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Nagoya Math. J.
Vol. 33 (1968), 75–83

SOME CONDITIONS UNDER WHICH SEQUENCES OF FUNCTIONS ARE UNIFORMLY BOUNDED

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§1. Introduction

After one introduces the theory of normal families in a course in complex analysis, the usual pattern is to give an example of a non-normal family. One of the simplest, of course, is the sequence $f_n(z) = nz$, $n = 1, 2, \cdots$. The very devastating effect of multiplying by zero insures the required abnormality! If one asks for a slightly more sophisticated example, we offer $f_n(z) = \dfrac{e^{nz}}{n}$, $n = 1, 2 \cdots$; here the $f_n(z)$ are zero free. However, the difference in behavior between the sequence $\left\{|f_n(0)| = \dfrac{1}{n}\right\}$ and $|f_n(z)| = \dfrac{e^{nx}}{n}$ $z = x + iy \neq 0$ is obvious.

In this paper we offer an explanation for this state of affairs. To be precise given a sequence $\{f_n\}$ of bounded holomorphic functions defined in the unit disk D in the complex plane we establish criteria based upon a comparison of $\{|f_n(0)|\}$ and $\{M(f_n)\}$, $M(f_n) = \max\limits_{|z|<1}|f_n(z)|$ insuring that the sequence $\{f_n\}$ will be uniformly bounded on certain compact subsets of D. We then extend the result in several directions to entire and harmonic functions.

Let $D(r)$, $0 < r < \infty$, denote the open disk in the complex plane with centre at the origin and radius r, while $\bar{D}(r)$ indicates the closure of $D(r)$; and $D(\infty)$ indicates the finite complex plane. If f is a complex valued function defined in $D(R)$, for $0 \leq r \leq R$ let

$$M(r, f) = \sup_{|z|<r} |f(z)|$$

and

$$m(r, f) = \inf_{|z|<r} |f(z)|.$$

If E is a set contained in $D(R)$ set

Received January 11, 1968.

If we need not display the dependence on f we write $M(r)$, and $m(r)M_E(r)$ respectively.

§2. Basic Theorem

The results of this paper hinge on the solution of the Carleman-Milloux problem which we give as formulated by Tsuji [1 p. 307]. For this formulation we suppose that E is a set contained in $D(R)$, $0<R<\infty$, with the properties that

i) each circle $|z|=r, 0<r<R$, meets E;

ii) $E\cap\bar{D}(r)$ is a closed set for each $0<r<R$.

We call such a set an intersecting set.

THEOREM A. *Let E be an intersecting set in $D(R)$ and f be holomorphic in $D(R)-E$, with $|f(z)|\leq M$, $z\in D(R)-E$, and*

$$\overline{\lim_{\substack{z\to E\\ z\in D(R)-E}}}\ |f(z)|\leq m<M.$$

Then for $z\in D(R)-E$

$$\log|f(z)|\leq\frac{2}{\pi}\Big(\arcsin\frac{R-|z|}{R+|z|}\Big)\log m+\Big(1-\frac{2}{\pi}\arcsin\frac{R-|z|}{R+|z|}\Big)\log M.$$

Actually this is not precisely the formulation of Tsuji who requires that E be closed in $\bar{D}(R)$. However it is evident that if one applies Tsuji's formulation to the disk $D(r)$, $0\leq r<R$, and let $r\to R$ one obtains Theorem A.

We now look for functions f with intersecting sets on which f is bounded by some useful number. There is one rather simple condition which insures the existence of an interesting intersecting set and it is that f takes on some finite value α only a finite number of times. Before giving a formal statement of this fact we introduce some standard notation.

If f takes on the value α only a finite number of times in $D(R)$, $0<R<\infty$, let $\{z_i\}_{i=1}^{n}$ be the non-zero points, counted with proper multiplicity, at which $f(z_i)=\alpha$; and let $z=0$ be a root of $f(z)-\alpha$ of multiplicity $p\geq 0$ where $p=0$ means $f(0)\neq\alpha$.

Set

$$B_\alpha(z) = \frac{z^p}{R^p} \prod_{i=1}^{n} \frac{R z_i}{|z_i|} \left(\frac{z_i - z}{R^2 - \bar{z}_i z} \right).$$

LEMMA 1. *Suppose f is holomorphic in $D(R)$, $0 < R < \infty$, and there takes on the value α only a finite number of times. Then the set*

$$E_\alpha = \left\{ z \in D(R) \mid |f(z)| \leq \left| \frac{f(0) - \alpha}{B_\alpha(0)} \right| + |\alpha| \equiv M_\alpha \right\}$$

is an intersecting set.

Proof. We first consider the case in which f omits the value 0. That $E_0 = \{z \in D(R) \mid |f(z)| \leq |f(0)|\}$ is an intersecting set can be seen in several ways. We can use Cauchy's integral formula as follows: Suppose E_0 does not meet some circle $|z| = r_0$, $0 \leq r_0 < R$. Since $\frac{1}{f}$ is also holomorphic in $D(R)$

$$\left| \frac{1}{f(0)} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|f(r_0 e^{i\theta})|}$$
$$< \frac{1}{2\pi} \int_{|z| = r_0} \frac{d\theta}{|f(0)|} = \frac{1}{|f(0)|},$$

which is absurd. Since $E_0 \cap \{|z| \leq r\}$, $0 \leq r < R$, is a closed set E_0 is intersecting.

For future exploitation we observe that if $R = \infty$ and f is an entire function which omits 0 then E_0 is also an intersecting set relative to $D(\infty)$.

For the general situation in which f takes on α a finite number of times in $D(R)$, $0 < R < \infty$, we let $g(z) = \frac{f(z) - \alpha}{B_\alpha(z)}$, which omits the value 0. Therefore

$$E_0^* = \{z \in D(R) \mid |g(z)| \leq |g(0)|\}$$

is intersecting. Now for $z \in E_0^*$,

$$|g(0)| = \left| \frac{f(0) - \alpha}{B_\alpha(0)} \right| \geq |g(z)|$$
$$= \left| \frac{f(z) - \alpha}{B_\alpha(z)} \right|$$
$$\geq |f(z)| - |\alpha|.$$

https://doi.org/10.1017/S0027763000012861 Published online by Cambridge University Press

Hence the set E_α contains $E_0{}^*$ which implies that E_α is also intersecting and the lemma is proved.

§3. Applications to functions omitting the value 0.

In this section we assume that the sequence of function $\{f_n\}$ all omit the value 0.

For $0<\delta<\infty$, let $B(\delta)=\sin\dfrac{\pi}{2(1+\delta)}$. An easy calculation gives that the subset of $D(R)$, $0<R<\infty$, given by $\dfrac{2}{\pi}$ arcsin $\dfrac{R-|z|}{R+|z|}>\dfrac{1}{1+\delta}$ is the disk $|z|\leq R\left(\dfrac{1-B(\delta)}{1+B(\delta)}\right)$. With this observation and the fact, already noted, that for a holomorphic function f omitting 0 in $D(R)$ the set

$$E_0=\{z\in D(R)\,|\,|f(z)|\leq|f(0)|\}$$

is intersecting in $D(R)$ we use Theorem A to give the key theorem.

THEOREM I. *Let f be holomorphic and bounded in $D(R)$, $0<R<\infty$, and omit there the value 0. Then for $|z|<R$ we have*

$$|f(z)|\leq|f(0)|^{\frac{2}{\pi}\arcsin\frac{R-|z|}{R+|z|}}(M(R,f))^{\left(1-\frac{2}{\pi}\arcsin\frac{R-|z|}{R+|z|}\right)}; \tag{3. 0}$$

and thus for $|z|<R\left(\dfrac{1-B(\delta)}{1+B(\delta)}\right)$

$$|f(z)|\leq|f(0)|^{\frac{1}{1+\delta}}(M(R,f))^{\frac{\delta}{1+\delta}}. \tag{3. 1}$$

Proof. Since $|f(z)|$ is bounded by $|f(0)|$ on the intersecting set E_0 a direct application of Theorem A gives (3. 0). The convexity of the right side of (3. 0) together with the remarks preceding this theorem validate (3. 1).

By way of illustration, if f is holomorphic, bounded by 1 in $D(1)$, and omits 0 there, then

$$|f(z)|\leq\sqrt[n+1]{|f(0)|},\quad |z|\leq\left(\frac{1-B(n)}{1+B(n)}\right).$$

There are some obvious results to be gathered from Theorem I about sequences of holomorphic functions omitting zero.

COROLLARY 1. *Let $\{f_n\}$ be a sequence of bounded holomorphic functions in $D(R)$ with each f_n omitting the value 0. If for $\delta>0$*

$$|f_n(0)|(M(R, f_n))^\delta \leq A, \quad n = 1, 2, \cdots,$$

then for
$$|z| \leq \left(\frac{1 - B(\delta)}{1 + B(\delta)}\right)R,$$

$$|f_n(z)| \leq A^{\frac{1}{1+\delta}}, \quad n = 1, 2, \cdots.$$

Proof. This is an immediate application of Theorem I.

One can avoid the hypothesis that each f_n be bounded.

COROLLARY 2. *Let $\{f_n\}$ be a sequence of holomorphic functions in $D(R)$, each omitting the value 0. If, for some sequence $\{r_n\}$, $0 < r_n < r_{n+1} < R$, $r_n \to R$, $n \to \infty$, and some value $0 < \delta < \infty$,*

$$|f_n(0)|(M(r_n, f_n))^\delta \leq A, \quad n = 1, 2, \cdots,$$

then for
$$|z| \leq \left(\frac{1 - B(\delta)}{1 + B(\delta)}\right) r_n, \quad n = 1, 2, \cdots,$$

$$|f_n(z)| \leq A^{\frac{1}{1+\delta}}.$$

Proof. Setting $g_n(\xi) = f_n\left(\frac{r_n}{R}\xi\right)$ and invoking Corollary 1 gives this result.

We now make several comments on Corollary 1. That each f_n must omit zero is needed. If we let $\delta = 1$, $R = 1$, and set $f_n(z) = n\left(z + \frac{1}{n^2}\right)$, $n = 1, 2, \cdots$, then $f_n(0) = \frac{1}{n}$ while $M(1, f_n) = n + \frac{1}{n}$. Thus $|f_n(0)| M(1, f_n) \leq 2$, but the sequence is not uniformly bounded in any $D(r)$, $0 < r < 1$. On the other hand the example given in the introduction $f_n(z) = \frac{e^{nz}}{n}$ is without zeroes for each $n = 1, 2, \cdots$; has $|f_n(0)| M(1, f_n) \to \infty$, and is not uniformly bounded on any subdisk $|z| < r$, $0 < r < 1$.

If $\{f_n\}$ is a sequence satisfying the hypothesis of Corollary 1, with $\delta = 1$, $R = 1$, we are guaranteed by this result that on the closed disk $D\left(\frac{1 - B(1)}{1 + B(1)}\right)$ the sequence $\{f_n\}$ is uniformly bounded. We ask for the largest possible closed disk $\bar{D}(R^*)$ on which *any* sequence satisfying the hypothesis of Corollary 1 (with $\delta = 1$) is uniformly bounded. We do not know the exact value of R^*. The sequence

$$f_n(z) = 2^{\frac{-n}{2}}(z + 1)^n, \quad n = 1, 2, \cdots,$$

satisfies the hypotheses of Corollary 1. Further $|f_n(z)|\leq 1$, for $|z|\leq(\sqrt{2}-1)$; but for any $\varepsilon>0$

$$f_n(\sqrt{2}-1+\varepsilon)\to\infty,$$

so that we can only estimate that

$$\frac{1-B(1)}{1+B(1)}=(\sqrt{2}-1)^2\leq R^*\leq(\sqrt{2}-1).$$

Suppose again that $\{f_n\}$ is a sequence of holomorphic functions in $D(R)$, each of which omits the value 0. Suppose further that for each $0<\delta<\infty$ there is a sequence $\{r_n\}$, $r_n\to R$, (with $\{r_n\}$ depending on δ) such that $\overline{\lim_{n\to\infty}}|f_n(0)|(M(r_n,f_n))^\delta<\infty$, then $\{f_n\}$ form a normal family. That this is so follows from Corollary 5. Given any disk $D(r)$, $0<r<R$, choose a value δ and an N_0 so that $r<\left(\frac{1-B(\delta)}{1+B(\delta)}\right)\frac{R^2}{r_n}$, $n>N_0$. Then $\{f_n\}$ is uniformly bounded in $D(r)$ and so is a normal family.

In passing we mention that Corollary 1 can be used to give a proof of Hurwitz's theorem. Suppose $\{f_n\}$ is a sequence of holomorphic functions defined, for simplicity, in $D(1)$ such that $\{f_n\}$ converges uniformly (in the spherical metric) on compact subsets of $D(1)$. Suppose $f_n(0)\to\alpha$, $n\to\infty$ but on some $D(r)$, $0<r<1$, no f_n takes on the value α. If α is finite, and since each f_n is continuous, we can conclude that for some $0<r_1<r$, $\{f_n\}$ is uniformly bounded, say by M, on $D(r_1)$. According to Corollary 4, applied to $\{f_n-\alpha\}$ on $D(r_1)$

$$|f_n(z)-\alpha|\leq\sqrt{|f_n(0)-\alpha|M}\,,\ |z|\leq(\sqrt{2}-1)^2r_1.$$

Thus $f_n(z)\to\alpha$ for all $|z|\leq(\sqrt{2}-1)^2r_1$ and therefore the limit function is identically α in $D(1)$. In the situation for which α is infinite we consider $\left\{\frac{1}{f_n}\right\}$.

§4. Bounded functions

With little effort we can extend Theorem I to cover arbitrary bounded functions by invoking the usual factorization by a Blaschke product. For a given holomorphic function f bounded by 1 in $D(1)$ let $\{z_n\}$ be the set of all non-zero roots of f with each root repeated as often as its multiplicity. Also suppose f has a zero of order p at the origin (where $p=0$ means $f(0)\neq 0$). Then the Blaschke product

$$B(z, f) = z^p \prod_{n=1}^{\infty} \frac{z_n}{|z_n|} \frac{z_n - z}{1 - z\bar{z}_n}$$

is known to converge in $D(1)$ to a holomorphic function and f can be factored into $f = Bg$, where g is a holomorphic function in D without zeros. Using this fact we give

THEOREM II. *Let $f(z)$ be a holomorphic function in $D(1)$ with*

$$|f(z)| < 1, \ z \in D(1).$$

Then

$$|f(z)| \leq |B(z; f)| \, |g(0)|^{\frac{2}{\pi} \arcsin \frac{1-|z|}{1+|z|}}.$$

Proof. According to Theorem I applied to the function g (and noting that $|g(z)| < 1, \ z \in D(1)$)

$$|g(z)| < |g(0)|^{\frac{2}{\pi} \arcsin \frac{1-|z|}{1+|z|}}.$$

Since $g = \frac{f}{B}$ the proof is complete.

§5. Entire functions omitting 0.

Our next application is to entire functions omitting 0.

THEOREM III. *Suppose f is an entire function omitting the value* 0 *such that for some sequence $\{r_n\}$, $0 < r_n < r_{n+1} < \infty$, $r_n \to \infty$, $n \to \infty$; some choice of $0 < \delta < \infty$; and some arbitrary but fixed $T > \frac{2}{1-B(\delta)}$*

$$\lim_{n \to \infty} m(r_n, f) \, [M(Tr_n, f)]^{\delta} < \infty.$$

Then f is a constant function.

Proof. For $n = 1, 2, \cdots$ let $|f(z_n)| = m(r_n, f)$; also set

$$\xi = \xi_n(z) = \frac{z - z_n}{(T-1)r_n},$$

and

$$g_n(\xi) = f(\xi_n^{-1}(\xi)).$$

A trivial calculation shows that the disk about z_n of radius $r_n(T-1)$ is contained in $D(Tr_n)$. Consequently $M(1, g_n) \leq M(Tr_n, f)$, $n = 1, 2, \cdots$. Since $|g_n(0)| = |f(z_n)| = m(r_n, f)$ we conclude that

$$\lim_{n\to\infty} |g_n(0)|(M(1, g_n))^\delta < \infty.$$

If we choose $A > 0$ so that $|g_n(0)|(M(1, g_n))^\delta < A$, all n, and notice that each $g_n(z)$ omits 0, Theorem I allows us to conclude that for

$$|\xi| \leq \frac{1 - B(\delta)}{1 + B(\delta)}, \quad n = 1, 2, \cdots,$$

$$|g_n(\xi)| \leq [|g_n(0)|(M(1, g_n))^\delta]^{\frac{1}{1+\delta}} \leq A^{\frac{1}{1+\delta}}.$$

Thus in this ξ disk the sequence $\{g_n\}$ is uniformly bounded. But this says that f is bounded by $A^{\frac{1}{1+\delta}}$ on each disk $\frac{|z - z_n|}{(T-1)r_n} \leq \frac{1 - B(\delta)}{1 + B(\delta)}$, $n = 1, 2, \cdots$. It can be seen that each of these disks contain the corresponding disk about the origin of radius $r_n\left[\frac{1 - B(\delta)}{1 + B(\delta)}(T-1) - 1\right] \equiv r_n p$. However the quantity inside the brackets is seen to be fixed positive number on account of our choice of $T > \frac{2}{1 - B(\delta)}$. Hence

$$|f(z)| \leq A^{\frac{1}{1+\delta}}$$

for

$$z \in \bigcup_{n=1}^{\infty} D(pr_n),$$

which union covers the plane and so application of Liouville's theorem completes the proof.

Observe that as $\delta \to 0$, $\frac{2}{1-B(\delta)} \to +\infty$ and as $\delta \to \infty$, $\frac{2}{1-B(\delta)} \to 2$, so if one wants a smaller multiple of r_n in computing the maximum modulus one must increase the exponent of the maximum modulus and vice-versa. If we take $\delta = 1$ then $\frac{2}{1-B(1)}$ is about 6.82 so if f is an entire function omitting 0 and

$$\varliminf_{r\to\infty} m(r, f) M(7r, f) < \infty$$

then f is a constant function.

§6. Harmonic functions

By the usual device of calling forth the exponential function we can rework Theorem I to yield some results on harmonic functions. The theorem becomes

THEOREM IV. *Let* $u(z)$ *be harmonic and bounded above by* M *in* $D(R)$, $0 < R < \infty$. *Then for* $|z| \leq R\left(\frac{1-B(\delta)}{1+B(\delta)}\right)$ *we have*

$$u(z) \leq \frac{1}{1+\delta}[u(0) + \delta M].$$

Proof. If we let v be a complex conjugate of $u(z)$ and form $g = e^{u+iv}$ then g is holomorphic and zero-free in $D(R)$. Hence Theorem I prevails and the result is immediate.

We have parallel results to Corollaries 1 and 2. We give only the analogous result to Corollary 1.

COROLLARY 3. *Let* $\{u_n\}$ *be a sequence of bounded harmonic functions in* $D(R)$, $0 < R < \infty$, *such that for some* $0 < \delta < \infty$ $\overline{\lim}_{n\to\infty} (u_n(0) + \delta M(R, u_n)) < \infty$. *Then the sequence* $\{u_n\}$ *is uniformly bounded from above on* $|z| \leq R\left(\frac{1-B(\delta)}{1+B(\delta)}\right)$.

Proof. This follows from Theorem IV in the same way Corollary 1 follows from Theorem I.

Remark. Generalizations of our theorems in the case f (or $\{f_n\}$) take a value α only finitely often are possible by using the full force of Lemma 1. However the results seem a bit technical so we do not list them.

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https://doi.org/10.1017/S0027763000012861 Published online by Cambridge University Press