \sum_{k=m+1}^{n} \mu_k u(x)^{p_k} = \lambda u(x)^q, \quad x \in I := (-1, 1), \\ u(x) > 0, \quad x \in I, \\ u(-1) = u(1) = 0, \end{gathered}$$

where $\mu = (\mu_1, \mu_2, \ldots, \mu_m, \mu_{m+1}, \ldots, \mu_n) \in \bar{R}_+^m \times R_+^{n-m}$ $(R_+ := (0, \infty), \bar{R}_+ := [0, \infty))$ and $\lambda \in R$ are parameters. We assume

$$(1.2)\qquad 1 \le q \le p_1 < p_2 < \cdots < p_n < 2q + 3.$$

The purpose of this paper is to extend the asymptotic formula of variational eigenvalue $\lambda = \lambda(\mu, \alpha)$ obtained in Shibata [4] by using Ljusternik-Schnirelman theory on general

Received by the editors June 27, 1996.
AMS subject classification: Primary: 34B15, 34B25.

level set $N_{\mu,\alpha}$ due to Zeidler [7], where

$$N_{\mu,\alpha} := \{u \in W_0^{1,2}(I) : B(\mu,u) := -\alpha\},$$
$$B(\mu,u) := \frac{1}{2}\int_I u'(x)^2\,dx + \sum_{k=1}^{m} \frac{1}{p_k+1}\mu_k \int_I |u(x)|^{p_i+1}\,dx$$
$$- \sum_{k=m+1}^{n} \frac{1}{p_k+1}\mu_k \int_I |u(x)|^{p_k+1}\,dx,$$

and $\alpha > 0$ is a parameter. Furthermore, we shall give the optimal condition of $\{(\mu,\alpha)\}$, under which $\mu_i (m+1 \le i \le n :$ fixed) dominates the asymptotic behavior of $\lambda(\mu,\alpha)$.

Differential equations which involve several parameters have been extensively investigated and numerous references are available. In linear case, searching the *asymptotic direction of eigenvalues* (the limit of the ratio of two eigenvalues) has drawn most of the attention in the literature. We refer to Faierman [2] and Turyn [6] for further information. As for the asymptotic properties of eigenvalues of nonlinear multiparameter problems, however, a few results seems to have been given. Recently, Shibata [4] studied a simple two-parameter problem

$$\text{(1.3)}\qquad \begin{gathered} u''(x) + \mu u(x)^p = \lambda u(x)^q, \quad 0 < x < 1,\\ u(x) > 0, \quad 0 < x < 1,\\ u(0) = u(1) = 0, \end{gathered}$$

where $\mu, \lambda \in R_+$ are parameters and $1 \le q < p < q+2$ are constants. By using Ljusternik-Schnirelman theory on general level set

$$S_{\mu,\alpha} := \left\{u \in W_0^1\big((0,1)\big) : \frac{1}{2}\int_0^1 u'(x)^2\,dx - \frac{1}{p+1}\mu\int_0^1 |u(x)|^{p+1}\,dx = -\alpha\right\}$$

($\alpha > 0$: a fixed constant), the following asymptotic formula for variational eigenvalue $\lambda = \lambda(\mu,\alpha)$ as $\mu \longrightarrow \infty$ was obtained:

$$\text{(1.4)}\qquad \lambda(\mu,\alpha) = C_1 \alpha^{\frac{2(p-q)}{p+3}} \mu^{\frac{q+3}{p+3}} + o\Big(\mu^{\frac{q+3}{p+3}}\Big),$$

where

$$\text{(1.5)}\qquad C_1 = \left\{\left(\frac{q+1}{p+1}\right)^{\frac{q+3}{2(p-q)}} \frac{(p+3)(q+1)(p-q)}{2(2q-p+3)} \sqrt{\frac{2}{\pi(q+1)}}\, \frac{\Gamma\Big(\frac{p+3}{2(p-q)}\Big)}{\Gamma\Big(\frac{q+3}{2(p-q)}\Big)}\right\}^{\frac{2(p-q)}{p+3}}.$$

In this paper, we shall extend this asymptotic formula (1.4) to our problem (1.1) under the following condition (B.1) for $\{(\mu,\alpha)\}$: Let $m+1 \le i \le n$ be fixed. Furthermore, let $E := \bar{R}_+^m \times R_+^{n-m} \times R_+$. A sequence $\{(\mu,\alpha)\} \subset E$ is said to satisfy the condition (B.1) if the following conditions hold:

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

(B.1)

$$\alpha\mu_i^{\frac{2}{p_i-1}} \longrightarrow \infty, \quad \alpha\mu_i^{-1/2} \longrightarrow 0. \tag{1.6}$$

$$\mu_k \alpha^{\frac{2(p_k-p_i)}{p_i+3}} \mu_i^{-\frac{p_k+3}{p_i+3}} \longrightarrow 0 \quad (k \neq i). \tag{1.7}$$

The typical example of $\{(\mu,\alpha)\} \subset E$ which satisfies (B.1) is:

$$\mu_i = s \longrightarrow \infty, \quad \mu_k = s^{\nu_k}\Big(k \neq i,\ 0 < \nu_k < \frac{p_k+3}{p_i+3}\Big), \quad \alpha = \alpha_0 (= \text{constant}).$$

The important point is that, without the condition (B.1), the asymptotic formula of $\lambda(\mu,\alpha)$ is not dominated by μ_i ($m+1 < i \le n$: fixed) any more. For example, if we assume a simple and natural condition

$$\mu_j = 0\ (1 \le j \le m), \quad \mu_{m+1} \longrightarrow \infty, \quad C^{-1}\mu_j \le \mu_{m+1} \le C\mu_j\ (m+1 \le j \le n), \tag{1.8}$$

where $C > 0$ is a constant, then we find that μ_{m+1} is the dominant term of the asymptotic behavior of $\lambda(\mu,\alpha)$ automatically. The reason why is that, roughly speaking, the maximum norm of the associated eigenfunction tends to 0 under the condition (1.8). Hence, the dominant nonlinear term should be $u^{p_{m+1}}$, the lowest term.

We shall show that, under the condition (B.1), $\mu_i (m+1 \le i \le n$: fixed) dominates the asymptotic behavior of $\lambda(\mu,\alpha)$, and establish an asymptotic formula of $\lambda(\mu,\alpha)$ in Theorem 2.1.

Secondly, we consider the simple condition (B.2), in which the condition (B.1) fails:
(B.2)
$\{(\mu,\alpha)\} \subset E$ satisfies (1.6), (1.7) for $1 < k \le n$ ($k \neq i$) and

$$\mu_1 \alpha^{\frac{2(q-p_i)}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}} \longrightarrow C_2, \tag{1.9}$$

where $C_2 > 0$ is a constant.

Under the condition (B.2), we shall establish the different kind of asymptotic formula for $\lambda(\mu,\alpha)$ in Theorem 2.2.

Finally, we consider (1.1) under the following condition (B.3) and give an asymptotic formula of $\lambda(\mu,\alpha)$ in Theorem 2.3:
(B.3)
$\{(\mu,\alpha)\} \subset E$ satisfies

$$\alpha\mu_i^{\frac{2}{p_i-1}} \longrightarrow 0. \tag{1.10}$$

Furthermore, for $k \neq i$

$$\mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \longrightarrow 0. \tag{1.11}$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

2. **Main Results.** We explain notations before stating our results. Let $X := W_0^{1,2}(I)$ be the usual real Sobolev space. For $u \in X$, let

$$\|u\|_X^2 := \int_I u'(x)^2\,dx,\ \|u\|_d^d = \int_I |u(x)|^d\,dx \quad (d \geq 1),$$
$$(u,v)_2 := \int_I u(x)v(x)\,dx,$$
$$\|u\|_\infty := \max_{x\in I} |u(x)|.$$

$\lambda = \lambda(\mu,\alpha)$ is called the variational eigenvalue of (1.1) when the associated eigenfunction $u(\mu,\alpha,x) \in N_{\mu,\alpha}$ satisfies the following conditions (2.1)–(2.2):

$$\big(\mu,\alpha,\lambda(\mu,\alpha),u(\mu,\alpha,x)\big) \in E \times R_+ \times N_{\mu,\alpha} \text{ satisfies (1.1)}. \tag{2.1}$$

$$\frac{1}{q+1}\|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1} = \beta(\mu,\alpha) := \frac{1}{q+1}\inf_{u\in N_{\mu,\alpha}}\|u\|_{q+1}^{q+1}. \tag{2.2}$$

We note that $\lambda(\mu,\alpha)$ is obtained as a Lagrange multiplier and explicitly represented as follows:

$$\lambda(\mu,\alpha) = \frac{2\alpha - \sum_{k=1}^m \frac{p_k-1}{p_k+1}\mu_k\|u(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} + \sum_{k=m+1}^n \frac{p_k-1}{p_k+1}\mu_k\|u(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1}}{\|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1}}. \tag{2.3}$$

Actually, multiplying (1.1) by $u(\mu,\alpha,x)$, we obtain by integration by parts that

$$\begin{aligned} -\|u(\mu,\alpha,\cdot)\|_X^2 - \sum_{k=1}^m \mu_k\|u(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} + \sum_{k=m+1}^n \mu_k\|u(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} \\ = \lambda(\mu,\alpha)\|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1}; \end{aligned} \tag{2.4}$$

this along with the fact that $u(\mu,\alpha,\cdot) \in N_{\mu,\alpha}$ implies (2.3).

Now we state our main results.

THEOREM 2.1. *Assume that a sequence $\{(\mu,\alpha)\} \subset E$ satisfies (B.1). Then the following asymptotic formula holds:*

$$\lambda(\mu,\alpha) = C_3\Big(\alpha\mu_i^{\frac{q+3}{2(p_i-q)}}\Big)^{\frac{2(p_i-q)}{p_i+3}} + o\bigg(\Big(\alpha\mu_i^{\frac{q+3}{2(p_i-q)}}\Big)^{\frac{2(p_i-q)}{p_i+3}}\bigg), \tag{2.5}$$

where

$$C_3 = \left\{\frac{(p_i+3)(q+1)(p_i-q)}{2(2q+3-p_i)}\sqrt{\frac{2}{\pi(q+1)}}\Big(\frac{q+1}{p_i+1}\Big)^{\frac{q+3}{2(p_i-q)}}\frac{\Gamma\big(\frac{p_i+3}{2(p_i-q)}\big)}{\Gamma\big(\frac{q+3}{2(p_i-q)}\big)}\right\}^{\frac{2(p_i-q)}{p_i+3}}. \tag{2.6}$$

Next, we shall show that the condition (B.1) is optimal to obtain Theorem 2.1. More precisely, we consider the case where the condition (B.1) does not hold:

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

THEOREM 2.2. *Assume that $p_1 = q$. Furthermore, suppose that a sequence $\{(\mu, \alpha)\} \subset E$ satisfies the condition (B.2). Then the following asymptotic formula holds:*

$$\lambda(\mu, \alpha) = C_4 \Big(\alpha \mu_i^{\frac{q+3}{2(p_i-q)}} \Big)^{\frac{2(p_i-q)}{p_i+3}} + o\Big(\Big(\alpha \mu_i^{\frac{q+3}{2(p_i-q)}} \Big)^{\frac{2(p_i-q)}{p_i+3}} \Big), \tag{2.7}$$

where $C_4 > 0$ is a unique positive solution x of the following equation:

$$\begin{aligned} &\frac{2}{(p_i+3)(q+1)} \{(2q+3-p_i)x - 2(p_i-q)C_2\} = L_1 (x + C_2)^{-\frac{2q+3-p_i}{2(p_i-q)}}, \\ &L_1 = (p_i - q) \sqrt{\frac{2}{\pi(q+1)}} \Big(\frac{q+1}{p_i+1} \Big)^{\frac{q+3}{2(p_i-q)}} \frac{\Gamma\big(\frac{p_i+3}{2(p_i-q)}\big)}{\Gamma\big(\frac{q+3}{2(p_i-q)}\big)}. \end{aligned} \tag{2.8}$$

We see from (2.8) that if $C_2 = 0$, then $C_3 = C_4$. Therefore, the formulas (2.5) and (2.7) are connected continuously.

THEOREM 2.3. *Suppose that $\{(\mu, \alpha)\} \subset E$ satisfies (B.3). Then the following asymptotic formula holds:*

$$\lambda(\mu, \alpha) = C_5 \mu_i^{\frac{q-1}{p_i-1}} + o\Big(\mu_i^{\frac{q-1}{p_i-1}} \Big), \tag{2.9}$$

where $C_5 = (p_i - 1)\|v_0\|_{p_i+1}^{p_i+1} / \{(p_i+1)\|v_0\|_{q+1}^{q+1}\}$ and v_0 is the unique minimizer of the problem

$$\begin{aligned} &\text{minimize } \|v\|_{q+1}^{q+1} \text{ under the constraint} \\ &v \in V_0 := \Big\{ v \in X : \frac{1}{2}\|v\|_X^2 = \frac{1}{p_i+1}\|v\|_{p_i+1}^{p_i+1},\ v \not\equiv 0 \Big\}. \end{aligned} \tag{2.10}$$

The remainder of this paper is organized as follows. In Section 3, we shall show the existence of variational eigenvalues. We prepare some fundamental lemmas in Section 4. Section 5 and Section 6 are devoted to the Proof of Theorem 2.1 and Theorem 2.2, respectively. Finally, we shall show Theorem 2.3 in Section 7.

3. **Existence of Variational Eigenvalues.** In what follows, let C denote various positive constants independent of $\{(\mu, \alpha)\}$. Furthermore, for a subsequence, we use the same notation as that of original sequence for convenience. To obtain the existence of variational eigenvalue $\lambda(\mu, \alpha)$, we shall apply the result of Zeidler [7, Proposition 6a].

LEMMA 3.1. *Let $(\mu,\alpha)\in E$ be fixed. Then $N_{\mu,\alpha}\neq\emptyset$.*

PROOF. We put for $t\geq 0$

$$m(t):=B\big(\mu,t\cos(\pi x/2)\big)=\frac{1}{2}t^2\|\cos(\pi x/2)\|_X^2+\sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k t^{p_k+1}\|\cos(\pi x/2)\|_{p_k+1}^{p_k+1}-\sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k t^{p_k+1}\|\cos(\pi x/2)\|_{p_k+1}^{p_k+1}.$$

Then, $m(0)=0$ and $m(t)\to-\infty$ as $t\to\infty$. Hence, there exists $t_\alpha>0$ such that $m(t_0)=-\alpha$, that is, $t_0\cos(\pi x/2)\in N_{\mu,\alpha}$. ∎

LEMMA 3.2. *Let $(\mu,\alpha)\in E$ be fixed. Then*

$$\inf_{u\in N_{\mu,\alpha}}\Big|-\|u\|_X^2-\sum_{k=1}^{m}\mu_k\|u\|_{p_k+1}^{p_k+1}+\sum_{k=m+1}^{n}\mu_k\|u\|_{p_k+1}^{p_k+1}\Big|>0. \tag{3.1}$$

Furthermore, for all constants $C_6>0$, the set $M_{\mu,\alpha}:=\{u\in N_{\mu,\alpha}:\|u\|_{q+1}<C_6\}\subset X$ is bounded.

PROOF. For $u\in N_{\mu,\alpha}$

$$\sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}=-\frac{1}{2}\|u\|_X^2+\sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}-\alpha\leq\sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}. \tag{3.2}$$

Hence, by (1.2) and (3.2)

$$\begin{aligned}\sum_{k=1}^{m}\frac{p_k-1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}&\leq(p_{m+1}-1)\sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}\\&\leq(p_{m+1}-1)\sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}\\&\leq\sum_{k=m+1}^{n}\frac{p_k-1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}.\end{aligned} \tag{3.3}$$

Then by (3.3), we obtain that for $u\in N_{\mu,\alpha}$

$$\begin{aligned}-\|u\|_X^2-\sum_{k=1}^{m}\mu_k\|u\|_{p_k+1}^{p_k+1}+\sum_{k=m+1}^{n}\mu_k\|u\|_{p_k+1}^{p_k+1}&=2\alpha+\sum_{k=m+1}^{n}\frac{p_k-1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}-\sum_{k=1}^{m}\frac{p_k-1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1}\\&\geq 2\alpha>0.\end{aligned} \tag{3.4}$$

Therefore, we obtain (3.1). Next, since we know Gagliardo-Nirenberg inequality

$$\|u\|_{p_k+1}\leq C\|u\|_{q+1}^{1-\gamma_k}\|u\|_X^{\gamma_k}, \tag{3.5}$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

for $u \in X$, where $\gamma_k = 2(p_k - q)/\{(p_k+1)(q+3)\}(1 \le k \le n)$, we obtain for $u \in M_{\mu,\alpha}$

$$\begin{aligned}\frac{1}{2}\|u\|_X^2 &= -\sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1} + \sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1} - \alpha \\ &\le \sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k\|u\|_{p_k+1}^{p_k+1} \\ &\le C\sum_{k=m+1}^{n}\mu_k\|u\|_{q+1}^{(p_k+1)(1-\gamma_k)}\|u\|_X^{(p_k+1)\gamma_k} \\ &\le C\sum_{k=m+1}^{n}\mu_k\|u\|_X^{\frac{2(p_k-q)}{q+3}}.\end{aligned}$$

Since $2(p_k - q)/(q+3) < 2$, we obtain our conclusion. ∎

By Lemma 3.1 and Lemma 3.2, we now apply [7, Proposition 6a] to (1.1) and obtain the following lemma:

LEMMA 3.3. *For a fixed* $(\mu,\alpha) \in E$, *there exists* $u_{\mu,\alpha}(x) \in N_{\mu,\alpha}$ *which satisfies (2.2).*

By Lagrange multiplier theory, there exists $\lambda(\mu,\alpha) \in R$ such that $\big(\mu, \lambda(\mu,\alpha), u_{\mu,\alpha}(x)\big)$ satisfies the equation in (1.1). Furthermore, by (2.3) and (3.3), we obtain

$$\lambda(\mu,\alpha) \ge \frac{2\alpha}{\|u_{\mu,\alpha}\|_{q+1}^{q+1}} > 0. \tag{3.6}$$

The existence of positive solution is obtained as follows.

LEMMA 3.4. *There exists* $\big(\mu,\alpha,\lambda(\mu,\alpha),u(\mu,\alpha,x)\big) \in E \times R_+ \times N_{\mu,\alpha}$ *which satisfies (2.1)–(2.2) for a fixed* $(\mu,\alpha) \in E$.

PROOF. Let $u(\mu,\alpha,x) = |u_{\mu,\alpha}(x)|$, where $u_{\mu,\alpha} \in N_{\mu,\alpha}$ is a function obtained in Lemma 3.3. Clearly,

$$\begin{aligned}\|u(\mu,\alpha,\cdot)\|_X &= \|u_{\mu,\alpha}\|_X, \\ \|u(\mu,\alpha,\cdot)\|_{q+1} &= \|u_{\mu,\alpha}\|_{q+1}, \\ \|u(\mu,\alpha,\cdot)\|_{p_k+1} &= \|u_{\mu,\alpha}\|_{p_k+1}.\end{aligned}$$

Hence, $u(\mu,\alpha,x) \in N_{\mu,\alpha}$. Moreover, by (2.3), we find that $\big(\mu,\alpha,\lambda(\mu,\alpha),u(\mu,\alpha,x)\big) \in E \times R_+ \times N_{\mu,\alpha}$ satisfies the equation in (1.1) and (2.2) for the same Lagrange multiplier $\lambda(\mu,\alpha)$ as that of $u_{\mu,\alpha}$. If there exists $x_0 \in I$ such that $u(\mu,\alpha,x_0) = 0$, then $u'(\mu,\alpha,x_0) = 0$, since $u(\mu,\alpha,x) \ge 0$ for $x \in I$. Then by the uniqueness theorem of ODE, we obtain that $u(\mu,\alpha,x) \equiv 0$ in I. However, this is impossible, since $u(\mu,\alpha,x) \in N_{\mu,\alpha}$ and $0 \notin N_{\mu,\alpha}$. Thus, $u(\mu,\alpha,x) > 0$ for $x \in I$. ∎

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

4. **Preliminaries.** In what follows, for the usual L^d-norm of $g \in L^d(J)$ ($d \geq 1$, $J \subset R$: open set), we write $\|g\|_d$ for simplicity. We put $\sigma_{\mu,\alpha} := \max_{x\in I} u(\mu,\alpha,x)$. By Gidas, Ni and Nirenberg [3], we know that $u(\mu,\alpha,x)$ satisfies the following properties:

$$u(\mu,\alpha,-x) = u(\mu,\alpha,x), \quad u'(\mu,\alpha,x) \leq 0, \quad x \in (0,1), \tag{4.1}$$

$$u'(\mu,\alpha,0) = 0, \quad \sigma_{\mu,\alpha} = u(\mu,\alpha,0). \tag{4.2}$$

LEMMA 4.1. *For a fixed $(\mu,\alpha) \in E$, the following equality holds for $x \in \overline{I}$:*

$$\begin{aligned}
&\frac{1}{2}u'(\mu,\alpha,x)^2 - \sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k u(\mu,\alpha,x)^{p_k+1} + \sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k u(\mu,\alpha,x)^{p_k+1}\\
&\qquad - \frac{1}{q+1}\lambda(\mu,\alpha)u(\mu,\alpha,x)^{q+1}\\
&\quad = -\sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k\sigma_{\mu,\alpha}^{p_k+1}\\
&\qquad + \sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k\sigma_{\mu,\alpha}^{p_k+1} - \frac{1}{q+1}\lambda(\mu,\alpha)\sigma_{\mu,\alpha}^{q+1}\\
&\quad = \frac{1}{2}u'(\mu,\alpha,1)^2 > 0.
\end{aligned} \tag{4.3}$$

PROOF. Multiplying (1.1) by $u'(\mu,\alpha,x)$, we obtain for $x \in \overline{I}$

$$\begin{aligned}
&\Big\{u''(\mu,\alpha,x) - \sum_{k=1}^{m}\mu_k u(\mu,\alpha,x)^{p_k} + \sum_{k=m+1}^{n}\mu_k u(\mu,\alpha,x)^{p_k} - \lambda(\mu,\alpha)u(\mu,\alpha,x)^q\Big\}\\
&\qquad u'(\mu,\alpha,x) = 0;
\end{aligned}$$

namely,

$$\begin{aligned}
&\frac{d}{dx}\Big\{\frac{1}{2}u'(\mu,\alpha,x)^2 - \sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k u(\mu,\alpha,x)^{p_k+1} + \sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k u(\mu,\alpha,x)^{p_k+1}\\
&\qquad - \frac{1}{q+1}\lambda(\mu,\alpha)u(\mu,\alpha,x)^{q+1}\Big\} \equiv 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{1}{2}u'(\mu,\alpha,x)^2 - \sum_{k=1}^{m}\frac{1}{p_k+1}\mu_k u(\mu,\alpha,x)^{p_k+1} + \sum_{k=m+1}^{n}\frac{1}{p_k+1}\mu_k u(\mu,\alpha,x)^{p_k+1}\\
&\qquad - \frac{1}{q+1}\lambda(\mu,\alpha)u(\mu,\alpha,x)^{q+1} \equiv \text{constant}.
\end{aligned} \tag{4.4}$$

Now put $x = 0, 1$ in (4.4). Then (4.3) follows immediately from (4.2). ∎

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

LEMMA 4.2. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.1). Then*

$$\lambda(\mu,\alpha) \geq C\alpha^{\frac{2(p_i-q)}{p_i+3}} \mu_i^{\frac{q+3}{p_i+3}}. \tag{4.5}$$

PROOF. Let $\eta = \eta_{\mu,\alpha} := \left(\alpha\mu_i^{\frac{2}{p_i-1}}\right)^{\frac{p_i-1}{p_i+3}}$. Furthermore, let w_η satisfy

$$\begin{aligned} w_\eta''(s) + w_\eta(s)^{p_i} - w_\eta(s)^q &= 0, \quad -\eta_{\mu,\alpha} < s < \eta_{\mu,\alpha}, \\ w_\eta(s) &> 0, \quad -\eta_{\mu,\alpha} < s < \eta_{\mu,\alpha}, \\ w_\eta(\pm\eta_{\mu,\alpha}) &= 0. \end{aligned}$$

The existence of w_η is obtained easily, for instance, by direct variational method. We put

$$U_{\mu,\alpha}(x) := d_{\mu,\alpha}(\alpha^2\mu_i^{-1})^{\frac{1}{p_i+3}} w_\eta(s), \quad x = \eta_{\mu,\alpha}^{-1}s,$$

where

$$d_{\mu,\alpha} := \inf\{t > 0 : t(\alpha^2\mu_i^{-1})^{\frac{1}{p_i+3}} w_\eta(\eta_{\mu,\alpha}x) \in N_{\mu,\alpha}\}.$$

For a fixed (μ,α), we know that

$$e(t) := B\left(\mu, t(\alpha^2\mu_i^{-1})^{\frac{1}{p_i+3}} w_\eta(\eta_{\mu,\alpha}x)\right) \longrightarrow -\infty \text{ as } t \longrightarrow \infty, \quad e(0) = 0.$$

Hence, $d_{\mu,\alpha} > 0$ exists. We shall show that

$$C^{-1} \leq d_{\mu,\alpha} \leq C. \tag{4.6}$$

Since $\eta_{\mu,\alpha} \longrightarrow \infty$ by (1.6), we know from Shibata [5, Lemma 4.7, Proof of Theorem 2.2] that $w_\eta \longrightarrow w_\infty$ uniformly on any compact subset in R, and furthermore, $w_\eta \longrightarrow w_\infty$ in $L^p(R)$ $(p \geq 1)$. Here, w_∞ is the ground state solution of

$$\begin{aligned} w''(t) + w(t)^{p_i} - w(t)^q &= 0, \quad t \in R, \\ w(t) &> 0, \quad t \in R, \\ \lim_{|t|\to\infty} w(t) &= 0. \end{aligned} \tag{4.7}$$

We have by (1.7)

$$\begin{aligned} \eta_{\mu,\alpha}(\alpha^2\mu_i^{-1})^{\frac{2}{p_i+3}} &= \alpha, \\ \mu_k\eta_{\mu,\alpha}^{-1}(\alpha^2\mu_i^{-1})^{\frac{p_k+1}{p_i+3}} &= \left(\mu_k\alpha^{\frac{2(p_k-p_i)}{p_i+3}}\mu_i^{-\frac{p_k+3}{p_i+3}}\right)\alpha = o(1)\alpha \quad (k \neq i). \end{aligned} \tag{4.8}$$

Then we obtain by (4.8) that

$$
\begin{aligned}
-\alpha &= e(d_{\mu,\alpha}) \\
&= d_{\mu,\alpha}^2 \alpha \Big\{ \frac{1}{2} \|w_\eta\|_X^2 - \frac{1}{p_i+1} d_{\mu,\alpha}^{p_i-1} \|w_\eta\|_{p_i+1}^{p_i+1} \\
&\qquad - o(1) \sum_{k=m+1,\, k\neq i}^{n} d_{\mu,\alpha}^{p_k-1} \frac{1}{p_k+1} \|w_\eta\|_{p_k+1}^{p_k+1} \\
&\qquad + o(1) \sum_{k=1}^{m} d_{\mu,\alpha}^{p_k-1} \frac{1}{p_k+1} \|w_\eta\|_{p_k+1}^{p_k+1} \Big\}.
\end{aligned} \tag{4.9}
$$

Hence, $d_{\mu,\alpha} \longrightarrow 0$ and $d_{\mu,\alpha} \longrightarrow \infty$ are impossible. Hence, we obtain (4.6). Then by (3.6)

$$
\begin{aligned}
\frac{2\alpha}{\lambda(\mu,\alpha)} &\leq \frac{1}{q+1} \|u_{\mu,\alpha}\|_{q+1}^{q+1} \\
&\leq \frac{1}{q+1} \|U_{\mu,\alpha}\|_{q+1}^{q+1} \\
&\leq C(\alpha^2 \mu_i^{-1})^{\frac{q+1}{p_i+3}} \eta_{\mu,\alpha}^{-1} \|w_\eta\|_{q+1}^{q+1} \\
&\leq C\alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}}.
\end{aligned} \tag{4.10}
$$

Then (4.5) follows from (4.10). ∎

LEMMA 4.3. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.1). Then $\sigma_{\mu,\alpha} \longrightarrow 0$.*

PROOF. We have by (4.3)

$$
\begin{aligned}
\frac{1}{2} u'(\mu,\alpha,x)^2 &= \sum_{k=m+1}^{n} \frac{1}{p_k+1} \mu_k \big(\sigma_{\mu,\alpha}^{p_k+1} - u(\mu,\alpha,x)^{p_k+1} \big) \\
&\qquad - \sum_{k=1}^{m} \frac{1}{p_k+1} \mu_k \big(\sigma_{\mu,\alpha}^{p_k+1} - u(\mu,\alpha,x)^{p_k+1} \big) \\
&\qquad - \frac{1}{q+1} \lambda(\mu,\alpha) \big(\sigma_{\mu,\alpha}^{q+1} - u(\mu,\alpha,x)^{q+1} \big) \\
&\leq \sum_{k=m+1}^{n} \frac{1}{p_k+1} \mu_k \sigma_{\mu,\alpha}^{p_k+1}.
\end{aligned} \tag{4.11}
$$

Let $x_1 = x_{1,\mu,\alpha} \in [0,1]$ satisfy $u(\mu,\alpha,x_1) = (1-\epsilon)\sigma_{\mu,\alpha}$, where $0 < \epsilon \ll 1$ is a fixed constant. By mean value theorem and (4.11) we have for $y_{1,\mu,\alpha} \in [0, x_{1,\mu,\alpha}]$

$$
\frac{\epsilon \sigma_{\mu,\alpha}}{x_1} = \left| \frac{u_{\mu,\alpha}(0) - u_{\mu,\alpha}(x_1)}{x_1} \right| = |u'_{\mu,\alpha}(y_{1,\mu,\alpha})| \leq \sqrt{C \sum_{k=m+1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k+1}}. \tag{4.12}
$$

By (4.10) and (4.12)

$$
\begin{aligned}
C\epsilon(1-\epsilon)^{q+1} \sigma_{\mu,\alpha}^{q+2} \Big(\sum_{k=m+1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k+1} \Big)^{-1/2} &\leq C(1-\epsilon)^{q+1} \sigma_{\mu,\alpha}^{q+1} x_1 \\
&\leq C \int_0^{x_1} u_{\mu,\alpha}(x)^{q+1}\, dx \\
&\leq C \|u_{\mu,\alpha}\|_{q+1}^{q+1} \\
&\leq C\alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}};
\end{aligned} \tag{4.13}
$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

this implies that

$$\sigma_{\mu,\alpha}^{q+2} \leq \sum_{k=m+1}^{n} \left(\mu_k \alpha^{\frac{2(p_k-p_i)}{p_i+3}} \mu_i^{-\frac{p_k+3}{p_i+3}}\right)^{1/2} \left(\alpha \mu_i^{-1/2}\right)^{\frac{2q+3-p_k}{p_i+3}} \sigma_{\mu,\alpha}^{\frac{p_k+1}{2}}. \tag{4.14}$$

Our conclusion follows from (1.6), (1.7) and (4.14), since $q+2 > (p_k+1)/2$ by (1.2). ■

LEMMA 4.4. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.1). Then for* $m+1 \leq k \leq n$

$$\mu_k \sigma_{\mu,\alpha}^{p_k+1} \leq C\mu_i \sigma_{\mu,\alpha}^{p_i+1}. \tag{4.15}$$

PROOF. For a fixed (μ,α), there exists $m+1 \leq j(\mu,\alpha) \leq n$ which satisfies

$$\max_{m+1\leq k\leq n} \mu_k \sigma_{\mu,\alpha}^{p_k+1} = \mu_{j(\mu,\alpha)} \sigma_{\mu,\alpha}^{p_{j(\mu,\alpha)}+1}. \tag{4.16}$$

Then there exists a subsequence of $\{(\mu,\alpha)\}$ and $m+1 \leq j \leq n$ such that $j = j(\mu,\alpha)$ for this subsequence. We consider this subsequence. Then by (4.14), we obtain

$$\sigma_{\mu,\alpha}^{q+2} \leq C\sqrt{\sum_{k=m+1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k+1}} \alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}} \leq C\mu_j^{1/2} \sigma_{\mu,\alpha}^{\frac{p_j+1}{2}} \alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}},$$

that is,

$$\sigma_{\mu,\alpha} \leq C\left(\mu_j^{1/2} \alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}}\right)^{\frac{2}{2q+3-p_j}}. \tag{4.17}$$

By Lemma 4.2, (4.2) and (4.17)

$$\begin{aligned}
\alpha^{\frac{2(p_i-q)}{p_i+3}} \mu_i^{\frac{q+3}{p_i+3}} &\leq C\lambda(\mu,\alpha) \\
&\leq C \sum_{k=m+1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k-q} \\
&\leq C\mu_j \sigma_{\mu,\alpha}^{p_j-q} \\
&\leq C\mu_j \left(\mu_j^{1/2} \alpha^{\frac{2q+3-p_i}{p_i+3}} \mu_i^{-\frac{q+3}{p_i+3}}\right)^{\frac{2(p_j-q)}{2q+3-p_j}};
\end{aligned}$$

namely,

$$\mu_j \geq C\alpha^{\frac{2(p_i-p_j)}{p_i+3}} \mu_i^{\frac{p_j+3}{p_i+3}}.$$

This along with (1.7) implies that (4.16) holds for $j = j(\mu,\alpha) = i$ except finite elements of $\{(\mu,\alpha)\}$. Thus, we obtain (4.15). ■

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

5. **Proof of Theorem 2.1.**

Let

$$\xi := \xi_{\mu,\alpha} = \big(\lambda(\mu,\alpha)/\mu_i\big)^{p_i-q}, \quad \nu := \nu_{\mu,\alpha} = \mu_i^{\frac{1-q}{2(p_i-q)}} \lambda(\mu,\alpha)^{\frac{p_i-1}{2(p_i-q)}},$$
$$t := \nu x, \quad w_{\mu,\alpha}(t) := \xi^{-1} u(\mu,\alpha,x).$$

Then by (1.1), $w_{\mu,\alpha}(t)$ satisfies the following equation:

$$\begin{aligned} &w''_{\mu,\alpha}(t) + w_{\mu,\alpha}(t)^{p_i} - w_{\mu,\alpha}(t)^q + \sum_{k=m+1,\, k\neq i}^{n} \nu^{-2}\mu_k \xi^{p_k-1} w_{\mu,\alpha}(t)^{p_k} \\ &\quad - \sum_{k=1}^{m} \nu^{-2}\mu_k \xi^{p_k-1} w_{\mu,\alpha}(t)^{p_k} = 0, \quad t \in I_{\mu,\alpha} := (-\nu,\nu), \\ &w_{\mu,\alpha}(t) > 0, \quad t \in I_{\mu,\alpha}, \\ &w_{\mu,\alpha}(\pm\nu) = 0. \end{aligned} \tag{5.1}$$

It follows from (1.6) and Lemma 4.2 that

$$\nu_{\mu,\alpha} \geq C\mu_i^{\frac{1-q}{2(p_i-q)}} \Big(\alpha^{\frac{2(p_i-q)}{p_i+3}} \mu_i^{\frac{q+3}{p_i+3}}\Big)^{\frac{p_i-1}{2(p_i-q)}} = C\Big(\alpha\mu_i^{\frac{2}{p_i-1}}\Big)^{\frac{p_i-1}{p_i+3}} \longrightarrow \infty. \tag{5.2}$$

Hence, we expect that $w_{\mu,\alpha}(t) \longrightarrow w_\infty(t)$ if $\{(\mu,\alpha)\} \subset E$ satisfies (B.1), where w_∞ is the ground state of (4.7). We recall here some important properties of w_∞. We know from Berestycki and Lions [1] that there uniquely exists a solution w_∞ of (4.7), which is called the *ground state solution* of (4.7), and satisfies the following properties:

$$\begin{aligned} &w_\infty(0) = \zeta := \Big(\frac{p_i+1}{q+1}\Big)^{\frac{1}{p_i-q}}, \\ &w_\infty(t) = w_\infty(-t), \quad t \in R, \\ &w'_\infty(t) \leq 0, \quad t \geq 0, \end{aligned} \tag{5.3}$$

$$\frac{1}{2} w'_\infty(t)^2 + \frac{1}{p_i+1} w_\infty(t)^{p_i+1} - \frac{1}{q+1} w_\infty(t)^{q+1} = 0, \quad t \in R, \tag{5.4}$$

$$w_\infty(t) \leq Ce^{-C|t|}, \quad t \in R. \tag{5.5}$$

We shall show that $w_{\mu,\alpha}(t) \longrightarrow w_\infty(t)$ in $L^{q+1}(R)$. To do this, we need some preparations. Let $\zeta_{\mu,\alpha} := \max_{t\in I_{\mu,\alpha}} w_{\mu,\alpha}(t) = \xi^{-1}\sigma_{\mu,\alpha}$.

LEMMA 5.1. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.1). Then*

$$C^{-1} \leq \zeta_{\mu,\alpha} \leq C.$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

PROOF. *By (4.3) and (4.15), we have*

$$\lambda(\mu,\alpha) \le C \sum_{k=m+1}^{n} \mu_k \sigma_{\mu,\alpha}^{p_k-q} \le C\mu_i \sigma_{\mu,\alpha}^{p_i-q}. \tag{5.6}$$

This implies the first inequality. Next, since (4.17) holds for $j = i$, we see from Lemma 4.2 that

$$\sigma_{\mu,\alpha}^{p_i-q} \le C\Big(\alpha\mu_i^{-1/2}\Big)^{\frac{2(p_i-q)}{p_i+3}} \le C\frac{\lambda(\mu,\alpha)}{\mu_i}. \tag{5.7}$$

Thus we obtain the second inequality. ∎

It follows from Lemma 4.1 that

$$\frac{1}{2}w_{\mu,\alpha}'(t)^2 + R\big(\mu,\alpha,t,w_{\mu,\alpha}(t)\big) = R(\mu,\alpha,0,\zeta_{\mu,\alpha}), \tag{5.8}$$

where

$$\begin{aligned} R(\mu,\alpha,t,w) := {}& \frac{1}{p_i+1}w^{p_i+1} - \frac{1}{q+1}w^{q+1} \\ &+ \frac{1}{p_k+1}\sum_{k=m+1,\,k\neq i}^{n} \nu^{-2}\mu_k\xi^{p_k-1}w^{p_k+1} \\ &- \frac{1}{p_k+1}\sum_{k=1}^{m}\nu^{-2}\mu_k\xi^{p_k-1}w^{p_k+1}. \end{aligned}$$

LEMMA 5.2. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.1). Then*

$$\lambda(\mu,\alpha)\nu^{-2}\xi^{q-1} = 1, \tag{5.9}$$

$$\nu^{-2}\mu_k\xi^{p_k-1} \longrightarrow 0\,(k \neq i). \tag{5.10}$$

PROOF. (5.9) follows from the definition of ν and ξ. We shall show (5.10). It follows from (5.6) and (5.7) that

$$\lambda(\mu,\alpha) \le C\Big(\alpha\mu_i^{\frac{q+3}{2(p_i-q)}}\Big)^{\frac{2(p_i-q)}{p_i+3}}. \tag{5.11}$$

Then by (1.7), Lemma 4.2 and (5.11), we obtain that for $k \neq i$

$$\nu^{-2}\mu_k\xi^{p_k-1} = \lambda(\mu,\alpha)^{\frac{p_k-p_i}{p_i-q}}\mu_k\mu_i^{-\frac{p_k-q}{p_i-q}} \le C\mu_k\alpha^{\frac{2(p_k-p_i)}{p_i+3}}\mu_i^{-\frac{p_k+3}{p_i+3}} \longrightarrow 0. \tag{5.12}$$

Thus the proof is complete. ∎

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

LEMMA 5.3. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.1). Then* $\|w_{\mu,\alpha}\|_{q+1} \le C$.

PROOF. By (4.10) and Lemma 4.2

$$\begin{aligned}\|w_{\mu,\alpha}\|_{q+1}^{q+1} &= \xi^{-(q+1)}\nu\|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1}\\ &\le C\Big(\lambda(\mu,\alpha)^{-1}\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}\Big)^{\frac{2q+3-p_i}{2(p_i-q)}}\\ &\le C.\end{aligned}$$

∎

LEMMA 5.4. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.1). Then* $w_{\mu,\alpha}(t) \longrightarrow w_\infty(t)$ *uniformly on any compact subsets on* R.

By using Lemma 5.1–Lemma 5.3 and a standard limiting argument, we can easily prove this lemma. Hence, we omit the proof.

LEMMA 5.5. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.1). Then there exists* $y(t) \in L^{q+1}(R)$ *such that* $w_{\mu,\alpha}(t) \le y(t) \le C$ *for* $t \in R$.

PROOF. Let $q < r < 2q+3$ be fixed. Then $y_1(t) := (t+1)^{-2/(r-1)}$ satisfies

$$\begin{aligned} y_1'(t) &= -\sqrt{Y_0\big(t, y_1(t)\big)}, \quad t>0,\\ y_1(0) &= 1,\end{aligned} \tag{5.13}$$

where $Y_0(t,y) := 4(r-1)^{-2}y^{r+1}$. Since $r < 2q+3 < 2p_k+3$, we see that $y_1(t) \in L^{q+1}(R_+) \cap L^{p_k+1}(R_+)$. We put $y(\mu,\alpha,t) := \zeta_{\mu,\alpha}^{-1}w(\mu,\alpha,t)$. Then it follows from (5.8) that $y(\mu,\alpha,t)$ satisfies

$$\begin{aligned} y'(\mu,\alpha,t) &= -\sqrt{Y_1\big(t, y(\mu,\alpha,t)\big)}, \quad 0<t<\nu,\\ y(\mu,\alpha,0) &= 1,\end{aligned} \tag{5.14}$$

where

$$Y_1(t,y) = 2\zeta_{\mu,\alpha}^{-2}\Big\{R(\mu,\alpha,0,\zeta_{\mu,\alpha}) - R\big(\mu,\alpha,t,\zeta_{\mu,\alpha}y(\mu,\alpha,t)\big)\Big\}. \tag{5.15}$$

Fix $0 < \epsilon \ll 1$. We shall show that for $0 \le y \le \epsilon$.

$$Y_1(t,y) - Y_0(t,y) > 0.$$

By Lemma 5.1 and Lemma 5.2

$$\nu^{-2}\mu_k\xi^{p_k-1}\zeta_{\mu,\alpha}^{p_k+1}y^{p_k+1} = o(1)y^{p_k+1} \quad (k \ne i). \tag{5.16}$$

We note that $R(\mu,\alpha,0,\zeta_{\mu,\alpha}) > 0$ by (4.3). Then we obtain by (5.16) that for $0 \le y \le \epsilon$

$$\begin{aligned} Y_1(t,y) - Y_0(t,y) &\ge 2\zeta_{\mu,\alpha}^{-2}R(\mu,\alpha,0,\zeta_{\mu,\alpha})\\ &\quad + \frac{2}{q+1}\zeta_{\mu,\alpha}^{q-1}y^{q+1} - \frac{2}{p_1+1}\zeta_{\mu,\alpha}^{p_i-1}y^{p_i+1}\\ &\quad - o(1)\sum_{k=1,\,k\ne i}^{n} y^{p_k+1} - \frac{4}{(r-1)^2}y^{r+1} > 0.\end{aligned} \tag{5.17}$$

Let $t_0 \gg 1$ satisfy $y_1(t_0) < \epsilon$. Then by (5.5) and Lemma 5.4, we find that $y(\mu,\alpha,t_0) < y_1(t_0)$. Now, by (5.17), we apply the comparison theorem of ODE. Then we have $y(\mu,\alpha,t) \le y_1(t)$ for $t > t_0$. This together with Lemma 5.4 implies that $w_{\mu,\alpha}(t) = \zeta_{\mu,\alpha} y(\mu,\alpha,t) \le 2\zeta y_1(t)$ for $t > t_0$. Now, put

$$y(t) = \begin{cases} C & |t| \le t_0, \\ 2\zeta y_1(|t|), & |t| > t_0, \end{cases} \tag{5.18}$$

where $C \gg 1$ is a constant. Then $y(t)$ is the desired function. ∎

The following lemma is a consequence of Lemma 5.4, 5.5 and Lebesgue's convergence theorem:

LEMMA 5.6. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.1). Then $w_{\mu,\alpha}(t) \longrightarrow w_\infty(t)$ in $L^{q+1}(R)$ and $L^{p_k+1}(R)$.*

LEMMA 5.7 ([4, LEMMA 4.6]). *Let $w_\infty(t)$ be the ground state of (4.7). Then*

$$\int_R w_\infty(t)^{q+1}\,dt = \frac{2}{p_i - q}\sqrt{\frac{\pi(q+1)}{2}}\,\zeta^{\frac{q+3}{2}}\frac{\Gamma\left(\frac{q+3}{2(p_i-q)}\right)}{\Gamma\left(\frac{p_i+3}{2(p_i-q)}\right)}. \tag{5.19}$$

Now we shall prove Theorem 2.1.

PROOF OF THEOREM 2.1. Multiply (4.7) by $w_\infty(t)$ and integrate it over R to obtain

$$\|w'_\infty\|_2^2 = \|w_\infty\|_{p_i+1}^{p_i+1} - \|w_\infty\|_{q+1}^{q+1}. \tag{5.20}$$

Integrate (5.4) over R to obtain

$$\frac{1}{2}\|w'_\infty\|_2^2 + \frac{1}{p_i+1}\|w_\infty\|_{p_i+1}^{p_i+1} - \frac{1}{q+1}\|w_\infty\|_{q+1}^{q+1} = 0;$$

this along with (5.20) implies that

$$\|w_\infty\|_{p_i+1}^{p_i+1} = \frac{(p_i+1)(q+3)}{(p_i+3)(q+1)}\|w_\infty\|_{q+1}^{q+1}. \tag{5.21}$$

By Lemma 5.6, we obtain

$$\|w_{\mu,\alpha}\|_{q+1}^{q+1} \longrightarrow \|w_\infty\|_{q+1}^{q+1},\ \|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1} \longrightarrow \|w_\infty\|_{p_k+1}^{p_k+1}. \tag{5.22}$$

Moreover, by (1.7) and (5.12)

$$\begin{aligned} \mu_i\|u(\mu,\alpha,\cdot)\|_{p_i+1}^{p_i+1} &= \lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1}, \\ \|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1} &= \lambda(\mu,\alpha)^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{q+1}^{q+1}, \\ \mu_k\|u(\mu,\alpha,\cdot)\|_{p_k+1}^{p_k+1} &= \mu_k\lambda(\mu,\alpha)^{\frac{p_k-p_i}{p_i-q}}\mu_i^{-\frac{p_k-q}{p_i-q}}\left\{\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\right\} \\ &= o(1)\left\{\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\right\} \quad (k \ne i). \end{aligned} \tag{5.23}$$

It follows from (2.3) and (5.23) that

$$\lambda(\mu,\alpha)=\frac{2\alpha+\sum_{k=m+1}^{n}\mu_k\frac{p_k-1}{p_k+1}\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1}-\sum_{k=1}^{m}\mu_k\frac{p_k-1}{p_k+1}\|u_{\mu,\alpha}\|_{p_k+1}^{p_k+1}}{\|u_{\mu,\alpha}\|_{q+1}^{q+1}}$$
$$=\frac{2\alpha+\frac{p_i-1}{p_i+1}\lambda(\mu,\alpha)^{\frac{p_i+3}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\{\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1}+o(1)\}}{\lambda(\mu,\alpha)^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|w_{\mu,\alpha}\|_{q+1}^{q+1}};$$

this along with (5.22) implies that

$$\frac{\lambda(\mu,\alpha)}{\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}}=\Big(\frac{2}{H_{\mu,\alpha}}\Big)^{\frac{2(p_i-q)}{p_i+3}}\longrightarrow\Big(\frac{2}{H}\Big)^{\frac{2(p_i-q)}{p_i+3}}, \tag{5.24}$$

where

$$\begin{aligned}H_{\mu,\alpha}&=\big(1+o(1)\big)\|w_{\mu,\alpha}\|_{q+1}^{q+1}-\frac{p_i-1}{p_i+1}\Big(\big(1+o(1)\big)\|w_{\mu,\alpha}\|_{p_i+1}^{p_i+1}+o(1)\Big),\\ H&=\|w_\infty\|_{q+1}^{q+1}-\frac{p_i-1}{p_i+1}\|w_\infty\|_{p_i+1}^{p_i+1}=\frac{2(2q+3-p_i)}{(p_i+3)(q+1)}\|w_\infty\|_{q+1}^{q+1}.\end{aligned} \tag{5.25}$$

Now, Theorem 2.1 is a direct consequence of (5.24), (5.25) and Lemma 5.7. ∎

6. **Proof of Theorem 2.2.** We first note that by Lemma 4.2 and (5.11) still holds under the condition (B.2), namely,

$$C^{-1}\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}\le\lambda(\mu,\alpha)\le C\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}. \tag{6.1}$$

In fact, to prove Lemma 4.3, we used the condition (B.1) for (4.8) to derive (4.9). If we assume (B.2), then instead of (4.9), we obtain

$$\begin{aligned}-\alpha&=e(d_{\mu,\alpha})\\&=d_{\mu,\alpha}^2\alpha\Big\{\frac12\|w_\eta\|_X^2-\frac{1}{p_i+1}d_{\mu,\alpha}^{p_i-1}\|w_\eta\|_{p_i+1}^{p_i+1}+\big(C_2+o(1)\big)\frac{1}{q+1}d_{\mu,\alpha}^{q-1}\|w_\eta\|_{q+1}^{q+1}\\&\qquad-o(1)\sum_{k=m+1,\,k\neq i}^{n}d_{\mu,\alpha}^{p_k-1}\frac{1}{p_k+1}\|w_\eta\|_{p_k+1}^{p_k+1}+o(1)\sum_{k=2}^{m}d_{\mu,\alpha}^{p_k-1}\frac{1}{p_k+1}\|w_\eta\|_{p_k+1}^{p_k+1}\Big\}.\end{aligned}$$

Then it is easy to see that neither the case where $d_{\mu,\alpha}\longrightarrow 0$ nor the case where $d_{\mu,\alpha}\longrightarrow\infty$ occur. Hence, Lemma 4.3 holds under the condition (B.2). Moreover, to obtain (5.11), Lemma 4.3 and Lemma 4.4 were used, and we find that these lemmas were proved without using the condition for μ_1. Hence, (5.11) also holds under the condition (B.2). Then we see that

$$\begin{aligned}D:=\Big\{&a>0:\ \text{there exists a subsequence of }\{(\mu,\alpha)\}\text{ such that}\\&\lambda(\mu,\alpha)/\Big\{\alpha^{\frac{2(p_i-q)}{p_i+3}}\mu_i^{\frac{q+3}{p_i+3}}\Big\}\longrightarrow a\Big\}\neq\emptyset.\end{aligned} \tag{6.2}$$

We shall show that $D = \{C_4\}$. Let $a_1 \in D$. We put

$$\eta = \eta_\mu := (\mu_1/\mu_i)^{\frac{1}{p_i-q}}, \quad \tau = \tau_\mu := \mu_1^{1/2}\eta^{\frac{q-1}{2}}, \quad s := \tau x, \quad z_{\mu,\alpha}(s) := \eta^{-1}u(\mu,\alpha,x).$$

Then by (1.1), we see that $z_{\mu,\alpha}$ satisfies

$$\text{(6.3)} \quad \begin{aligned} & z''_{\mu,\alpha}(s) + z^{p_i}_{\mu,\alpha} - z^q_{\mu,\alpha} - \sum_{k=2}^{m} \mu_k\eta^{p_k-1}\tau^{-2}z_{\mu,\alpha}(s)^{p_k} + \sum_{k=m+1,\,k\neq i}^{n} \mu_k\eta^{p_k-1}\tau^{-2}z_{\mu,\alpha}(s)^{p_k} \\ &\qquad = \lambda(\mu,\alpha)\eta^{q-1}\tau^{-2}z_{\mu,\alpha}(s)^q, \quad -\tau < s < \tau, \\ &\qquad z_{\mu,\alpha}(s) > 0, \quad -\tau < s < \tau, \quad z_{\mu,\alpha}(\pm\tau) = 0. \end{aligned}$$

We obtain by (1.9) and (6.2) that for $b_1 := a_1/C_2$

$$\text{(6.4)} \quad \lambda(\mu,\alpha) = \big(b_1 + o(1)\big)\mu_1.$$

LEMMA 6.1. *Assume that a sequence $\{(\mu,\alpha)\} \subset E$ satisfies (B.2). Then for $2 \le k \le n\,(k \neq i)$*

$$\text{(6.5)} \quad \begin{aligned} \lambda(\mu,\alpha)\eta^{q-1}\tau^{-2} &= b_1 + o(1) \longrightarrow b_1, \\ \mu_k\eta^{p_k-1}\tau^{-2} &\longrightarrow 0. \end{aligned}$$

PROOF. The first assertion is a direct consequence of (6.4). Further, we obtain by (1.7) and (1.9) that for $2 \le k \le n\,(k \neq i)$

$$\text{(6.6)} \quad \mu_k\eta^{p_k-1}\tau^{-2} = \mu_k\mu_i^{-\frac{p_k-q}{p_i-q}}\mu_1^{\frac{p_k-p_i}{p_i-q}} \le C\mu_k\alpha^{\frac{2(p_k-p_i)}{p_i+3}}\mu_i^{-\frac{p_k+3}{p_i+3}} \longrightarrow 0.$$

Thus the proof is complete. ∎

By repeating the same arguments as those used in Lemma 5.4–Lemma 5.6, we see from (6.3) and (6.4) that if $\{(\mu,\alpha)\} \subset E$ satisfies (B.2), then $z_{\mu,\alpha}(s) \longrightarrow z(s)$ uniformly on any compact subsets in R, and furthermore, $z_{\mu,\alpha}(s) \longrightarrow z(s)$ in $L^{p_k+1}(R)$, $L^q(R)$, where $z(s)$ is the ground state solution of the following equation

$$\text{(6.7)} \quad \begin{aligned} z''(s) + z(s)^{p_i} - b_2 z(s)^q = 0, &\quad s \in R, \\ z(s) > 0, &\quad s \in R, \\ \lim_{|s|\to\infty} z(s) = 0, & \end{aligned}$$

where $b_2 := b_1 + 1$. By definition, we have

$$\text{(6.8)} \quad \begin{aligned} \|u(\mu,\alpha,\cdot)\|^{q+1}_{q+1} &= \eta^{q+1}\tau^{-1}\|z_{\mu,\alpha}\|^{q+1}_{q+1} = \mu_1^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|z_{\mu,\alpha}\|^{q+1}_{q+1}, \\ \|u(\mu,\alpha,\cdot)\|^{p_k+1}_{p_k+1} &= \eta^{p_k+1}\tau^{-1}\|z_{\mu,\alpha}\|^{p_k+1}_{p_k+1} = \mu_1^{\frac{2p_k+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{2p_k+3-q}{2(p_i-q)}}\|z_{\mu,\alpha}\|^{p_k+1}_{p_k+1}. \end{aligned}$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

Then we obtain by (2.3), (6.4) and (6.8) that

$$\begin{aligned}\big(b_1+o(1)\big)\mu_1\mu_1^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|z_{\mu,\alpha}\|_{q+1}^{q+1}\\ &= 2\alpha+\frac{p_i-1}{p_i+1}\mu_i\mu_1^{\frac{2p_i+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{2p_i+3-q}{2(p_i-q)}}\|z_{\mu,\alpha}\|_{p_i+1}^{p_i+1}\\ &\quad-\frac{q-1}{q+1}\mu_1\mu_1^{\frac{2q+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{q+3}{2(p_i-q)}}\|z_{\mu,\alpha}\|_{q+1}^{q+1}\\ &\quad-\sum_{k=2}^{m}\frac{p_k-1}{p_k+1}\mu_k\mu_1^{\frac{2p_k+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{2p_k+3-q}{2(p_i-q)}}\|z_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\\ &\quad+\sum_{k=m+1,\,k\neq i}^{n}\frac{p_k-1}{p_k+1}\mu_k\mu_1^{\frac{2p_k+3-p_i}{2(p_i-q)}}\mu_i^{-\frac{2p_k+3-q}{2(p_i-q)}}\|z_{\mu,\alpha}\|_{p_k+1}^{p_k+1};\end{aligned}$$

that is,

$$\begin{aligned}\Big\{b_1+\frac{q-1}{q+1}+o(1)\Big\}\|z_{\mu,\alpha}\|_{q+1}^{q+1}-\frac{p_i-1}{p_i+1}\|z_{\mu,\alpha}\|_{p_i+1}^{p_i+1}\\ &= 2\alpha\mu_1^{-\frac{p_i+3}{2(p_i-q)}}\mu_i^{\frac{q+3}{2(p_i-q)}}\\ &\quad-\sum_{k=2}^{m}\frac{p_k-1}{p_k+1}\mu_k\mu_1^{\frac{p_k-p_i}{p_i-q}}\mu_i^{-\frac{p_k-q}{p_i-q}}\|z_{\mu,\alpha}\|_{p_k+1}^{p_k+1}\\ &\quad+\sum_{k=m+1,\,k\neq i}^{n}\frac{p_k-1}{p_k+1}\mu_k\mu_1^{\frac{p_k-p_i}{p_i-q}}\mu_i^{-\frac{p_k-q}{p_i-q}}\|z_{\mu,\alpha}\|_{p_k+1}^{p_k+1}.\end{aligned}\tag{6.9}$$

Hence, by passing to the limit in (6.9), we obtain by (1.9) and (6.6) that

$$\Big(b_1+\frac{q-1}{q+1}\Big)\|z\|_{q+1}^{q+1}-\frac{p_i-1}{p_i+1}\|z\|_{p_i+1}^{p_i+1}=2C_2^{-\frac{p_i+3}{2(p_i-q)}}.\tag{6.10}$$

Hence, it is necessary to investigate $\|z\|_{q+1}$ and $\|z\|_{p_i+1}$ precisely. To do this, we recall some properties of z. Similar to (5.3) and (5.4), we have

$$z(0)=\theta:=\Big(\frac{(p_i+1)b_2}{q+1}\Big)^{\frac{1}{p_i-q}},\quad z(t)=z(-t),\quad t\in R,\quad z'(t)\le 0,\quad t\ge 0,\tag{6.11}$$

$$\frac{1}{2}z'(t)^2+\frac{1}{p_i+1}z(t)^{p_i+1}-\frac{1}{q+1}b_2z(t)^{q+1}=0,\quad t\in R.\tag{6.12}$$

LEMMA 6.2. *Let z be the ground state solution of (6.7). Then*

$$\|z\|_{q+1}^{q+1}=\frac{2}{p_i-q}\sqrt{\frac{(q+1)\pi}{2b_2}}\,\theta^{\frac{q+3}{2}}\frac{\Gamma\big(\frac{q+3}{2(p_i-q)}\big)}{\Gamma\big(\frac{p_i+3}{2(p_i-q)}\big)},\tag{6.13}$$

$$\|z\|_{p_i+1}^{p_i+1}=\frac{(p_i+1)(q+3)b_2}{(p_i+3)(q+1)}\|z\|_{q+1}^{q+1}.\tag{6.14}$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

PROOF. By (6.11) and (6.12), we obtain for $t \geq 0$

$$z'(t) = -z(t)\sqrt{\frac{2b_2}{q+1}z(t)^{q-1} - \frac{2}{p_i+1}z(t)^{p_i-1}}. \tag{6.15}$$

Then by putting $w = z(t)$, $y = \theta^{-1}w = \sin^{\frac{2}{p_i-q}} x$ and using (6.15) we obtain

$$\begin{aligned}
\frac{1}{2}\|z\|_{q+1}^{q+1} &= \int_0^\infty z(t)^q \cdot z(t)\,dt \\
&= \int_0^\infty z(t)^q \frac{-z'(t)}{\sqrt{\frac{2b_2}{q+1}z(t)^{q-1} - \frac{2}{p_i+1}z(t)^{p_i-1}}}\,dt \\
&= \sqrt{\frac{q+1}{2b_2}}\int_0^\theta \frac{w^{\frac{q+1}{2}}}{\sqrt{1-\theta^{q-p_i}w^{p_i-q}}}\,dw \\
&= \sqrt{\frac{q+1}{2b_2}}\theta^{\frac{q+3}{2}}\int_0^1 \frac{y^{\frac{q+1}{2}}}{\sqrt{1-y^{p_i-q}}}\,dy \\
&= \sqrt{\frac{q+1}{2b_2}}\frac{2}{p_i-q}\theta^{\frac{q+3}{2}}\int_0^{\pi/2}\sin^{\frac{2q+3-p_i}{p_i-q}} x\,dx \\
&= \sqrt{\frac{q+1}{2b_2}}\frac{\sqrt{\pi}}{p_i-q}\theta^{\frac{q+3}{2}}\frac{\Gamma\left(\frac{q+3}{2(p_i-q)}\right)}{\Gamma\left(\frac{p_i+3}{2(p_i-q)}\right)}.
\end{aligned}$$

Thus, we obtain (6.13). Similarly, we obtain

$$\begin{aligned}
\frac{1}{2}\|z\|_{p_i+1}^{p_i+1} &= \sqrt{\frac{q+1}{2b_2}}\frac{\sqrt{2}}{p_i-q}\theta^{\frac{2p_i+3-q}{2}}\frac{\Gamma\left(\frac{q+3}{2(p_i-q)}+1\right)}{\Gamma\left(\frac{p_i+3}{2(p_i-q)}+1\right)} \\
&= \frac{(p_i+1)(q+3)b_2}{(p_i+3)(q+1)}\sqrt{\frac{q+1}{2b_2}}\frac{\sqrt{\pi}}{p_i-q}\theta^{\frac{q+3}{2}}\frac{\Gamma\left(\frac{q+3}{2(p_i-q)}\right)}{\Gamma\left(\frac{p_i+3}{2(p_i-q)}\right)} \\
&= \frac{(p_i+1)(q+3)b_2}{2(p_i+3)(q+1)}\|z\|_{q+1}^{q+1}.
\end{aligned} \tag{6.16}$$

Thus, (6.14) follows from (6.17). ∎

Now we are ready to prove Theorem 2.2.

PROOF OF THEOREM 2.2. Now, a simple calculation with the aid of (6.10), (6.13) and (6.14) shows that $x = a_1$ satisfies:

$$\frac{2}{(p_i+3)(q+1)}\{(2q+3-p_i)x - 2(p_i-q)C_2\} = L_1(x+C_2)^{-\frac{2q+3-p_i}{2(p_i-q)}}, \tag{6.17}$$

where $L_1 > 0$ is defined in (2.8). Since $2q+3-p_i, p_i-q > 0$, it is clear that the positive solution $x = C_4$ of equation (6.17) uniquely exists. Hence, we obtain that $a_1 = C_4$. Now, full assertion follows from a standard compactness argument. Thus the proof is complete. ∎

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

7. **Proof of Theorem 2.3.** Let

$$V_{\mu,\alpha} := \Big\{ v \in X : \frac{1}{2}\|v\|_X^2 + \sum_{k=1}^{m} \frac{1}{p_k+1}\mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \|v\|_{p_k+1}^{p_k+1} - \sum_{k=m+1}^{n} \frac{1}{p_k+1}\mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \|v\|_{p_k+1}^{p_k+1} = -\alpha \mu_i^{\frac{2}{p_i-1}} \Big\}.$$

Since $u(\mu,\alpha,x) \in N_{\mu,\alpha}$, we see that $v_{\mu,\alpha}(x) := \mu_i^{1/(p_i-1)} u(\mu,\alpha,x) \in V_{\mu,\alpha}$. Using (1.1), we find that $v_{\mu,\alpha}$ satisfies the following equation:

$$\begin{aligned} v''_{\mu,\alpha}(x) - \sum_{k=1}^{m} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} v_{\mu,\alpha}(x)^{p_k} + \sum_{k=m+1}^{n} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} v_{\mu,\alpha}(x)^{p_k} \\ = \lambda(\mu,\alpha) \mu_i^{-\frac{q-1}{p_i-1}} v_{\mu,\alpha}(x)^q, \quad x \in I, \\ v_{\mu,\alpha}(x) > 0, \quad x \in I, \\ v_{\mu,\alpha}(\pm 1) = 0. \end{aligned} \tag{7.1}$$

LEMMA 7.1. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.3). Then $\|v_{\mu,\alpha}\|_{q+1} \le C$.*

PROOF. We choose $u_0 \in X$ ($u_0 \not\equiv 0$) and define a function $m(t)$ for $t \ge 0$ by:

$$m(t) := \frac{1}{2} t^2 \|u_0\|_X^2 + \sum_{k=1}^{m} \frac{1}{p_k+1} \mu_k t^{p_k+1} \|u_0\|_{p_k+1}^{p_k+1} - \sum_{k=m+1}^{n} \frac{1}{p_k+1} \mu_k t^{p_k+1} \|u_0\|_{p_k+1}^{p_k+1}.$$

Since $m(0) = 0$ and $m(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$, we have

$$G_{\mu,\alpha} := \{ t > 0 : m(t) = -\alpha \} \ne \emptyset.$$

Let $t_{\mu,\alpha} = \inf\{ t > 0 : t \in G_{\mu,\alpha} \}$. Then $m(t_{\mu,\alpha}) = -\alpha$, namely, $t_{\mu,\alpha} u_0 \in N_{\mu,\alpha}$, and by definition, $m(t) < -\alpha$ implies that $t_{\mu,\alpha} < t$. We see from (1.11) that $m(t) < -\alpha$ implies:

$$\begin{aligned} &\frac{1}{2}\|u_0\|_X^2 - \frac{1}{p_i+1}\mu_i t^{p_i-1} \|u_0\|_{p_i+1}^{p_i+1} + \sum_{k=1}^{m} \frac{1}{p_k+1} \mu_k t^{p_k-1} \|u_0\|_{p_k+1}^{p_k+1} \\ &\quad - \sum_{k=m+1,\, k\ne i}^{n} \frac{1}{p_k+1} \mu_k t^{p_k-1} \|u_0\|_{p_k+1}^{p_k+1} \\ &= \frac{1}{2}\|u_0\|_X^2 - \frac{1}{p_i+1} \big(t\mu_i^{\frac{1}{p_i-1}}\big)^{p_i-1} \|u_0\|_{p_i+1}^{p_i+1} \\ &\quad + \sum_{k=1}^{m} \frac{1}{p_k+1} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \big(t\mu_i^{\frac{1}{p_i-1}}\big)^{p_k-1} \\ &\quad - \sum_{k=m+1,\, k\ne i}^{n} \frac{1}{p_k+1} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \big(t\mu_i^{\frac{1}{p_i-1}}\big)^{p_k-1} < -\alpha t^{-2}. \end{aligned} \tag{7.2}$$

For $C \gg 1$, we put $t = C\mu_i^{-\frac{1}{p_i-1}}$. Then it follows from (1.11) and (7.2) that

$$\begin{aligned} &\frac{1}{2}\|u_0\|_X^2 - \frac{1}{p_i+1} C^{p_i-1} \|u_0\|_{p_i+1}^{p_i+1} + o(1) \sum_{k=1}^{m} C^{p_k-1} - o(1) \sum_{k=m+1,\, k\ne i}^{n} C^{p_k-1} \\ &\quad < -\alpha \mu_i^{\frac{2}{p_i-1}} C^{-2}. \end{aligned} \tag{7.3}$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

Therefore, we obtain that $t_{\mu,\alpha} \le C\mu_i^{-\frac{1}{p_i-1}}$. Now, by using (2.2), we obtain

$$\frac{1}{q+1}\|u(\mu,\alpha,\cdot)\|_{q+1}^{q+1} \le \frac{1}{q+1} t_{\mu,\alpha}^{q+1}\|u_0\|_{q+1}^{q+1} \le C\mu_i^{-\frac{q+1}{p_i-1}}. \tag{7.4}$$

Thus the proof is complete. ∎

LEMMA 7.2. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.3). Then* $\|v_{\mu,\alpha}\|_X \le C$.

PROOF. Since $v_{\mu,\alpha} \in V_{\mu,\alpha}$, we obtain by (1.11), (3.5) and Lemma 7.1 that

$$\begin{aligned} \|v_{\mu,\alpha}\|_X^2 &\le \frac{2}{p_i+1}\|v_{\mu,\alpha}\|_{p_i+1}^{p_i+1} + \sum_{k=m+1,\,k\ne i}^{n} \frac{2}{p_k+1}\mu_k\mu_i^{-\frac{p_k-1}{p_i-1}}\|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} \\ &\le C\|v_{\mu,\alpha}\|_{q+1}^{(p_i+1)(1-\gamma_k)}\|v_{\mu,\alpha}\|_X^{(p_i+1)\gamma_k} \\ &\quad + o(1)\sum_{k=m+1,\,k\ne i}^{n}\frac{2}{p_k+1}\|v_{\mu,\alpha}\|_{q+1}^{(p_k+1)(1-\gamma_k)}\|v_{\mu,\alpha}\|_X^{(p_k+1)\gamma_k} \\ &\le C\|v_{\mu,\alpha}\|_X^{\frac{2(p_i-q)}{q+3}} + o(1)\sum_{k=m+1,\,k\ne i}^{n}\|v_{\mu,\alpha}\|_X^{\frac{2(p_k-q)}{q+3}}, \end{aligned} \tag{7.5}$$

where $\gamma_k = 2(p_k-q)/\{(p_k+1)(q+1)\}$. Since we know from (1.2) that $2(p_k-q)/(q+3) < 2$ for $1 \le k \le n$, our assertion follows from (7.5). Thus the proof is complete. ∎

LEMMA 7.3. *Assume that* $\{(\mu,\alpha)\} \subset E$ *satisfies (B.3). Then* $\|v_{\mu,\alpha}\|_{q+1} \ge C$.

PROOF. By Sobolev's embedding theorem, (3.5) and Lemma 7.2

$$\begin{aligned} \|v_{\mu,\alpha}\|_{q+1}^2 &\le C\|v_{\mu,\alpha}\|_X^2 \\ &\le C\|v_{\mu,\alpha}\|_{p_i+1}^{p_i+1} + o(1)\sum_{k=m+1,\,k\ne i}^{n}\|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} \\ &\le C\|v_{\mu,\alpha}\|_{q+1}^{\frac{(q+1)(p_i+3)}{q+3}}\|v_{\mu,\alpha}\|_X^{\frac{2(p_i-q)}{q+3}} \\ &\quad + o(1)\sum_{k=m+1,\,k\ne i}^{n}\|v_{\mu,\alpha}\|_{q+1}^{\frac{(q+1)(p_k+3)}{q+3}}\|v_{\mu,\alpha}\|_X^{\frac{2(p_k-q)}{q+3}} \\ &\le C\|v_{\mu,\alpha}\|_{q+1}^{\frac{(q+1)(p_i+3)}{q+3}} + o(1)\sum_{k=m+1,\,k\ne i}^{n}\|v_{\mu,\alpha}\|_{q+1}^{\frac{(q+1)(p_k+3)}{q+3}}; \end{aligned}$$

that is,

$$1 \le C\|v_{\mu,\alpha}\|_{q+1}^{\frac{p_iq+p_i+q-3}{q+3}} + o(1)\sum_{k=m+1,\,k\ne i}^{n}\|v_{\mu,\alpha}\|_{q+1}^{\frac{p_kq+p_k+q-3}{q+3}}. \tag{7.6}$$

Since $p_kq + p_k + q - 3 > 0$ $(1 \le k \le n)$ by (1.2), we obtain our conclusion. ∎

We introduce the uniqueness lemma of the minimizer of the problem (2.10).

LEMMA 7.4 ([5, PROPOSITION 3.9]). *There uniquely exists the minimizer v_0 of the problem (2.10).*

Next, we shall show that $v_{\mu,\alpha} \longrightarrow v_0$ in X. Let

$$T(\mu,\alpha) := \inf_{v \in V_{\mu,\alpha}} \|v\|_{q+1}^{q+1} (= \|v_{\mu,\alpha}\|_{q+1}^{q+1}), \quad T_0 := \inf_{v \in V_0} \|v\|_{q+1}^{q+1}. \tag{7.7}$$

LEMMA 7.5. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.3). Then* $\limsup T(\mu,\alpha) \le T_0$.

PROOF. We define $s_{\mu,\alpha} > 0$ by $s_{\mu,\alpha} v_0 \in V_{\mu,\alpha}$. We shall show that $s_{\mu,\alpha} \longrightarrow 1$. By definition of $s_{\mu,\alpha}$ and the fact that $v_0 \in V_0$, we have

$$\begin{aligned} &\frac{1}{2}\|v_0\|_X^2 s_{\mu,\alpha}^2 \{1 - s_{\mu,\alpha}^{p_i-1}\} + \sum_{k=1}^m \frac{1}{p_k+1} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} s_{\mu,\alpha}^{p_k+1} \|v_0\|_{p_k+1}^{p_k+1} \\ &\qquad - \sum_{k=m+1,\, k\neq i}^n \frac{1}{p_k+1} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} s_{\mu,\alpha}^{p_k+1} \|v_0\|_{p_k+1}^{p_k+1} \\ &= -\alpha \mu_i^{\frac{2}{p_i-1}}, \end{aligned} \tag{7.8}$$

that is,

$$\begin{aligned} &\frac{1}{2}\|v_0\|_X^2 s_{\mu,\alpha}^2 + \sum_{k=1}^m \frac{1}{p_k+1} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} s_{\mu,\alpha}^{p_k+1} \|v_0\|_{p_k+1}^{p_k+1} + \alpha \mu_i^{\frac{2}{p_i-1}} \\ &\quad = \frac{1}{2}\|v_0\|_X^2 s_{\mu,\alpha}^{p_i+1} + \sum_{k=m+1,\, k\neq i}^n \frac{1}{p_k+1} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} s_{\mu,\alpha}^{p_k+1} \|v_0\|_{p_k+1}^{p_k+1}. \end{aligned}$$

This along with (1.10) and (1.11) implies that $s_{\mu,\alpha} \le C$. Next, assume that there exists a subsequence of $\{s_{\mu,\alpha}\}$ such that $s_{\mu,\alpha} \longrightarrow 0$. Then we obtain by (7.7) that

$$\|v_{\mu,\alpha}\|_{q+1}^{q+1} = T(\mu,\alpha) \le \|s_{\mu,\alpha} v_0\|_{q+1}^{q+1} \longrightarrow 0.$$

This is a contradiction, since we have Lemma 7.3. Hence, $C^{-1} \le s_{\mu,\alpha} \le C$. Then it follows from (1.10), (1.11) and (7.8) that $s_{\mu,\alpha} \longrightarrow 1$. Now,

$$T(\mu,\alpha) = \|v_{\mu,\alpha}\|_{q+1}^{q+1} \le s_{\mu,\alpha}^{q+1} \|v_0\|_{q+1}^{q+1} = s_{\mu,\alpha}^{q+1} T_0. \tag{7.9}$$

By passing to the limit in (7.9), we obtain our assertion. ∎

LEMMA 7.6. *Assume that $\{(\mu,\alpha)\} \subset E$ satisfies (B.3). Then $v_{\mu,\alpha} \longrightarrow v_0$ in X.*

PROOF. By Lemma 7.2, we can extract a subsequence of $\{v_{\mu,\alpha}\}$ such that $v_{\mu,\alpha} \longrightarrow v_\infty$ weakly in X, strongly in $C(I)$, $L^{p_k+1}(I)$, $L^{q+1}(I)$. By Lemma 7.3, we see that $v_\infty \not\equiv 0$. Since $v_{\mu,\alpha} \in V_{\mu,\alpha}$, it follows from (1.10), (1.11), (7.1) and Lemma 7.2 that

$$\begin{aligned} \lambda(\mu,\alpha)\mu^{-\frac{q-1}{p_i-1}} &= \frac{2\alpha \mu_i^{\frac{2}{p_i-1}} + \frac{p_i-1}{p_i+1}\|v_{\mu,\alpha}\|_{p_i+1}^{p_i+1} + h(\mu,\alpha)}{\|v_{\mu,\alpha}\|_{q+1}^{q+1}} \longrightarrow C_5 \\ &= \frac{p_i-1}{p_i+1} \frac{\|v_\infty\|_{p_i+1}^{p_i+1}}{\|v_\infty\|_{q+1}^{q+1}}, \end{aligned} \tag{7.10}$$

https://doi.org/10.4153/CJM-1997-053-0 Published online by Cambridge University Press

where

$$h(\mu,\alpha) := -\sum_{k=1}^{m} \frac{p_k-1}{p_k+1}\mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} + \sum_{k=m+1,\, k\neq i}^{n} \frac{p_k-1}{p_k+1}\mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1}.$$

We shall show that $v_\infty \in V_0$. By (1.11), (7.1) and (7.10), v_∞ is a weak solution of the following equation, namely, for $\psi \in X$

$$(7.11) \qquad -\int_I v_\infty'(x)\psi'(x)\,dx + \int_I v_\infty(x)^{p_i}\psi(x)\,dx = C_5 \int_I v_\infty(x)^q \psi(x)\,dx.$$

Multiplying (7.1) by $v_{\mu,\alpha}$ along with integration by parts, we have

$$(7.12) \qquad \|v_{\mu,\alpha}\|_X^2 = F(\mu,\alpha,v_{\mu,\alpha}) := -\sum_{k=1}^{m} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} + \sum_{k=m+1}^{n} \mu_k \mu_i^{-\frac{p_k-1}{p_i-1}} \|v_{\mu,\alpha}\|_{p_k+1}^{p_k+1} - \lambda(\mu,\alpha)\mu_i^{-\frac{q-1}{p_i-1}} \|v_{\mu,\alpha}\|_{q+1}^{q+1}.$$

We put $\psi = v_\infty$ in (7.10). Then we obtain by (1.11), (7.10) and (7.12) that

$$\|v_\infty\|_X^2 = \|v_\infty\|_{p_i+1}^{p_i+1} - C_5\|v_\infty\|_{q+1}^{q+1} = \lim F(\mu,\alpha,v_{\mu,\alpha}) = \lim \|v_{\mu,\alpha}\|_X^2.$$

Hence, $v_{\mu,\alpha} \longrightarrow v_\infty$ in X and this implies that $v_\infty \in V_0$. Now, it follows from Lemma 7.5 that

$$(7.13) \qquad T_0 \le \|v_\infty\|_{q+1}^{q+1} = \lim \|v_{\mu,\alpha}\|_{q+1}^{q+1} \le T_0.$$

This along with Lemma 7.4 implies that $v_\infty \equiv v_0$. Now full assertion follows from a standard compactness argument. Thus the proof is complete. ∎

PROOF OF THEOREM 2.3. Theorem 2.3 follows immediately from (7.10) and Lemma 7.6. ∎

REFERENCES

1. H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, I, Existence of a ground state*, Arch. Rational Mech. Analysis **82**(1983), 313–345.

2. M. Faierman, *Two-parameter eigenvalue problems in ordinary differential equations*, Longman House, Essex, UK.

3. B. Gidas, W. M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commn. Math. Phys. **68**(1979), 209–243.

4. T. Shibata, *Asymptotic behavior of eigenvalues of two-parameter nonlinear Sturm-Liouville problems*, J. Analyse Math. **66**(1995), 277–294.

5. ———, *Variational eigencurve and bifurcation for two-parameter nonlinear Sturm-Liouville equations*, Topol. Methods Nonlinear Anal. **8**(1996) 79–93.

6. L. Turyn, *Sturm-Liouville problems with several parameters*, J. Differential Equations **38**(1980), 239–259.

7. E. Zeidler, *Ljusternik-Schnirelman theory on general level sets*, Math. Nachr. **129**(1986), 235–259.

The Division of Mathematical and Information Sciences
Faculty of Integrated Arts and Sciences
Hiroshima University
Higashi-Hiroshima, 739 Japan