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Exceptional Sequences in Hall Algebras and Quantum Groups ⋆

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Abstract. Let Λ be a finite-dimensional hereditary algebra over a finite field k, $\mathcal{H}(\Lambda)$ and $\mathcal{C}(\Lambda)$ be, respectively, the Hall algebra and the composition algebra of Λ, $\mathcal{P}$ be the isomorphism classes of finite-dimensional Λ-modules and I the isomorphism classes of simple Λ-modules. We define δ_α and ${}_\alpha\delta$, $\alpha \in \mathcal{P}$, to be the right and left derivations of $\mathcal{H}(\Lambda)$, respectively. By using these derivations and the action of the braid group on the set of exceptional sequences of Λ-mod, we provide an effective algorithm of calculating the root vectors of real Schur roots. This means that we get an inductive method to express u_λ as the combinations of elements u_i in the Hall algebra, where $i \in I$ and $\lambda \in \mathcal{P}$ is any exceptional Λ-module. Because of the canonical isomorphism between the Drinfeld–Jimbo quantum group and the generic composition algebra, our algorithm is applicable directly to quantum groups. In particular, all the root vectors are obtained in this way in the finite type cases.

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1. Introduction

1.1. Let $\mathbf{g}$ be a semisimple Lie algebra over the complex field $\mathbf{C}$, $\mathbf{U}(\mathbf{g})$ be the universal enveloping algebra of $\mathbf{g}$. It is well-known that $\mathbf{g}$ has a root space decomposition

$$\mathbf{g} = \mathbf{g}_0 \bigoplus \coprod_{\alpha\in\Phi} \mathbf{g}_\alpha$$

with the root system Φ. Moreover, for any $\alpha \in \Phi$, $\dim_{\mathbf{C}}\mathbf{g}_\alpha = \mathbf{1}$. Choose a complete set of simple roots in Φ and denote it by I, then the root system Φ is divided into two parts: $\Phi = \Phi^+ \bigcup \Phi^-$, where Φ^+ and Φ^- are positive and negative roots, respectively. Therefore $\mathbf{g}$ has a triangular decomposition

$$\mathbf{g} = \mathbf{g}^- \bigoplus \mathbf{g}_0 \bigoplus \mathbf{g}^+.$$

For any $\alpha \in \Phi^+$, choose nonzero $x_\alpha \in \mathbf{g}_\alpha$ such that $\{x_\alpha \in \mathbf{g}_\alpha | \alpha \in \Phi^+\}$ constitute a basis of $\mathbf{g}^+$. Moreover, it is possible to choose $x_\alpha \in \mathbf{g}_\alpha (\alpha \in \Phi^+)$ such that

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$\{x_\alpha \in \mathbf{g}_\alpha | \alpha \in \Phi^+\}$ is a Chevalley basis of $\mathbf{g}^+$. The vectors $\{x_\alpha \in \mathbf{g}_\alpha | \alpha \in \Phi^+\}$ are called *root vectors*. Of course $\mathbf{g} \subset \mathbf{U}(\mathbf{g})$ as Lie subalgebra and $\mathbf{U}(\mathbf{g})$ has the triangular decomposition $\mathbf{U}(\mathbf{g}) = \mathbf{U}^- \otimes \mathbf{U}^0 \otimes \mathbf{U}^+$ which satisfies $\mathbf{g}^+ \subset \mathbf{U}^+$, $\mathbf{g}^- \subset \mathbf{U}^-$ and $\mathbf{g}_0 \subset \mathbf{U}^0$.

It is well known that the semisimple Lie algebra $\mathbf{g}$ itself cannot be quantized, but according to Drinfeld and Jimbo, the universal enveloping algebra $\mathbf{U}(\mathbf{g})$ of $\mathbf{g}$ admits a nontrivial quantization with q as a parameter, that is the so-called quantum group or the quantum enveloping algebra $\mathbf{U_q}(\mathbf{g})$ of $\mathbf{g}$. When $q \neq 1$, there is no longer $\mathbf{g} \subset \mathbf{U_q}(\mathbf{g})$. However according to Lusztig [L], there is an action of the braid group on $\mathbf{U_q}(\mathbf{g})$. It is still possible to obtain a family of linearly independent elements of $\mathbf{U_q^+}(\mathbf{g})$ by applying the braid group action in an admissible order on the generators of $\mathbf{U_q^+}(\mathbf{g})$. Those elements obtained are degenerated into a basis of $\mathbf{g}$ along $q \to 1$. So we also call those elements the *root vectors* of $\mathbf{U_q^+}(\mathbf{g})$.

1.2. Given any Dynkin diagram Δ of type $A_n, B_n, C_n, D_n, E_i (i = 6, 7, 8), F_4$ and G_2 and any finite field k, there exists a finite-dimensional hereditary k-algebra Λ corresponding to Δ. Let $\mathcal{P}$ be the set of isomorphism classes of finite-dimensional Λ-modules. The Hall algebra $\mathcal{H} = \mathcal{H}(\Lambda)$ is by definition the free $\mathbf{Z}[v, v^{-1}]$-module with the basis $\{u_\alpha | \alpha \in \mathcal{P}\}$ where $v^2 = q$ and $q = |k|$; the multiplication is given by $u_\alpha u_\beta = v^{\langle\alpha,\beta\rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha\beta}^\lambda u_\lambda$ for all $\alpha, \beta \in \mathcal{P}$, where

$$\langle\alpha, \beta\rangle = \dim_k \operatorname{Hom}_\Lambda(V_\alpha, V_\beta) - \dim_k \operatorname{Ext}_\Lambda(V_\alpha, V_\beta)$$

with $V_\alpha \in \alpha, V_\beta \in \beta$ and $g_{\alpha\beta}^\lambda$ is the number of submodules X of V_λ such that V_λ / X and X lie in the isomorphism classes α and β, respectively. Let $I \subset \mathcal{P}$ be the set of isomorphism classes of simple Λ-modules. Ringel [R2,R3] has proved there exists a canonical isomorphism η between the Lusztig's quantum group f and $\mathcal{H}(\Lambda)$ such that $\eta(\theta_i) = u_i, i \in I$, if they enjoy a common Dynkin diagram. Furthermore, the generic form $\mathcal{H}(\Delta)$ of $\mathcal{H}(\Lambda)$ is canonically isomorphic to the Drinfeld-Jimbo quantum group $\mathbf{U_q^+}(\mathbf{g})$ if they both enjoy a common Dynkin diagram. Under the canonical isomorphism, Ringel has shown that the set $\{u_\lambda | \lambda \in \mathcal{P}, V_\lambda \text{ indecomposable}\}$ just provide a complete set of root vectors of $\mathbf{U_q^+}(\mathbf{g})$.

1.3. One may ask the question: how to decompose the root vectors into the combinations of the generators of $\mathbf{U_q^+}(\mathbf{g})$? If we apply the Lusztig's braid group operations on $\mathbf{U_q^+}(\mathbf{g})$ to deal with this question, a trouble appears immediately. Namely, we cannot expect that the computation is undertaken in the interior of $\mathbf{U_q^+}(\mathbf{g})$. Oppositely, it often goes across into the negative part $\mathbf{U_q^-}(\mathbf{g})$, although the final result of the calculation lies in $\mathbf{U_q^+}(\mathbf{g})$. Ringel has pointed out in [R4] that the braid group action on the exceptional sequences of Λ-mod in the sense of Crawley-Boevey [CB] may provide an inductive constructure of root vectors. We will give a refinement of his idea in the present paper. Namely, we will give an accurate recursive formula to express u_λ as the combinations of $u_i (i \in I)$ for any exceptional module $\lambda \in \mathcal{P}$. According to our formula, the whole calculation is going on in the interior of $\mathbf{U_q^+}(\mathbf{g})$ moreover, it does not depend on the constructure of the exceptional sequences.

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It is well known that any indecomposable Λ-module is exceptional if Λ is finite representation type. So our result is suitable for the quantum enveloping algebra of semisimple Lie algebra. In fact our research is made in the Hall algebras and the composition algebras of any type. Because of the fundamental theorem of Green and Ringel, namely, there exists a canonical isomorphism between $\mathbf{U}_{\mathbf{q}}^{+}(\mathbf{g})$ and the generic composition algebra (see [Gr], [R5]), our result is also suitable for the quantum enveloping algebra of any symmetrizable Kac–Moody algebra.

A complete statement of our result is Theorem 5.1.

2. The Hall Algebra of a Hereditary Algebra

2.1. Let Λ be a finite-dimensional hereditary algebra over a finite field k, $\mathcal{P}$ be the set of isomorphism classes of finite-dimensional Λ-modules, $I \subset \mathcal{P}$ the set of isomorphism classes of simple Λ-modules. We choose a representative $V_\alpha \in \alpha$ for any $\alpha \in \mathcal{P}$. Given Λ-modules M, N, let

$$\langle M, N\rangle = \dim_k \operatorname{Hom}_\Lambda(M, N) - \dim_k \operatorname{Ext}_\Lambda(M, N).$$

Since Λ is hereditary, $\langle M, N\rangle$ depends only on the dimension vectors $\underline{\dim}\, M$ and $\underline{\dim}\, N$. For $\alpha, \beta \in \mathcal{P}$ we write $\langle \alpha, \beta\rangle = \langle V_\alpha, V_\beta\rangle$. So the Ringel form $\langle -, -\rangle$ is defined on $\mathbf{Z}[I]$. The Ringel symmetric form $(-, -)$ is given by $(\alpha, \beta) = \langle \alpha, \beta\rangle + \langle \beta, \alpha\rangle$ on $\mathbf{Z}[I]$.

2.2. Let R be a commutative integral domain containing $\mathbf{Q}(v)$, where $v^2 = q, q = |k|$ and $\mathbf{Q}(v)$ is the rational function field of v. The Hall algebra $\mathcal{H}(\Lambda)$ is by definition the free R-module with the basis $\{u_\alpha | \alpha \in \mathcal{P}\}$ and the multiplication given as $u_\alpha u_\beta = v^{\langle \alpha, \beta\rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha\beta}^{\lambda} u_\lambda$ for all $\alpha, \beta \in \mathcal{P}$. It is easy to verify that $\mathcal{H}(\Lambda)$ is an associative $\mathbf{N}[I]$-graded R-algebra with the identity element u_0. The grading $\mathcal{H}(\Lambda) = \oplus_{r \in \mathbf{N}[I]} \mathcal{H}_r$ is defined as follows: for each $r \in \mathbf{N}[I]$, $\mathcal{H}_r$ is the R-span of the set $\{u_\lambda | \lambda \in \mathcal{P}, \underline{\dim}\, V_\lambda = r\}$. Ringel [R2, R3] has proved that the elements $u_i, i \in I$ satisfy the quantum Serre relations

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1 - a_{ij} \\ t \end{bmatrix}_{\varepsilon_i} u_i^t u_j u_i^{1-a_{ij}-t} = 0$$

for any $i \neq j$ in I, where $a_{ij} = 2\frac{(i,j)}{(i,i)}$, $\varepsilon_i = \dim_k \operatorname{End}(V_i)$.

2.3. Of course, a strong feature of a quantum group is its Hopf algebra structure. According to Ringel–Green theory, we can get the Hopf algebra structure on the Hall algebra by adding the torus algebra $\mathbf{T}$ to it. Let $\mathcal{H}(\Lambda, \mathbf{T})$ be a free R-module with the basis $\{K_\alpha u_\lambda | \alpha \in \mathbf{Z}[I], \lambda \in \mathcal{P}\}$, it is a Hopf algebra under the following operations (see [X]).

(1) Multiplication:

$$u_\alpha u_\beta = v^{\langle \alpha, \beta\rangle} \sum_{\lambda \in \mathcal{P}} g_{\alpha\beta}^{\lambda} u_\lambda \quad \text{for any } \alpha, \beta \in \mathcal{P},$$

$$K_\alpha u_\beta = v^{(\alpha,\beta)} u_\beta K_\alpha \quad \text{for } \alpha \in \mathbf{Z}[I], \beta \in \mathcal{P},$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \quad \text{for } \alpha, \beta \in \mathbf{Z}[I],$$

with identity element $u_0 = K_0 = 1$.

(2) Comultiplication:

$$r(u_\lambda) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\alpha K_\beta \otimes u_\beta, \quad \text{where } \lambda \in \mathcal{P},$$

$$r(K_\alpha) = K_\alpha \otimes K_\alpha, \quad \text{where } \alpha \in \mathbf{Z}[I],$$

with counit $\epsilon u_\lambda = 0$ for any $\lambda \neq 0, \lambda \in \mathcal{P}$ and $\epsilon K_\alpha = 1$ for any $\alpha \in \mathbf{Z}[I]$.

(3) Antipode:

$$\sigma(u_\lambda) = \delta_{\lambda 0} + \sum_{m\geqslant 1} (-1)^m \sum_{\substack{\pi\in\mathcal{P} \\ \lambda_1,\ldots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} \times$$

$$\times g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} K_{-\lambda} u_\pi \quad \sigma(K_\alpha) = K_{-\alpha},$$

where $\lambda \in \mathcal{P}$, $\alpha \in \mathbf{Z}[I]$ and $\mathcal{P}_1 = \mathcal{P} - \{0\}$.

We denote by $\mathcal{C}(\Lambda)$ the R-subalgebra of $\mathcal{H}(\Lambda)$ which is generated by u_i $(i \in I)$, by $\mathcal{C}(\Lambda, \mathbf{T})$ the subalgebra of $\mathcal{H}(\Lambda, \mathbf{T})$ which is generated by u_i $(i \in I)$ and K_α $(\alpha \in \mathbf{Z}[I])$. Clearly $\mathcal{C}(\Lambda, \mathbf{T})$ is a Hopf subalgebra of $\mathcal{H}(\Lambda, \mathbf{T})$. The subalgebra $\mathcal{C}(\Lambda)$ is called the *composition algebra.*

As an important case, when Λ is finite representation type, i.e, the corresponding Lie algebra $\mathbf{g}$ is finite-dimensional complex semi-simple, the Hall algebra coincides with its composition subalgebra.

2.4. The above fact depends on the following Green formula.

For any $\alpha, \beta, \alpha_1, \beta_1 \in \mathcal{P}$, let

$$N_1 = N_1(\alpha, \beta, \alpha_1, \beta_1) = \sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha\beta} g^\lambda_{\alpha_1\beta_1} \frac{a_\alpha a_\beta a_{\alpha_1} a_{\beta_1}}{a_\lambda},$$

$$N_2 = N_2(\alpha, \beta, \alpha_1, \beta_1) = \sum_{\rho,\sigma,\sigma_1,\tau_1\in\mathcal{P}} q^{-\langle\rho,\tau\rangle} g^\alpha_{\rho\sigma} g^{\alpha_1}_{\rho\sigma_1} g^\beta_{\sigma_1\tau} g^{\beta_1}_{\sigma\tau} a_\rho a_\sigma a_{\sigma_1} a_\tau$$

then $N_1 = N_2$ (see [Gr])

2.5. Let $\bar{k}$ be the algebraic closure of k. For any $n \in \mathbf{N}$, let $F(n)$ be a subfield of $\bar{k}$ such that $[F(n){:}\,k] = n$. We define $\Lambda(n) = \Lambda \otimes_k F(n)$, then $\Lambda(n)$ is a finite-dimensional hereditary $F(n)$-algebra corresponding to the same Cartan datum as that of Λ. We also have the Hall algebra $\mathcal{H}_n = \mathcal{H}_n(\Lambda(n))$ of $\Lambda(n)$. Define $\Pi = \prod_{n>0} \mathcal{H}_n$. Let $v = (v_n)_n \in \Pi$ where $v_n = \sqrt{|F(n)|}$. Obviously v lies in the center of Π and is transcendental over the rational field $\mathbf{Q}$. Let $u_i = (u_i(n))_n \in \Pi$

satisfy that $u_i(n)$ is the element of $\mathcal{H}(\Lambda(n))$ corresponding to $V_i(n)$, where $V_i(n)$ is the simple $\Lambda(n)$-module which lies in the class i. The generic composition algebra $\mathcal{C}(\Delta)$ of the Cartan datum Δ is defined to be the subring of Π generated by the elements v, v^{-1} and u_i $(i \in I)$. Let $\mathbf{U}_\mathbf{q}^+(\mathbf{g})$ be the positive part of the Drinfeld–Jimbo quantum group corresponding to the Cartan datum Δ. A fundamental theorem of Green and Ringel concludes that the mapping $\eta\colon \mathbf{U}_\mathbf{q}^+(\mathbf{g}) \to \mathcal{C}(\Delta)$ with $\eta(E_i) = u_i$ $(i \in I)$ is a bijection as associative algebras. Moreover, if we extend $\mathbf{U}_\mathbf{q}^+(\mathbf{g})$ to the Borel part of $\mathbf{U}_\mathbf{q}(\mathbf{g})$ and $\mathcal{C}(\Delta)$ to $\mathcal{C}(\Delta, \mathbf{T})$, then η can be extended canonically to be a bijection as Hopf algebras.

In the following, our results are stated for the Hall algebra $\mathcal{H}(\Lambda)$ and for the composition algebra $\mathcal{C}(\Lambda)$. Without any changes, the same conclusions hold for the corresponding generic composition algebra $\mathcal{C}(\Delta)$. For any Cartan datum $\Delta = (I, (,))$ (see [L]) and any finite field k, there exists a finite-dimensional hereditary k-algebra Λ such that the symmetric Ringel form $(-,-)$ of Λ together with the index set I of simple Λ-modules gives a realization of Δ (see [R5]). So our result is really for the quantum group $\mathbf{U}_\mathbf{q}^+(\mathbf{g})$ of any symmetrizable Kac–Moody algebra $\mathbf{g}$.

2.6. An indecomposible Λ-module V_α is called *exceptional* provided $\mathrm{Ext}^1_\Lambda(V_\alpha, V_\alpha) = 0$. Note that the endomorphism ring of an exceptional Λ-module is always a division ring; in our case, it is a finite field. A pair (V_α, V_β) of exceptional Λ-modules is called an *exceptional pair* provided we have in addition

$$\mathrm{Hom}_\Lambda(V_\beta, V_\alpha) = \mathrm{Ext}^1_\Lambda(V_\beta, V_\alpha) = 0.$$

A sequence $(V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n})$ is called *exceptional* provided any pair $(V_{\alpha_i}, V_{\alpha_j})$ with $i < j$ is exceptional. An exceptional sequence $(V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n})$ is said to be *complete* provided $n = |I|$: the number of isomorphism classes of simple Λ-modules.

For any exceptional sequence $(V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_s})$, let $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ be the smallest full subcategory of Λ-mod which contains $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_s}$ and is closed under extensions, kernels of epimorphisms and cokernels of monomorphisms. By Crawley-Boevey [CB] and Ringel [R6], we know that $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is equivalent to the module category of a finite-dimensional hereditary algebra with precisely s isomorphism classes of simple modules. Moreover, the functor from $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ to this module category is exact and induce isomorphisms on both Hom and Ext. Thus we can talk about simple objects, projective and injective objects of $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, exceptional sequence for $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$, etc.

Since the endomorphism ring of any simple object of $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is a finite field, $\mathcal{C}(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is equivalent to the module category of a finite dimensional hereditary algebra over a finite field. In particular, if (V_α, V_β) is an exceptional pair, then $C(\alpha, \beta)$ is equivalent to the module category of a finite-dimensional hereditary algebra over a finite field with two isomorphism classes of simple modules. Since this kind of algebras have no regular exceptional module, (see [R1]), (V_α, V_β) must be one of the following cases:

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(1) (V_α, V_β) are slice modules in the preprojective component of $C(\alpha, \beta)$.
(2) (V_α, V_β) are slice modules in the preinjective component of $C(\alpha, \beta)$.
(3) (V_α, V_β) is orthogonal pair, i.e., $\mathrm{Hom}_\Lambda(V_\alpha, V_\beta) = \mathrm{Hom}_\Lambda(V_\beta, V_\alpha) = 0$. In this case, V_α is the simple injective object and V_β is the simple projective object of $\mathcal{C}(\alpha, \beta)$.

2.7. If (V_α, V_β) is an exceptional pair, then there are unique modules $L(\alpha, \beta)$ and $R(\alpha, \beta)$ with the property that $(L(\alpha, \beta), V_\alpha)$ and $(V_\beta, R(\alpha, \beta))$ are exceptional pair in $C(\alpha, \beta)$. Moreover, if $A = (V_{\alpha_1}, V_{\alpha_2}, \dots, V_{\alpha_n})$ is a complete exceptional sequence, let $1 \leqslant i < n$, there are uniquely determined exceptional sequences $B = (V_{\beta_1}, V_{\beta_2}, \dots, V_{\beta_n})$ and $D = (V_{\gamma_1}, V_{\gamma_2}, \dots, V_{\gamma_n})$ such that $V_{\beta_j} = V_{\gamma_j} = V_{\alpha_j}$ for all $j \notin \{i, i+1\}$ and $V_{\beta_{i+1}} = V_{\alpha_i}, V_{\beta_i} = L(\alpha_i, \alpha_{i+1}), V_{\gamma_i} = V_{\alpha_{i+1}}, V_{\gamma_{i+1}} = R(\alpha_i, \alpha_{i+1})$. Recall that the braid group $\mathcal{B}_n$ in $n-1$ generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ is the free group with these generators and the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $1 \leqslant i < n-1$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $j \geqslant i+2$. We define $\sigma_i A = D, \sigma_i^{-1} A = B$, in this way, we obtain an action of the braid group $\mathcal{B}_{n-1}$ on the set of complete exceptional sequences (see [CB] [R6]). The above result is valid for arbitrary hereditary Artin algebra, it is not necessary to assume the basis field is an algebric closed field (see [R6]).

The effectiveness of our algorithm will base on the following three facts:

(1) Any exceptional module can be enlarged to a complete exceptional sequence. (Bongartz Lemma).

(2) The action of the braid group $\mathcal{B}_{n-1}$ on the set of complete exceptional sequences is transitive. (the theorem of Crawley-Boevey).

(3) Let V_λ and V_ρ be two exceptional modules, if $\underline{\dim} V_\lambda = \underline{\dim}\, V_\rho$, then $V_\lambda \simeq V_\rho$. (for example, see [Ke]).

2.8. In the quantum group and the Hall algebra, the following notations and relations are often used.

$$(1) \quad [s] = \frac{v^s - v^{-s}}{v - v^{-1}} = v^{s-1} + v^{s-3} + \cdots + v^{-s+1}, \qquad [s]! = \prod_{r=1}^{s} [r],$$

$$\begin{bmatrix} s \\ r \end{bmatrix} = \frac{[s]!}{[r]![s-r]!}, \qquad |s] = \frac{q^s - 1}{q - 1} = q^{s-1} + \cdots + q + 1,$$

$$|s]! = \prod_{r=1}^{s} |r], \qquad \left| \begin{matrix} s \\ r \end{matrix} \right] = \frac{|s]!}{|r]!|s-r]!}.$$

We set $q = v^2$, there are the following relations:

$$|s] = v^{s-1}[s], \qquad |s]! = v^{\frac{s(s-1)}{2}}[s]!, \qquad \left| \begin{matrix} s \\ r \end{matrix} \right] = v^{r(s-r)} \begin{bmatrix} s, \\ r \end{bmatrix}.$$

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If we assume $q = |k|$, then $\left[{s \atop r} \right]$ is just the number of r-dimensional subspaces of k^s. For given a polynomial $f \in \mathbf{Z}[v, v^{-1}]$ and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a.

The following formula is often used too.

$$(2) \qquad \sum_{t=0}^{s} (-1)^t v^{t(s-1)} \begin{bmatrix} s \\ t \end{bmatrix} = 0, \quad \text{for } s > 0.$$

(3) For any exceptional module V_λ, set $u_\lambda^{(t)} = (1/[t]!_{\varepsilon(\lambda)}) u_\lambda^t$ in Hall algebras, where $\varepsilon(\lambda) = \dim_k \operatorname{End}_\Lambda V_\lambda$. We have the following identities: $u_\lambda^{(t)} = (v^{\varepsilon(\lambda)})^{t(t-1)} u_{t\lambda}$, where $u_{t\lambda} = u_{\underbrace{\lambda \oplus \lambda \cdots \oplus \lambda}_{t}}$.

2.9. Let $A, B, X \in \Lambda$-mod, there is a homological formula to calculate the filtration number $g_{A,B}^X$ according to Riedtmann [Rie] and Peng [P].

LEMMA. *For any $X, A, B \in \Lambda$-mod, we have*

$$g_{A,B}^X = \frac{|\operatorname{Ext}^1_\Lambda(A, B)_X||\operatorname{Aut}_\Lambda X|}{|\operatorname{Aut}_\Lambda A||\operatorname{Aut}_\Lambda B||\operatorname{Hom}_\Lambda(A, B)|},$$

where $\operatorname{Ext}^1_\Lambda(A, B)_X$ is the set of all exact sequence in $\operatorname{Ext}^1_\Lambda(A, B)$ with middle term X.

3. Some Derivation of a Hall Algebra

3.1. For any $\alpha \in \mathcal{P}$, we denote by δ_α and ${}_\alpha\delta$ such that $\delta_\alpha, {}_\alpha\delta \in \operatorname{Hom}_R(\mathcal{H}(\Lambda), \mathcal{H}(\Lambda))$ given by respectively,

$${}_\alpha\delta(u_\lambda) = \sum_{\beta \in \mathcal{P}} v^{\langle \alpha, \beta \rangle} g_{\alpha\beta}^\lambda \frac{a_\beta}{a_\lambda} u_\beta = v^{\langle \alpha, \lambda \rangle - \langle \alpha, \alpha \rangle} \sum_{\beta \in \mathcal{P}} g_{\alpha\beta}^\lambda \frac{a_\beta}{a_\lambda} u_\beta,$$

$$\delta_\alpha(u_\lambda) = \sum_{\beta \in \mathcal{P}} v^{\langle \beta, \alpha \rangle} g_{\beta\alpha}^\lambda \frac{a_\beta}{a_\lambda} u_\beta = v^{\langle \lambda, \alpha \rangle - \langle \alpha, \alpha \rangle} \sum_{\beta \in \mathcal{P}} g_{\beta\alpha}^\lambda \frac{a_\beta}{a_\lambda} u_\beta,$$

where $\lambda, \alpha \in \mathcal{P}$. In particular, for any $i \in I$,

$${}_i\delta(u_\lambda) = \sum_{\beta \in \mathcal{P}} v^{\langle i, \beta \rangle} g_{i\beta}^\lambda \frac{a_\beta}{a_\lambda} u_\beta, \qquad \delta_i(u_\lambda) = \sum_{\beta \in \mathcal{P}} v^{\langle \beta, i \rangle} g_{\beta i}^\lambda \frac{a_\beta}{a_\lambda} u_\beta,$$

where $\lambda \in \mathcal{P}$. It is easily checked that for $i, j \in I$,

$${}_i\delta(u_j) = \frac{\delta_{ij}}{a_i} = \frac{\delta_{ij}}{(v^{(i,i)} - 1)}, \qquad \delta_i(u_j) = \frac{\delta_{ij}}{a_i} = \frac{\delta_{ij}}{(v^{(i,i)} - 1)}.$$

PROPOSITION 3.2. *For any* $i \in I$ *and* $\lambda_1, \lambda_2 \in \mathcal{P}$, *we have*

(1) $\quad {}_i\delta(u_{\lambda_1}u_{\lambda_2}) = {}_i\delta(u_{\lambda_1})u_{\lambda_2} + v^{(i,\lambda_1)}u_{\lambda_1}({}_i\delta)(u_{\lambda_2}),$

(2) $\quad \delta_i(u_{\lambda_1}u_{\lambda_2}) = v^{(i,\lambda_2)}\delta_i(u_{\lambda_1})u_{\lambda_2} + u_{\lambda_1}\delta_i(u_{\lambda_2}).$

Proof. We only prove (1), It is similar for (2).

$$\begin{aligned}
{}_i\delta(u_{\lambda_1}u_{\lambda_2}) &= {}_i\delta\left(\sum_{\lambda\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda}_{\lambda_1\lambda_2} u_\lambda\right)\\
&= \sum_{\lambda\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda}_{\lambda_1\lambda_2}\, {}_i\delta(u_\lambda)\\
&= \sum_{\lambda,\beta\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle+\langle i,\beta\rangle} g^{\lambda}_{\lambda_1\lambda_2} g^{\lambda}_{i\beta} \frac{a_\beta}{a_\lambda} u_\beta,\\
{}_i\delta(u_{\lambda_1})u_{\lambda_2} &= \left(\sum_{\beta_1\in\mathcal{P}} v^{\langle i,\beta_1\rangle} g^{\lambda_1}_{i\beta_1} \frac{a_{\beta_1}}{a_{\lambda_1}} u_{\beta_1}\right) u_{\lambda_2}\\
&= \sum_{\beta,\beta_1,\in\mathcal{P}} v^{\langle i,\beta_1\rangle+\langle\beta_1,\lambda_2\rangle} g^{\lambda_1}_{i\beta_1} g^{\beta}_{\beta_1\lambda_2} \frac{a_{\beta_1}}{a_{\lambda_1}} u_\beta,\\
u_{\lambda_1}({}_i\delta)(u_{\lambda_2}) &= u_{\lambda_1}\left(\sum_{\beta_2\in\mathcal{P}} v^{\langle i,\beta_2\rangle} g^{\lambda_2}_{i\beta_2} \frac{a_{\beta_2}}{a_{\lambda_2}} u_{\beta_2}\right)\\
&= \sum_{\beta_2,\beta\in\mathcal{P}} v^{\langle i,\beta_2\rangle+\langle\lambda_1,\beta_2\rangle} g^{\lambda_2}_{i\beta_2} g^{\beta}_{\lambda_1\beta_2} \frac{a_{\beta_2}}{a_{\lambda_2}} u_\beta.
\end{aligned}$$

In order to prove (1), we only need to prove the following equation $(*)$:

$$\begin{aligned}
&\sum_{\lambda\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle+\langle i,\beta\rangle} g^{\lambda}_{\lambda_1\lambda_2} g^{\lambda}_{i\beta} \frac{a_\beta}{a_\lambda}\\
&\quad = \sum_{\beta_1\in\mathcal{P}} v^{\langle i,\beta_1\rangle+\langle\beta_1,\lambda_2\rangle} g^{\lambda_1}_{i\beta_1} g^{\beta}_{\beta_1\lambda_2} \frac{a_{\beta_1}}{a_{\lambda_1}} + v^{(i,\lambda_1)} \sum_{\beta_2\in\mathcal{P}} v^{\langle i,\beta_2\rangle+\langle\lambda_1,\beta_2\rangle} g^{\lambda_2}_{i\beta_2} g^{\beta}_{\lambda_1\beta_2} \frac{a_{\beta_2}}{a_{\lambda_2}}. \qquad (*)
\end{aligned}$$

Let

$$N_1 = N_1(\lambda_1,\lambda_2,i,\beta) = \sum_{\lambda\in\mathcal{P}} g^{\lambda}_{\lambda_1\lambda_2} g^{\lambda}_{i\beta} \frac{a_{\lambda_1}a_{\lambda_2}a_i a_\beta}{a_\lambda},$$

$$N_2 = N_2(\lambda_1,\lambda_2,i,\beta) = \sum_{\rho,\sigma,\rho_1,\sigma_1\in\mathcal{P}} |k|^{-\langle\rho,\sigma_1\rangle} g^{\lambda_1}_{\rho\rho_1} g^{\lambda_2}_{\sigma\sigma_1} g^{i}_{\rho\sigma} g^{\beta}_{\rho_1\sigma_1} a_\rho a_\sigma a_{\rho_1} a_{\sigma_1}.$$

We know that $g^{i}_{\rho\sigma} \neq 0$ if and only if $\rho = 0, \sigma = i$ or $\rho = i, \sigma = 0$. $\rho = 0, \sigma = i$ implies $\rho_1 = \lambda_1$. $\rho = i, \sigma = 0$ implies $\sigma_1 = \lambda_2$. Thus

$$N_2(\lambda_1, \lambda_2, i, \beta) = \sum_{\rho_1 \in \mathcal{P}} |k|^{-\langle i,\lambda_2\rangle} g^{\lambda_1}_{i\rho_1} g^{\beta}_{\rho_1\lambda_2} a_i a_{\rho_1} a_{\lambda_1} + \sum_{\sigma_1 \in \mathcal{P}} g^{\lambda_2}_{i\sigma_1} g^{\beta}_{\lambda_1\sigma_1} a_i a_{\lambda_1} a_{\sigma_1}.$$

By replacing ρ_1 by β_1 and σ_1 by β_2, we have

$$N_2(\lambda_1, \lambda_2, i, \beta) = \sum_{\beta_1 \in \mathcal{P}} |k|^{-\langle i,\lambda_2\rangle} g^{\lambda_1}_{i\beta_1} g^{\beta}_{\beta_1\lambda_2} a_i a_{\beta_1} a_{\lambda_2} + \sum_{\beta_2 \in \mathcal{P}} g^{\lambda_2}_{i\beta_2} g^{\beta}_{\lambda_1\beta_2} a_i a_{\lambda_1} a_{\beta_2}.$$

In order to prove $(*)$, we need to show the following identity:

$$\begin{aligned}
&\sum_{\lambda \in \mathcal{P}} g^{\lambda}_{\lambda_1\lambda_2} g^{\lambda}_{i\beta} \frac{a_i a_\beta a_{\lambda_1} a_{\lambda_2}}{a_\lambda} \\
&\quad = \sum_{\beta_1 \in \mathcal{P}} v^{\langle i,\beta_1\rangle + \langle \beta_1,\lambda_2\rangle - \langle \lambda_1,\lambda_2\rangle - \langle i,\beta\rangle} g^{\lambda_1}_{i\beta_1} g^{\beta}_{\beta_1\lambda_2} a_i a_{\beta_1} a_{\lambda_2} + \\
&\qquad + \sum_{\beta_2 \in \mathcal{P}} v^{\langle i,\beta_2\rangle + \langle \lambda_1,\beta_2\rangle + (i,\lambda_1) - \langle \lambda_1,\lambda_2\rangle - \langle i,\beta\rangle} g^{\lambda_2}_{i\beta_2} g^{\beta}_{\lambda_1\beta_2} a_{\beta_2} a_{\lambda_1} a_i .
\end{aligned}$$

We may assume here that

$$\begin{aligned}
\underline{\dim}\, V_{\lambda_1} &= \underline{\dim}\, V_i + \underline{\dim}\, V_{\beta_1}, &\qquad \underline{\dim}\, V_\beta &= \underline{\dim}\, V_{\beta_1} + \underline{\dim}\, V_{\lambda_2}, \\
\underline{\dim}\, V_{\lambda_2} &= \underline{\dim}\, V_i + \underline{\dim}\, V_{\beta_2}, &\qquad \underline{\dim}\, V_\beta &= \underline{\dim}\, V_{\lambda_1} + \underline{\dim}\, V_{\beta_2},
\end{aligned}$$

thus

$$\begin{aligned}
&\langle i, \beta_1\rangle + \langle \beta_1, \lambda_2\rangle - \langle \lambda_1, \lambda_2\rangle - \langle i, \beta\rangle \\
&\quad = \langle i, \beta_1 - \beta\rangle + \langle \beta_1 - \lambda_1, \lambda_2\rangle \\
&\quad = \langle i, -\lambda_2\rangle + \langle -i, \lambda_2\rangle \\
&\quad = -2\langle i, \lambda_2\rangle \langle i, \beta_2\rangle + \langle \lambda_1, \beta_2\rangle + (i, \lambda_1) - \langle \lambda_1, \lambda_2\rangle - \langle i, \beta\rangle \\
&\quad = \langle i, \beta_2\rangle + \langle \lambda_1, \beta_2\rangle + \langle i, \lambda_1\rangle + \langle \lambda_1, i\rangle - \langle \lambda_1, \lambda_2\rangle - \langle i, \beta\rangle \\
&\quad = \langle i, \beta_2 + \lambda_1\rangle + \langle \lambda_1, \beta_2 + I\rangle - \langle \lambda_1, \lambda_2\rangle - \langle i, \beta\rangle \\
&\quad = \langle i, \beta\rangle + \langle \lambda_1, \lambda_2\rangle - \langle \lambda_1, \lambda_2\rangle - \langle i, \beta\rangle = 0.
\end{aligned}$$

So in order to prove $(*)$, we just verify the following identity

$$\sum_{\lambda \in \mathcal{P}} g^{\lambda}_{\lambda_1\lambda_2} g^{\lambda}_{i\beta} \frac{a_i a_\beta a_{\lambda_1} a_{\lambda_2}}{a_\lambda} = \sum_{\beta_1 \in \mathcal{P}} v^{-2\langle i,\lambda_2\rangle} g^{\lambda_1}_{i\beta_1} g^{\beta}_{\beta_1\lambda_2} a_i a_{\beta_1} a_{\lambda_2} + \sum_{\beta \in \mathcal{P}} g^{\lambda_2}_{i\beta_2} g^{\beta}_{\lambda_1\beta_2} a_i a_{\lambda_1} a_{\beta_2}.$$

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But we have set $v^2 = |k|$, so $v^{-2\langle i,\lambda_2\rangle} = |k|^{-\langle i,\lambda_2\rangle}$. This is just the equation $N_1(\lambda_1, \lambda_2, i, \beta) = N_2\ (\lambda_1, \lambda_2, i, \beta)$ by the Green formula in 2.4.

3.3. We call ${}_i\delta$ and δ_i the left and right derivation of the Hall algebra $\mathcal{H}(\Lambda)$ respectively. In general, if $\alpha \notin I$, ${}_\alpha\delta$, δ_α have not the property as the same as ${}_i\delta$, δ_i in Proposition 3.2. We call ${}_\alpha\delta$, δ_α the high order derivations of $\mathcal{H}(\Lambda)$.

3.4. We consider the following linear maps:

$$\begin{aligned} \phi_1\colon \mathcal{H}(\Lambda) &\longrightarrow \mathrm{Hom}_R(\mathcal{H}(\Lambda), \mathcal{H}(\Lambda)) \\ u_\lambda &\longrightarrow {}_\lambda\delta \\ \phi_2\colon \mathcal{H}(\Lambda) &\longrightarrow \mathrm{Hom}_R(\mathcal{H}(\Lambda), \mathcal{H}(\Lambda)) \\ u_\lambda &\longrightarrow \delta_\lambda, \end{aligned}$$

where $\lambda \in \mathcal{P}$.

PROPOSITION.

(1) ϕ_1 *is an anti-homomorphism, i.e* $\phi_1(u_{\lambda_1}u_{\lambda_2}) = \phi_1(u_{\lambda_2})\phi_1(u_{\lambda_1})$
(2) ϕ_2 *is a homomorphism, i.e* $\phi_2(u_{\lambda_1}u_{\lambda_2}) = \phi_2(u_{\lambda_1})\phi_2(u_{\lambda_2})$.

Proof. We only prove (1), the proof of (2) is similar.

$$\begin{aligned} \phi_1(u_{\lambda_1}u_{\lambda_2}) &= \phi_1\left(\sum_{\lambda_3\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} u_{\lambda_3}\right) \\ &= \sum_{\lambda_3\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} \phi_1(u_{\lambda_3}), \\ \phi_1(u_{\lambda_1}u_{\lambda_2})(u_\lambda) &= \sum_{\lambda_3\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} \phi_1(u_{\lambda_3})(u_\lambda) \\ &= \sum_{\lambda_3\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} ({}_{\lambda_3}\delta)(u_\lambda) \\ &= \sum_{\lambda_3\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} \left(\sum_{\beta_1\in\mathcal{P}} v^{\langle\lambda_3,\beta_1\rangle} g^{\lambda}_{\lambda_3,\beta_1} \frac{a_{\beta_1}}{a_\lambda} u_{\beta_1}\right) \\ &= \sum_{\lambda_3,\beta_1\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle+\langle\lambda_3,\beta_1\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} g^{\lambda}_{\lambda_3,\beta_1} \frac{a_{\beta_1}}{a_\lambda} u_{\beta_1}, \end{aligned}$$

$$\begin{aligned}
\phi_1(u_{\lambda_2})\phi_1(u_{\lambda_1})(u_\lambda) &= \phi_1(u_{\lambda_2})({}_{\lambda_1}\delta(u_\lambda)) \\
&= \phi_1(u_{\lambda_2})\left(\sum_{\beta\in\mathcal{P}} v^{\langle\lambda_1,\beta\rangle} g^{\lambda}_{\lambda_1,\beta}\frac{a_\beta}{a_\lambda}u_\beta\right) \\
&= \sum_{\beta\in\mathcal{P}} v^{\langle\lambda_1,\beta\rangle} g^{\lambda}_{\lambda_1,\beta}\frac{a_\beta}{a_\lambda}({}_{\lambda_2}\delta)(u_\beta) \\
&= \sum_{\beta,\beta_1\in\mathcal{P}} v^{\langle\lambda_1,\beta\rangle} g^{\lambda}_{\lambda_1,\beta}\frac{a_\beta}{a_\lambda}\left(v^{\langle\lambda_2,\beta_1\rangle} g^{\beta}_{\lambda_2,\beta_1}\frac{a_{\beta_1}}{a_\beta}u_{\beta_1}\right) \\
&= \sum_{\beta,\beta_1\in\mathcal{P}} v^{\langle\lambda_1,\beta\rangle+\langle\lambda_2,\beta_1\rangle} g^{\lambda}_{\lambda_1,\beta} g^{\beta}_{\lambda_2,\beta_1}\frac{a_{\beta_1}}{a_\lambda}u_{\beta_1}.
\end{aligned}$$

At the same time we have

$$\underline{\dim}\, V_\beta = \underline{\dim}\, V_{\lambda_2} + \underline{\dim}\, V_{\beta_1}, \qquad \underline{\dim}\, V_\lambda = \underline{\dim}\, V_{\lambda_1} + \underline{\dim}\, V_\beta,$$

$$\underline{\dim}\, V_\lambda = \underline{\dim}\, V_{\lambda_3} + \underline{\dim}\, V_{\beta_1}, \qquad \underline{\dim}\, V_{\lambda_3} = \underline{\dim}\, V_{\lambda_1} + \underline{\dim}\, V_{\lambda_2}.$$

So

$$\begin{aligned}
\langle\lambda_1,\beta\rangle + \langle\lambda_2,\beta_1\rangle &= \langle\lambda_1,\lambda_2+\beta_1\rangle + \langle\lambda_2,\beta_1\rangle \\
&= \langle\lambda_1,\lambda_2\rangle + \langle\lambda_1+\lambda_2,\beta_1\rangle \\
&= \langle\lambda_1,\lambda_2\rangle + \langle\lambda_3,\beta_1\rangle.
\end{aligned}$$

In order to prove (1), we only need to prove

$$\sum_{\lambda_3\in\mathcal{P}} v^{\langle\lambda_1,\lambda_2\rangle+\langle\lambda_3,\beta_1\rangle} g^{\lambda_3}_{\lambda_1,\lambda_2} g^{\lambda}_{\lambda_3,\beta_1}\frac{a_{\beta_1}}{a_\lambda} = \sum_{\beta\in\mathcal{P}} v^{\langle\lambda_1,\beta\rangle+\langle\lambda_2,\beta_1\rangle} g^{\lambda}_{\lambda_1,\beta} g^{\beta}_{\lambda_2,\beta_1}\frac{a_{\beta_1}}{a_\lambda}.$$

It is reduced to

$$\sum_{\lambda_3\in\mathcal{P}} g^{\lambda_3}_{\lambda_1,\lambda_2} g^{\lambda}_{\lambda_3,\beta_1} = \sum_{\beta\in\mathcal{P}} g^{\lambda}_{\lambda_1,\beta} g^{\beta}_{\lambda_2\beta_1}$$

The two sides of the above equal to $g^{\lambda}_{\lambda_1\lambda_2\beta_1}$. This completes the proof.

Remark 3.5. From Propositions 3.2 and 3.4, if $u_\alpha \in \mathcal{C}(\Lambda)$, then ${}_\alpha\delta\ (\mathcal{C}(\Lambda)) \subseteq \mathcal{C}(\Lambda)$ $\delta_\alpha(\mathcal{C}(\Lambda)) \subseteq \mathcal{C}(\Lambda)$. Moreover if u_α is expressed as the combination of u_i, $i \in I$, then ${}_\alpha\delta$ is immediately expressed as the corresponding combinations of ${}_i\delta$, $i \in I$, and completely same for the expression of δ_α by δ_i.

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4. Exceptional Pair

Let (V_α, V_β) be an exceptional pair of Λ-mod, we denote by

$$n(\alpha,\beta)=\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}=2\frac{(\alpha,\beta)}{(\alpha,\alpha)},\qquad m(\alpha,\beta)=\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}=2\frac{(\alpha,\beta)}{(\beta,\beta)}.$$

4.1. THE CASE OF $n(\alpha,\beta)\underline{\dim}\, v_\alpha > \underline{\dim}\, v_\beta$

LEMMA 4.1.1. *In $\mathcal{C}(\alpha,\beta)$ there exist the relative AR-sequence*

$$0\longrightarrow V_\gamma\longrightarrow n(\alpha,\beta)V_\alpha\longrightarrow V_\beta\longrightarrow 0,$$

where $L(\alpha,\beta)\simeq V_\gamma$

Proof. If $n(\alpha,\beta)\underline{\dim}\, V_\alpha > \underline{\dim}\, V_\beta$, then (V_α, V_β) are the slice modules in the preprojective component or in the preinjective component of $\mathcal{C}(\alpha,\beta)$. The relative irreducible map space $\mathrm{Irr}_{\mathcal{C}(\alpha,\beta)}(V_\alpha, V_\beta)$ equals to $\mathrm{Hom}_\Lambda(V_\alpha, V_\beta)$. Assume

$$0\longrightarrow V_\gamma\longrightarrow nV_\alpha\longrightarrow V_\beta\longrightarrow 0$$

to be an relative AR-sequence in $\mathcal{C}(\alpha,\beta)$, then we know that $n=[\mathrm{Irr}_{\mathcal{C}(\alpha,\beta)}(V_\alpha, V_\beta)\colon \mathrm{End}_\Lambda(V_\alpha)]=[\mathrm{Hom}_\Lambda(V_\alpha,V_\beta)\colon \mathrm{End}_\Lambda(V_\alpha)]$. Since $\mathrm{Ext}^1_\Lambda(V_\alpha,V_\alpha)=0$, $\mathrm{Ext}^1_\Lambda(V_\alpha,V_\beta)=0$, we have $n=n(\alpha,\beta)$. This completes the proof.

LEMMA 4.1.2. *Let $V_\alpha, V_\beta, V_\gamma$ be the same as those in Lemma* 4.1.1, *if $f\colon n(\alpha,\beta)V_\alpha\to V_\beta$ is an epimorphism, then $\ker f\simeq V_\gamma$ or $\ker f\simeq V_{\lambda_i}\oplus iV_\alpha$ with $1\leqslant i\leqslant n-1$, moreover here V_{λ_i} can be embedded into V_γ and no longer contain direct summands which are isomorphic to V_α.*

Proof. We know that there exists a relative AR-sequence in $\mathcal{C}(\alpha,\beta)$

$$0\longrightarrow V_\gamma\xrightarrow{\;b\;} n(\alpha,\beta)V_\alpha\xrightarrow{\;a\;} V_\beta\longrightarrow 0$$

and f is a nonsplit epimorphism, so the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker f & \longrightarrow & n(\alpha,\beta)V_\alpha & \xrightarrow{\;f\;} & V_\beta & \longrightarrow & 0\\
 & & \big\downarrow{\scriptstyle d} & & \big\downarrow{\scriptstyle c} & & \big\| & & \\
0 & \longrightarrow & V_\gamma & \xrightarrow{\;b\;} & n(\alpha,\beta)V_\alpha & \xrightarrow{\;a\;} & V_\beta & \longrightarrow & 0
\end{array}$$

commutates. The morphism c has the form $c=(c_{ij})_{n\times n}$ with $n=n(\alpha,\beta)$ and $c_{ij}\in \mathrm{End}_\Lambda V_\alpha$. But $\mathrm{End}_\Lambda V_\alpha$ is a finite field ,so there exist inverse transformation

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$h_1, h_2 \in M_n(\mathrm{End}_\Lambda V_\alpha)$ such that $h_1 c h_2 = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where I is an unit matrix. We consider the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker f & \xrightarrow{gh_1^{-1}} & n(\alpha,\beta)V_\alpha & \xrightarrow{h_1 f} & V_\beta & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)} & & \Big\| & & \\
0 & \longrightarrow & V_\gamma & \xrightarrow{bh_2} & n(\alpha,\beta)V_\alpha & \xrightarrow{h_2^{-1}a} & V_\beta & \longrightarrow & 0.
\end{array}$$

The second exact sequence is also a relative AR-sequence in $\mathcal{C}(\alpha, \beta)$. If c is non-degenerate, i.e $I = I_{n\times n}$, then $\ker f \simeq V_\gamma$. If c is degenerate, then $\ker f \simeq \ker h_1 f \simeq \ker(\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right))h_2^{-1}a)$. This completes the proof.

LEMMA 4.1.3. *Let $V_\alpha, V_\beta, V_\gamma, V_{\lambda_i}$ be the same as those in Lemma* 4.1.2, *then we have*

$$\begin{aligned}
&[\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma) : \mathrm{End}_\Lambda V_\beta] \\
&\quad = [\mathrm{Ext}^1_\Lambda(V_\beta, V_{\lambda_i} \oplus i V_\alpha) : \mathrm{End}_\Lambda V_\beta] = [\mathrm{Ext}^1_\Lambda(V_\beta, V_{\lambda_i}) : \mathrm{End}_\Lambda V_\beta] = 1.
\end{aligned}$$

Proof. We know that $\mathrm{Hom}_\Lambda(V_\beta, V_\alpha) = \mathrm{Hom}_\Lambda(V_\beta, V_\gamma) = \mathrm{Hom}_\Lambda(V_\beta, V_{\lambda_i} \oplus i V_\alpha) = 0$, $\mathrm{Ext}^1_\Lambda(V_\beta, V_\alpha) = \mathrm{Ext}^1_\Lambda(V_\beta, V_\beta) = \mathrm{Ext}^1_\Lambda(V_\alpha, V_\alpha) = 0$, so $\dim_k \mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma) = -\langle \beta, \gamma \rangle = -\langle \beta, n(\alpha,\beta)\alpha + \beta \rangle = -\langle \beta, n(\alpha,\beta)\alpha \rangle + \langle \beta, \beta \rangle = \dim_k \mathrm{End}_\Lambda V_\beta$. It follows $[\mathrm{Ext}^1_\Lambda(V_\beta, V\gamma) : \mathrm{End}_\Lambda V_\beta] = 1$. By the same reason, we have $[\mathrm{Ext}^1_\Lambda(V_\beta, V_{\lambda_i} \oplus i V_\alpha) : \mathrm{End}_\Lambda V_\beta] = 1$. Since $\mathrm{Ext}^1_\Lambda(V_\beta, V_\alpha) = 0$, we also have $[\mathrm{Ext}^1_\Lambda(V_\beta, V_{\lambda_i}) : \mathrm{End}_\Lambda V_\beta)] = 1$.

For any exceptional module V_λ, we denote by $\varepsilon(\lambda) = \langle \lambda, \lambda \rangle = \frac{1}{2}(\lambda, \lambda) = \dim_k \mathrm{End}_\Lambda V_\lambda$ and for any exceptional pair (V_α, V_β), $n = n(\alpha, \beta)$ for convenience.

LEMMA 4.1.4. *Let $V_\alpha, V_\beta, V_\gamma$ be the same as those in Lemma* 4.1.1, *then in the Hall algebra*

$$u_\beta u_\gamma = v^{\langle \beta,\gamma \rangle} \frac{(q^{\varepsilon(\beta)} - 1)a_{n\alpha}}{a_\beta a_\gamma} u_{n\alpha} + v^{\langle \beta,\gamma \rangle} \frac{a_{\beta\oplus\gamma}}{a_\beta a_\gamma} u_{\beta\oplus\gamma},$$

where $a_\pi = |\mathrm{Aut}_\Lambda V_\pi|$ for $\pi \in \mathcal{P}$.

Proof. By definition $u_\beta u_\gamma = v^{\langle \beta,\gamma \rangle} \sum_{\lambda \in \mathcal{P}} g^\lambda_{\beta\gamma} u_\lambda$. According to the homological formula in 2.9,

$$g^\lambda_{\beta\gamma} = \frac{|\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma)_{V_\lambda}| a_\lambda}{a_\beta a_\gamma |\mathrm{Hom}_\Lambda(V_\beta, V_\gamma)|},$$

where $\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma)_{V_\lambda}$ means the set of all exact sequences in $\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma)$ with middle term V_λ. We know that $[\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma) : \mathrm{End}_\Lambda V_\beta] = 1$ by Lemma 4.1.3 and

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$\mathrm{End}_\Lambda V_\beta$ is a finite field, thus any nonsplit exact sequence is equivalent to an exact sequence of the form

$$0 \longrightarrow V_\gamma \longrightarrow nV_\alpha \longrightarrow V_\beta \longrightarrow 0.$$

So we have $|\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma)_{nV_\alpha}| = |\mathrm{Ext}^1_\Lambda(V_\beta, V_\gamma)| - 1 = q^{\varepsilon(\beta)} - 1$. Also by the fact $\mathrm{Hom}_\Lambda(V_\beta, V_\gamma) = 0$. This completes the proof.

LEMMA 4.1.5. *In the Hall algebra $\mathcal{H}(\Lambda)$, we have*

$${}_\beta\delta(u_{n\alpha}) = v^{-\varepsilon(\beta)} \sum_{\substack{\lambda\in\mathcal{P}\\ g^{n\alpha}_{\beta\lambda}\neq 0}} u_\lambda = v^{-\varepsilon(\beta)} \left(u_\gamma + \sum_{i=1}^{n-1} \sum_{\substack{\lambda_i\in\mathcal{P}\\ g^{n\alpha}_{\beta\lambda_i\oplus i\alpha}\neq 0}} u_{\lambda_i\oplus i\alpha} \right),$$

where λ_i for $1 \leqslant i \leqslant n-1$ and $\alpha, \beta, \gamma, \lambda_i$ are the same as those in Lemma 4.1.2.

Proof. By definition of the derivation, we know that

$${}_\beta\delta(u_{n\alpha}) = v^{\langle\beta,n\alpha\rangle - \langle\beta,\beta\rangle} \sum_{\lambda\in\mathcal{P}} g^{n\alpha}_{\beta\lambda} \frac{a_\lambda}{a_{n\alpha}} u_\lambda.$$

Since (V_α, V_β) is an exceptional pair, i.e $\mathrm{Hom}_\Lambda(V_\beta, V_\alpha) = 0$ and $\mathrm{Ext}^1_\Lambda(V_\beta, V_\alpha) = 0$, it follows that $\langle\beta, n\alpha\rangle = 0$. Because $\mathcal{C}(\alpha, \beta)$ is equivalent to the module category of a finite-dimensional hereditary algebra with two simple modules and (V_α, V_β) are slice modules, then $\mathrm{Hom}_\Lambda(V_\beta, V_\lambda) = 0$ if $g^{n\alpha}_{\beta\lambda} \neq 0$. From Lemma 4.1.3 we know that $[\mathrm{Ext}^1_\Lambda(V_\beta, V_\lambda) \colon \mathrm{End}_\Lambda V_\beta] = 1$. As same as in the proof of Lemma 4.1.4, any nonsplit exact sequence in $\mathrm{Ext}^1_\Lambda(V_\beta, V_\lambda)$ is equivalent to an exact sequence of the form

$$0 \longrightarrow V_\lambda \longrightarrow nV_\alpha \longrightarrow V_\beta \longrightarrow 0.$$

By the homological formula in 2.9 We have

$$g^{n\alpha}_{\beta\lambda} \frac{a_\lambda}{a_{n\alpha}} = \frac{|\mathrm{Ext}^1_\Lambda(V_\beta, V_\lambda)_{n\alpha}|}{a_\beta |\mathrm{Hom}_\Lambda(V_\beta, V_\lambda)|} = \frac{q^{\varepsilon(\beta)} - 1}{a_\beta} = 1,$$

then

$${}_\beta\delta(u_{n\alpha}) = v^{-\varepsilon(\beta)} \sum_{\substack{\lambda\in\mathcal{P}\\ g^{n\alpha}_{\beta\lambda}\neq 0}} u_\lambda.$$

According to Lemma 4.1.2,

$${}_\beta\delta(u_{n\alpha}) = v^{-\varepsilon(\beta)} \left(u_\gamma + \sum_{i=1}^{n-1} \sum_{\substack{\lambda_i\in\mathcal{P}\\ g^{n\alpha}_{\beta,\lambda_i\oplus i\alpha}\neq 0}} u_{\lambda_i\oplus i\alpha} \right).$$

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This completes the proof.

In general, if $g^{n\alpha}_{\beta,\lambda\oplus i\alpha} \neq 0$, then $[\mathrm{Ext}^1_\Lambda(V_\beta, V_\lambda)\colon \mathrm{End}_\Lambda V_\beta] = 1$ by Lemma 4.1.3. In fact $g^{n\alpha}_{\beta\lambda\oplus i\alpha} \neq 0$ if and only if $g^{(n-i)\alpha}_{\beta\lambda} \neq 0$. We also have

$$g^{(n-i)\alpha}_{\beta\lambda}\frac{a_\lambda}{a_{(n-i)\alpha}} = 1,$$

accordingly we have the following results.

LEMMA 4.1.6. *In the Hall algebra $\mathcal{H}(\Lambda)$, we have*

$$
\begin{aligned}
{}_\beta\delta(u_\alpha) &= v^{-\varepsilon(\beta)} \sum_{\substack{\lambda_{n-1}\in\mathcal{P}\\ g^{\alpha}_{\beta,\lambda_{n-1}}\neq 0}} u_{\lambda_{n-1}},\\
{}_\beta\delta(u_{2\alpha}) &= v^{-\varepsilon(\beta)} \left(\sum_{\substack{\lambda_{n-1}\in\mathcal{P}\\ g^{\alpha}_{\beta,\lambda_{n-1}}\neq 0}} u_{\lambda_{n-1}\oplus\alpha} + \sum_{\substack{\lambda_{n-2}\in\mathcal{P}\\ g^{2\alpha}_{\beta,\lambda_{n-2}}\neq 0}} u_{\lambda_{n-2}} \right),\\
&\dots\\
{}_\beta\delta(u_{i\alpha}) &= v^{-\varepsilon(\beta)} \sum_{j=1}^{i} \sum_{\substack{\lambda_{n-j}\in\mathcal{P}\\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}} u_{\lambda_{n-j}\oplus(i-j)\alpha},
\end{aligned}
$$

for $1 \leqslant i \leqslant n$ *and* $\alpha, \beta, \lambda_{n-i}$ *are the same as those in Lemma* 4.1.2, *in particular we take* $\lambda_0 = \gamma$.

Proof. It is the direct consequence of the above consideration.

The following Lemma is well known.

LEMMA 4.1.7.

(1) *Let V_λ be an indecomposable Λ-module with* $\dim_k \mathrm{End}_\Lambda V_\lambda = s$ *and* $\dim_k \mathrm{rad}\,\mathrm{End}_\Lambda V_\lambda = t$, *then* $a_\lambda = (q^{s-t}-1)q^t$.
(2) *Let* $V_\lambda \simeq s_1V_{\lambda_1} \oplus \cdots \oplus s_tV_{\lambda_t}$ *such that* $V_{\lambda_i} \not\simeq V_{\lambda_j}$ *for any* $i \neq j$, *then* $a_\lambda = q^s a_{s_1\lambda_1}\cdots a_{s_t\lambda_t}$, *where* $s = \sum_{i\neq j} s_is_j \dim_k \mathrm{Hom}_\Lambda(V_{\lambda_i}, V_{\lambda_j})$.
(3) *Let* $V_\lambda = sV_\rho$ *with* $\mathrm{End}_\Lambda V_\rho = F$ *and F is a field, then* $a_\lambda = |\mathrm{GL}_s(F)| = \prod_{1\leqslant t\leqslant s}(d^s - d^{t-1})$, *where* $d = |F| = q^{[F:k]}$.

4.1.8. We follow Lemma 4.1.6, if $g^{j\alpha}_{\beta\lambda_{n-j}} \neq 0$, then $V_{\lambda_{n-j}}$ is isomorphic to a submodule of V_γ. We have $\mathrm{Ext}^1_\Lambda(V_{\lambda_{n-j}\oplus(i-j)\alpha}, V_\alpha) \simeq \mathrm{Ext}^1_\Lambda(V_{\lambda_{n-j}}, V_\alpha) = 0$ by Auslander–Reiten formula. By definition

$$u_{\lambda_{n-j}\oplus(i-j)\alpha}u_{(n-i)\alpha} = v^{\langle\lambda_{n-j},(n-i)\alpha\rangle+\langle(i-j)\alpha,(n-i)\alpha\rangle} g^{\lambda_{n-j}\oplus(n-j)\alpha}_{\lambda_{n-j}\oplus(i-j)\alpha,(n-i)\alpha} u_{\lambda_{n-j}\oplus(n-j)\alpha}.$$

Since $\underline{\dim}\, V_{\lambda_{n-j}} + \underline{\dim}\, V_{(i-j)\alpha} = \underline{\dim}\, V_{i\alpha} - \underline{\dim}\, V_{\beta}$ and $\langle \beta, \alpha \rangle = 0$, we obtain

$$\langle \lambda_{n-j}, (n-i)\alpha \rangle + \langle (i-j)\alpha, (n-i)\alpha \rangle$$
$$= \langle i\alpha, (n-i)\alpha \rangle - \langle \beta, (n-i)\alpha \rangle = i(n-i)\varepsilon(\alpha).$$

We claim that

$$g^{\lambda_{n-j}\oplus(n-j)\alpha}_{\lambda_{n-j}\oplus(i-j)\alpha,(n-i)\alpha} = \frac{a_{\lambda_{n-j}\oplus(n-j)\alpha}}{a_{\lambda_{n-j}\oplus(i-j)\alpha} a_{(n-i)\alpha} |\mathrm{Hom}_{\Lambda}(V_{\lambda_{n-j}\oplus(i-j)\alpha}, V_{(n-i)\alpha})|}.$$

By using the fact $\mathrm{Ext}^1_{\Lambda}(V_{\lambda_{n-j}}, V_{\alpha}) = 0$ and $\mathrm{Hom}_{\Lambda}(V_{\alpha}, V_{\lambda_{n-j}}) = 0$, it follows from Lemma 4.1.7 that

$$a_{\lambda_{n-j}\oplus(n-j)\alpha} = q^{\langle \lambda_{n-j}, (n-j)\alpha \rangle} a_{\lambda_{n-j}} a_{(n-j)\alpha}.$$

Also

$$\langle \lambda_{n-j}, (n-j)\alpha \rangle = \langle j\alpha - \beta, (n-j)\alpha \rangle = j(n-j)\varepsilon(\alpha),$$

then $q^{\langle \lambda_{n-j}, (n-j)\alpha \rangle} = (q^{\varepsilon(\alpha)})^{j(n-j)}$. Thus

$$a_{\lambda_{n-j}\oplus(n-j)\alpha} = (q^{\varepsilon(\alpha)})^{j(n-j)} a_{\lambda_{n-j}} a_{(n-j)\alpha}.$$

Similarly

$$a_{\lambda_{n-j}\oplus(i-j)\alpha} = (q^{\varepsilon(\alpha)})^{j(i-j)} a_{\lambda_{n-j}} a_{(i-j)\alpha}.$$

Recall that $|\mathrm{Hom}_{\Lambda}\,(V_{\lambda_{n-j}\oplus(i-j)\alpha}, V_{(n-i)\alpha})| = q^{\langle \lambda_{n-j}+(i-j)\alpha, (n-i)\alpha \rangle} = q^{\langle j\alpha-\beta+(i-j)\alpha, (n-i)\alpha \rangle} = (q^{\varepsilon(\alpha)})^{i(n-i)}$, we obtain

$$\begin{aligned} g^{\lambda_{n-j}\oplus(n-j)\alpha}_{\lambda_{n-j}\oplus(i-j)\alpha,(n-i)\alpha} &= \frac{(q^{\varepsilon(\alpha)})^{j(n-j)} a_{\lambda_{n-j}} a_{(n-j)\alpha}}{(q^{\varepsilon(\alpha)})^{j(i-j)} a_{\lambda_{n-j}} a_{(i-j)\alpha} a_{(n-i)\alpha} (q^{\varepsilon(\alpha)})^{i(n-i)}} \\ &= \frac{a_{(n-j)\alpha}}{a_{(i-j)\alpha} a_{(n-i)\alpha} (q^{\varepsilon(\alpha)})^{(i-j)(n-i)}} \\ &= g^{(n-j)\alpha}_{(i-j)\alpha,(n-i)\alpha} \\ &= \left| \begin{matrix} n-j \\ i-j \end{matrix} \right]_{\varepsilon(\alpha)} \\ &= (v^{\varepsilon(\alpha)})^{(i-j)(n-i)} \left[\begin{matrix} n-j \\ i-j \end{matrix} \right]_{\varepsilon(\alpha)}. \end{aligned}$$

So we have the following Lemma.

LEMMA 4.1.9.

$$u_{\lambda_{n-j}\oplus(i-j)\alpha}u_{(n-i)\alpha} = (v^{\varepsilon(\alpha)})^{i(n-i)}\begin{bmatrix} n-j \\ i-j \end{bmatrix}_{\varepsilon(\alpha)} u_{\lambda_{n-j}\oplus(n-j)\alpha}.$$

PROPOSITION 4.1.10. *Assume* (V_α, V_β) *to be an exceptional pair of* Λ*-mod and*

$$n = n(\alpha, \beta) = \frac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} = 2\frac{(\alpha, \beta)}{(\alpha, \alpha)}.$$

If $n(\alpha, \beta)\underline{\dim} V_\alpha > \underline{\dim}\, V_\beta$ *and* $\gamma \in \mathcal{P}$ *such that* $L(\alpha, \beta) \simeq V_\gamma$*, then in the Hall algebra* $\mathcal{H}(\Lambda)$*, we have*

$$u_\gamma = \sum_{r=0}^{n-1}(-1)^r(v^{\varepsilon(\beta)})(v^{-\varepsilon(\alpha)})^{(n-r)(n-1)}{}_\beta\delta(u_\alpha^{(n-r)})u_\alpha^{(r)}.$$

Proof. Let

$$f(r) = (-1)^r(v^{\varepsilon(\beta)})(v^{-\varepsilon(\alpha)})^{(n-r)(n-1)}{}_\beta\delta(u_\alpha^{(n-r)})u_\alpha^{(r)},$$

we only need to prove $u_\gamma = \sum_{r=0}^{n-1} f(r)$. By using the notation in 2.8 $u_\alpha^{(t)} = (v^{\varepsilon(\alpha)})^{t(t-1)}u_{t\alpha}$, we have

$$f(r) = (-1)^r(v^{\varepsilon(\beta)})(v^{\varepsilon(\alpha)})^{-(n-r)(n-1)+(n-r)(n-r-1)+r(r-1)}{}_\beta\delta(u_{(n-r)\alpha})u_{r\alpha}.$$

By Lemmas 4.1.6 and 4.1.9, we see

$$f(r) = (-1)^r(v^{\varepsilon(\beta)})(v^{\varepsilon(\alpha)})^{-(n-r)(n-1)+(n-r)(n-r-1)+r(r-1)}v^{-\varepsilon(\beta)}\sum_{j=1}^{n-r}\sum_{\substack{\lambda_{n-j}\in\mathcal{P} \\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}}$$

$$(v^{\varepsilon(\alpha)})^{(n-r)r}\begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)} u_{\lambda_{n-j}\oplus(n-j)\alpha}$$

$$= (-1)^r(v^{\varepsilon(\alpha)})^{r(r-1)}\sum_{j=1}^{n-r}(v^{\varepsilon(\alpha)})^{(n-j-r)r}\times$$

$$\times\begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)}\sum_{\substack{\lambda_{n-j}\in\mathcal{P} \\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}} u_{\lambda_{n-j}\oplus(n-j)\alpha}.$$

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So

$$\sum_{r=0}^{n-1} f(r) = \sum_{r=0}^{n-1}\sum_{j=1}^{n-r}(-1)^r (v^{\varepsilon(\alpha)})^{(n-j-1)r} \begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\substack{\lambda_{n-j}\in\mathcal{P} \\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}} u_{\lambda_{n-j}\oplus(n-j)\alpha}.$$

Since

$$\sum_{r=0}^{n-1}\sum_{j=1}^{n-r}(-1)^r (v^{\varepsilon(\alpha)})^{(n-j-1)r} \begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\substack{\lambda_{n-j}\in\mathcal{P} \\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}} u_{\lambda_{n-j}\oplus(n-j)\alpha}$$

$$= \sum_{j=1}^{n}\sum_{r=0}^{n-j}(-1)^r (v^{\varepsilon(\alpha)})^{(n-j-1)r} \begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\substack{\lambda_{n-j}\in\mathcal{P} \\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}} u_{\lambda_{n-j}\oplus(n-j)\alpha}.$$

By using the equation in 2.8 (2):

$$\sum_{r=0}^{n-j}(-1)^r (v^{\varepsilon(\alpha)})^{(n-j-1)r} \begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)} = 0$$

for any $n-j>0$. We have $\lambda_0=\gamma$, so

$$\sum_{r=0}^{n-1} f(r) = u_\gamma + \sum_{j=1}^{n-1}\sum_{r=0}^{n-j}(-1)^r (v^{\varepsilon(\alpha)})^{(n-j-1)r} \begin{bmatrix} n-j \\ n-r-j \end{bmatrix}_{\varepsilon(\alpha)} \times$$

$$\times \sum_{\substack{\lambda_{n-j}\in\mathcal{P} \\ g^{j\alpha}_{\beta,\lambda_{n-j}}\neq 0}} (v^{\varepsilon(\alpha)})^{(n-r)r} u_{\lambda_{n-j}\oplus(n-j)\alpha} = u_\gamma + 0 = u_\gamma.$$

This completes the proof.

4.2. THE CASE OF $0 \leqslant n(\alpha,\beta)\underline{\dim}\, V_\alpha < \underline{\dim}\, V_\beta$

In fact, in this case V_α, V_β are the two projective objects in $\mathcal{C}(\alpha,\beta)$, moreover V_α is the simple projective object of $\mathcal{C}(\alpha,\beta)$, let $\gamma\in\mathcal{P}$ such that $L(\alpha,\beta)\simeq V_\gamma$, we also know that V_γ is the simple injective object of $\mathcal{C}(\alpha,\beta)$.

PROPOSITION. *Let (V_α, V_β) be an exceptional pair, denote by $n=n(\alpha,\beta)=2((\alpha,\beta)/(\alpha,\alpha))$. If $0\leqslant n\,\underline{\dim}\, V_\alpha < \underline{\dim}\, V_\beta$, let $\gamma\in\mathcal{P}$ such that $V_\gamma\simeq L(\alpha,\beta)$, then in the Hall algebra*

$$u_\gamma = \frac{(v^{\varepsilon(\alpha)})^n}{[n]!_{\varepsilon(\alpha)}}(\delta_\alpha)^n(u_\beta).$$

Proof. In $\mathcal{C}(\alpha, \beta)$, there exists an exact sequence

$$0 \longrightarrow nV_\alpha \longrightarrow V_\beta \longrightarrow V_\gamma \longrightarrow 0.$$

If there is another exact sequence of the form

$$0 \longrightarrow nV_\alpha \longrightarrow V_\beta \longrightarrow V_\lambda \longrightarrow 0$$

then $V_\lambda \simeq V_\gamma$. By defination, $\delta_{n\alpha}(u_\beta) = v^{\langle\gamma,n\alpha\rangle} g^\beta_{\gamma n\alpha} \frac{a_\gamma}{a_\beta} u_\gamma$. We also have $g^\beta_{\gamma n\alpha} = 1$ and $\langle\beta,\alpha\rangle = 0$, $\langle\gamma, n\alpha\rangle = \langle\beta - n\alpha, n\alpha\rangle = -n^2\langle\alpha,\alpha\rangle$, $n^2\langle\alpha,\alpha\rangle - n\langle\alpha,\beta\rangle = 0$, $a_\beta = q^{\varepsilon(\beta)} - 1$, $a_\gamma = q^{\varepsilon(\gamma)} - 1$. Thus $a_\gamma = q^{\varepsilon(\gamma)} - 1 = q^{\langle\gamma,\gamma\rangle} - 1 = q^{\langle\beta-n\alpha,\beta-n\alpha\rangle} = q^{\langle\beta,\beta\rangle} - 1$, therefore $a_\gamma = a_\beta$. So we have $u_\gamma = v^{-\langle\gamma,n\alpha\rangle}\delta_{n\alpha}(u_\beta) = (v^{\varepsilon(\alpha)})^{n^2}\delta_{n\alpha}(u_\beta)$. Since we have known

$$u_{n\alpha} = \frac{(v^{\varepsilon(\alpha)})^{-n(n-1)}}{[n]!_{\varepsilon(\alpha)}}(u_\alpha)^n,$$

thus we have

$$\delta_{n\alpha} = \frac{(v^{\varepsilon(\alpha)})^{-n(n-1)}}{[n]!_{\varepsilon(\alpha)}}(\delta_\alpha)^n$$

from Proposition 3.4, then

$$u_\gamma = \frac{(v^{\varepsilon(\alpha)})^n}{[n]!_{\varepsilon(\alpha)}}(\delta_\alpha)^n(u_\beta).$$

The proof is finished.

4.3. THE CASE OF $\langle\alpha,\beta\rangle \leqslant 0$

In this case V_α, V_β are the two simple modules in $\mathcal{C}(\alpha,\beta)$, V_β is the simple projective object and V_α the simple injective object. Without loss of generality, we may assume $\langle\alpha,\beta\rangle < 0$. In $\mathcal{C}(\alpha,\beta)$, there exists the standard exact sequence

$$0 \longrightarrow V_\beta \longrightarrow V_\gamma \longrightarrow nV_\alpha \longrightarrow 0,$$

where V_γ is the other indecomposible injective object and

$$n = -n(\alpha,\beta) = -\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -2\frac{(\alpha,\beta)}{(\alpha,\alpha)};$$

in fact $n = \langle\gamma,\alpha\rangle/\langle\alpha,\alpha\rangle$ and $\langle\alpha,\gamma\rangle = 0$, then $-\langle\alpha,\beta\rangle = -\langle\alpha,\gamma - n\alpha\rangle = n\langle\alpha,\alpha\rangle$. We also know $\langle\gamma,\alpha\rangle = \langle\beta + n\alpha,\alpha\rangle = n\langle\alpha,\alpha\rangle$, thus $-\langle\alpha,\beta\rangle = \langle\gamma,\alpha\rangle$.

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LEMMA 4.3.1. *For* $\langle\alpha,\beta\rangle < 0$ *and* $n = -n(\alpha,\beta)$, *if there exists an exact sequence*

$$0 \longrightarrow V_\beta \longrightarrow V \longrightarrow nV_\alpha \longrightarrow 0,$$

then $V \simeq V_{\lambda_{n-i}} \oplus iV_\alpha$, $0 \leqslant i \leqslant n$, where $V_{\lambda_{n-i}}$ does not contain direct summands which are isomorphic to V_α, moreover there exist monomorphisms $V_{\lambda_i} \longrightarrow V_\gamma$, for $0 \leqslant i \leqslant n$, in particular $V_{\lambda_n} \simeq V_\gamma$ and $V_{\lambda_0} \simeq V_\beta$.

Proof. Since V_γ is the other indecomposible injective object, We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_\beta & \longrightarrow & V & \longrightarrow & nV_\alpha & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & V_\beta & \longrightarrow & V_\gamma & \longrightarrow & nV_\alpha & \longrightarrow & 0. \end{array}$$

By the same reason as those in Lemma 4.1.2, without loss of generality we may assume $g = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, where I is the unit matrix with rank $n-i$, so $\ker g \simeq iV_\alpha$. The following diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & \ker f & \xrightarrow{\simeq} & iV_\alpha & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V_\beta & \longrightarrow & V & \longrightarrow & nV_\alpha & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}} & & \\ 0 & \longrightarrow & V_\beta & \longrightarrow & V_\gamma & \longrightarrow & nV_\alpha & \longrightarrow & 0 \end{array}$$

is commutative, where $\ker f \simeq iV_\alpha$, thus $V \simeq V_{\lambda_{n-i}} \oplus iV_\alpha$. This completes the proof.

4.3.2. By definition

$$u_{n\alpha}u_\beta = v^{n\langle\alpha,\beta\rangle} \sum_{i=0}^{n} \sum_{\substack{\lambda_{n-i}\in\mathcal{P} \\ g_{n\alpha,\beta}^{\lambda_{n-i}\oplus i\alpha} \neq 0}} g_{n\alpha\beta}^{\lambda_{n-i}\oplus i\alpha} u_{\lambda_{n-i}\oplus i\alpha}.$$

We also know $g_{n\alpha,\beta}^{\lambda_{n-i}\oplus i\alpha} = g_{(n-i)\alpha,\beta}^{\lambda_{n-i}}$ and $V_{\lambda_{n-i}}$ has a unique simple subobject which is isomorphic to V_β in $\mathcal{C}(\alpha,\beta)$, so $g_{n\alpha,\beta}^{\lambda_{n-i}\oplus i\alpha} = g_{(n-i)\alpha,\beta}^{\lambda_{n-i}} = 1$, therefore

$$u_{n\alpha}u_\beta = v^{n\langle\alpha,\beta\rangle} \sum_{i=0}^{n} \sum_{\substack{\lambda_{n-i}\in\mathcal{P} \\ g_{n\alpha,\beta}^{\lambda_{n-i}\oplus i\alpha} \neq 0}} u_{\lambda_{n-i}\oplus i\alpha}.$$

Moreover, in general, for $0 \leqslant j \leqslant n$ we have

$$u_{(n-j)\alpha}u_\beta = v^{(n-j)\langle\alpha,\beta\rangle}\sum_{i=0}^{n-j}\sum_{\substack{\lambda_{n-j-i}\in\mathcal{P}\\ g^{\lambda_{n-j-i}}_{(n-j-i)\alpha,\beta}\neq 0}} u_{\lambda_{n-j-i}\oplus i\alpha},$$

where α, β and λ_{n-j-i} are the same as in Lemma 4.3.1.

Since any extension of the form

$$0 \longrightarrow jV_\alpha \longrightarrow V' \longrightarrow V_{\lambda_{n-j-i}} \oplus iV_\alpha \longrightarrow 0$$

is split, then

$$u_{\lambda_{n-j-i}\oplus i\alpha}u_{j\alpha} = v^{\langle\lambda_{n-j-i}+i\alpha,j\alpha\rangle}g^{\lambda_{n-j-i}\oplus(i+j)\alpha}_{\lambda_{n-j-i}\oplus i\alpha,j\alpha}u_{\lambda_{n-j-i}\oplus(i+j)\alpha}.$$

Because that

$$\langle\lambda_{n-j-i}+i\alpha, j\alpha\rangle = \langle\beta+n\alpha-j\alpha, j\alpha\rangle = j(n-j)\langle\alpha,\alpha\rangle$$

and

$$g^{\lambda_{n-j-i}\oplus(i+j)\alpha}_{\lambda_{n-j-i}\oplus i\alpha,j\alpha} = g^{(i+j)\alpha}_{i\alpha,j\alpha} = \begin{bmatrix} i+j \\ j \end{bmatrix}_{\varepsilon(\alpha)},$$

therefore

$$u_{\lambda_{n-j-i}\oplus i\alpha}u_{j\alpha} = (v^{\varepsilon(\alpha)})^{j(n-j)}\begin{bmatrix} i+j \\ j \end{bmatrix}_{\varepsilon(\alpha)}u_{\lambda_{n-j-i}\oplus(i+j)\alpha}.$$

Thus

$$\begin{aligned} &u_{(n-j)\alpha}u_\beta u_{j\alpha}\\ &\quad= v^{(n-j)\langle\alpha,\beta\rangle}\sum_{i=0}^{n-j}\sum_{\substack{\lambda_{n-j-i}\in\mathcal{P}\\ g^{\lambda_{n-j-i}}_{(n-j-i)\alpha,\beta}\neq 0}} u_{\lambda_{n-j-i}\oplus i\alpha}u_{j\alpha},\\ &\quad= v^{(n-j)\langle\alpha,\beta\rangle}(v^{\varepsilon(\alpha)})^{j(n-j)}\sum_{i=0}^{n-j}\begin{bmatrix} i+j \\ j \end{bmatrix}_{\varepsilon(\alpha)}\sum_{\substack{\lambda_{n-j-i}\in\mathcal{P}\\ g^{\lambda_{n-j-i}}_{(n-j-i)\alpha,\beta}\neq 0}} u_{\lambda_{n-j-i}\oplus(i+j)\alpha},\\ &\quad= (v^{\varepsilon(\alpha)})^{-n(n-j)+j(n-j)}\sum_{i=0}^{n-j}(v^{\varepsilon(\alpha)})^{ij}\begin{bmatrix} i+j \\ j \end{bmatrix}_{\varepsilon(\alpha)}\sum_{\substack{\lambda_{n-j-i}\in\mathcal{P}\\ g^{\lambda_{n-j-i}}_{(n-j-i)\alpha,\beta}\neq 0}} u_{\lambda_{n-j-i}\oplus(i+j)\alpha}.\end{aligned}$$

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PROPOSITION 4.3.3. *Let (V_α, V_β) be an exceptional pair of Λ-mod. If $\langle\alpha,\beta\rangle \leqslant 0$ and $\gamma \in \mathcal{P}$ such that $V_\gamma \simeq L(\alpha,\beta)$, then in the Hall algebra $\mathcal{H}(\Lambda)$*

$$u_\gamma = \sum_{j=0}^{n}(-1)^j (v^{\varepsilon(\alpha)})^{n-j} u_\alpha^{(n-j)} u_\beta u_\alpha^{(j)},$$

where

$$n = -n(\alpha,\beta) = -\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -2\frac{(\alpha,\beta)}{(\alpha,\alpha)}.$$

Proof. Let

$$f = \sum_{j=0}^{n}(-1)^j (v^{\varepsilon(\alpha)})^{n-j} u_\alpha^{(n-j)} u_\beta u_\alpha^{(j)}.$$

Then by 4.3.2

$$\begin{aligned} f &= \sum_{j=0}^{n}(-1)^j (v^{\varepsilon(\alpha)})^{n-j} (v^{\varepsilon(\alpha)})^{(n-j)(n-j-1)} (v^{\varepsilon(\alpha)})^{j(j-1)} (u_{(n-j)\alpha} u_\beta u_{j\alpha}) \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n-j}(-1)^j (v^{\varepsilon(\alpha)})^{j(i+j-1)} \begin{bmatrix} i+j \\ j \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\substack{\lambda_{n-j-i}\in\mathcal{P} \\ g^{\lambda_{n-j-i}}_{(n-j-i)\alpha,\beta}\neq 0}} u_{\lambda_{n-j-i}\oplus(i+j)\alpha}, \end{aligned}$$

since $(n-j)+(n-j)(n-j-1)+j(j-1)-n(n-j)+j(n-j)+ij = j(i+j-1)$. Let $t = i+j$, since

$$\sum_{j=0}^{n}\sum_{i=0}^{n-j} = \sum_{t=0}^{n}\sum_{j=0}^{t},$$

then

$$f = \sum_{t=0}^{n}\sum_{j=0}^{t}(-1)^j (v^{\varepsilon(\alpha)})^{j(t-1)} \begin{bmatrix} t \\ j \end{bmatrix}_{\varepsilon(\alpha)} \sum_{\substack{\lambda_{n-t}\in\mathcal{P} \\ g^{\lambda_{n-t}}_{(n-t)\alpha,\beta}\neq 0}} u_{\lambda_{n-t}\oplus t\alpha}.$$

According to 2.8 (2),

$$\sum_{j=0}^{t}(-1)^j (v^{\varepsilon(\alpha)})^{j(t-1)} \begin{bmatrix} t \\ j \end{bmatrix}_{\varepsilon(\alpha)} = 0$$

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for $t > 0$, and $\lambda_n = \gamma$, we obtain $f = u_\gamma$. The proof is completed.

By completely similar proofs, we have a sequence of propositions which are dual to the above ones.

PROPOSITION 4.4. *Let (V_α, V_β) be an exceptional pair of Λ-mod,*

$$m = m(\alpha, \beta) = \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = 2\frac{(\alpha, \beta)}{(\beta, \beta)}.$$

Let $\lambda \in \mathcal{P}$ such that $V_\lambda \simeq R(\alpha, \beta)$. If $m\,\underline{\dim}\, V_\beta > \underline{\dim} V_\alpha$, then in the Hall algebra $\mathcal{H}(\Lambda)$,

$$u_\lambda = \sum_{r=0}^{m-1} (-1)^r v^{\varepsilon(\alpha)} (v^{-\varepsilon(\beta)})^{(m-r)(m-1)} u_\beta^{(r)} \delta_\alpha (u_\beta^{(m-r)}).$$

Proof. This is dual to Proposition 4.1.10.

PROPOSITION 4.5. *Let (V_α, V_β) be an exceptional pair of Λ-mod,*

$$m = m(\alpha, \beta) = \frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = 2\frac{(\alpha, \beta)}{(\beta, \beta)}.$$

Let $\lambda \in \mathcal{P}$ such that $V_\lambda \simeq R(\alpha, \beta)$. If $0 < m\,\underline{\dim}\, V_\beta < \underline{\dim}\, V_\alpha$, then in the Hall algebra $\mathcal{H}(\Lambda)$,

$$u_\lambda = \frac{(v^{\varepsilon(\beta)})^m}{[m]!_{\varepsilon(\beta)}} ({}_\beta\delta)^m (u_\alpha).$$

Proof. This is dual to Proposition 4.2.

PROPOSITION 4.6. *Let (V_α, V_β) be an exceptional pair of Λ-mod and $\lambda \in \mathcal{P}$ such that $V_\lambda \simeq R(\alpha, \beta)$. If $\langle \alpha, \beta \rangle < 0$, then in the Hall algebra $\mathcal{H}(\Lambda)$,*

$$u_\lambda = \sum_{j=0}^{m} (-1)^j (v^{\varepsilon(\beta)})^{m-j} u_\beta^{(j)} u_\alpha u_\beta^{(m-j)}$$

where

$$m = -m(\alpha, \beta) = -\frac{\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = -2\frac{(\alpha, \beta)}{(\beta, \beta)}.$$

Proof. This is dual to Proposition 4.3.3.

Remark 4.7. Let (V_i, V_j) be an exceptional pair consisting of two simple modules.

(1) In Proposition 4.3.3, since -$\dim_k V_\gamma + \varepsilon(\gamma) = -\varepsilon_i n$ where $\varepsilon_i = \varepsilon(i)$, then

$$v^{-\dim_k V_\gamma + \varepsilon(\gamma)} u_\gamma = \sum_{r=0}^{n} (-1)^r v^{-\varepsilon_i r} u_i^{(n-r)} u_j u_i^{(r)},$$

where $n = -a_{ij}$. It exactly corresponds to the braid group operation $T''_{i,1}$ on θ_j in the sense of Lusztig (see [L] Chapter 37).

(2) In Proposition 4.6, since $-\dim_k V_\lambda + \varepsilon(\lambda) = -\varepsilon_j m$, then

$$v^{-\dim_k V_\lambda + \varepsilon(\lambda)} u_\lambda = \sum_{r=0}^{m} (-1)^r v^{-\varepsilon_j r} u_j^{(r)} u_i u_j^{(m-r)},$$

where $m = -a_{ji}$. It exactly corresponds to the braid group operation $T'_{j,-1}$ on θ_i in the sense of Lusztig (see [L] Chapter 37).

These facts were first observed by Ringel in [R4].

5. Exceptional Sequences in a Hall Algebra

For $1 \leqslant s \leqslant |I|$, where $|I|$ is the number of isomorphism classes of simple Λ-modules, let $\mathcal{B}_{s-1} = \langle \sigma_1, \ldots, \sigma_{s-1} \rangle$ be the braid group on the $s-1$ generators, which define relations are the following:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for } 1 \leqslant i \leqslant s-1,$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i-j| \geqslant 2.$$

According to Crawley-Boevey and Ringel, there exists an action of the group $\mathcal{B}_{s-1}$ on the set of exceptional sequences of length s in Λ-mod (see [CB] and [R6]). The action is given as follows.

Let $A = (V_{\alpha_1}, \ldots, V_{\alpha_s})$ be an exceptional sequence in Λ-mod, then $\sigma_i A = (V_{\beta_1}, \ldots, V_{\beta_s})$ such that $V_{\beta_j} = V_{\alpha_j}$ for all $j \notin \{i, i+1\}$, $V_{\beta_i} = V_{\alpha_{i+1}}$ and $V_{\beta_{i+1}} \simeq R(\alpha_i, \alpha_{i+1})$; $\sigma_i^{-1} A = (V_{\gamma_1}, \ldots, V_{\gamma_s})$ such that $V_{\gamma_j} = V_{\alpha_j}$ for all $j \notin \{i, i+1\}$, $V_{\gamma_i} \simeq L(\alpha_i, \alpha_{i+1})$ and $V_{\gamma_{i+1}} = V_{\alpha_i}$. The action is transitive if $s = |I|$.

Our main result is the following.

THEOREM 5.1. *For* $1 \leqslant s \leqslant |I|$, *let* $\mathcal{B}_{s-1} = \langle \sigma_1, \ldots, \sigma_{s-1} \rangle$ *be the braid group on* $s-1$ *generators,* $A = (V_{\alpha_1}, \ldots, V_{\alpha_s})$ *any exceptional sequence of length* s *in* Λ*-mod. Denoted by*

$$m(i, i+1) = \frac{\langle \alpha_i, \alpha_{i+1} \rangle}{\langle \alpha_{i+1}, \alpha_{i+1} \rangle} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_{i+1}, \alpha_{i+1})}$$

and

$$n(i, i+1) = \frac{\langle \alpha_i, \alpha_{i+1} \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2\frac{(\alpha_i, \alpha_{i+1})}{(\alpha_i, \alpha_i)}$$

and assume that $\sigma_i A = (V_{\beta_1}, \ldots, V_{\beta_s})$ *and* $\sigma_i^{-1} A = (V_{\gamma_1}, \ldots, V_{\gamma_s})$ *for* $1 \leqslant i \leqslant s-1$. *Then, in the Hall algebra* $\mathcal{H}(\Lambda)$, *we have*

(1) *If* $m(i,i+1)\underline{\dim}\, V_{\alpha_{i+1}} > \underline{\dim}\, V_{\alpha_i}$, *then*

$$\begin{aligned} u_{\beta_{i+1}} &= \sum_{r=0}^{m(i,i+1)-1} (-1)^r v^{\varepsilon(\alpha_i)} \times \\ &\quad \times (v^{-\varepsilon(\alpha_{i+1})})^{(m(i,i+1)-r)(m(i,i+1)-1)} u_{\alpha_{i+1}}^{(r)} \delta_{\alpha_i} (u_{\alpha_{i+1}}^{(m(i,i+1)-r)}). \end{aligned}$$

(2) *If* $0 < m(i,i+1)\underline{\dim}\, V_{\alpha_{i+1}} < \underline{\dim}\, V_{\alpha_i}$, *then*

$$u_{\beta_{i+1}} = \frac{(v^{\varepsilon(\alpha_{i+1})})^{m(i,i+1)}}{[m(i,i+1)]!_{\varepsilon(\alpha_{i+1})}} ({}_{\alpha_{i+1}}\delta)^{m(i,i+1)} (u_{\alpha_i}).$$

(3) *If* $m(i,i+1) \leqslant 0$, *then*

$$u_{\beta_{i+1}} = \sum_{r=0}^{-m(i,i+1)} (-1)^r (v^{\varepsilon(\alpha_{i+1})})^{-m(i,i+1)-r} u_{\alpha_{i+1}}^{(r)} u_{\alpha_i} u_{\alpha_{i+1}}^{(-m(i,i+1)-r)}$$

(1′) *If* $n(i,i+1)\underline{\dim}\, V_{\alpha_i} > \underline{\dim}\, V_{\alpha_{i+1}}$, *then*

$$\begin{aligned} u_{\gamma_i} &= \sum_{r=0}^{n(i,i+1)-1} (-1)^r v^{\varepsilon(\alpha_{i+1})} \times \\ &\quad \times (v^{-\varepsilon(\alpha_i)})^{(n(i,i+1)-r)(n(i,i+1)-1)} ({}_{\alpha_{i+1}}\delta(u_{\alpha_i}^{(n(i,i+1)-r)})) u_{\alpha_i}^{(r)}. \end{aligned}$$

(2′) *If* $0 < n(i,i+1)\underline{\dim}\, V_{\alpha_i} < \underline{\dim}\, V_{\alpha_{i+1}}$, *then*

$$u_{\gamma_i} = \frac{(v^{\varepsilon(\alpha_i)})^{n(i,i+1)}}{[n(i,i+1)]!_{\varepsilon(\alpha_i)}} (\delta_{\alpha_i})^{n(i,i+1)} (u_{\alpha_{i+1}}).$$

(3′) *If* $n(i,i+1) \leqslant 0$, *then*

$$u_{\gamma_i} = \sum_{r=0}^{-n(i,i+1)} (-1)^r (v^{\varepsilon(\alpha_i)})^{-n(i,i+1)-r} u_{\alpha_i}^{(-n(i,i+1)-r)} u_{\alpha_{i+1}} u_{\alpha_i}^{(r)}.$$

Proof. It is the sum of the last section.

Obviously, all possibilities are listed by Theorem 5.1. Combining with Proposition 3.4, we obtain an algorithm which starts from the complete exceptional sequence consisting of simple modules. The algorithm being effective and complete also depends on the properties (1), (2) and (3) in 2.7. We stress here that our algorithm in fact only depends on the Cartan datum, in the language of representation theory of Λ, that is, the symmetric Ringel form and dimension vectors.

COROLLARY 5.2 (Ringel, see [Z]). *If $\lambda \in \mathcal{P}$ is an exceptional module, then the corresponding element u_λ lies in the composition subalgebra $\mathcal{C}(\Lambda)$.*

Proof. Because the action of the braid group on the set of complete exceptional sequences is transitive and any exceptional module belongs to some complete exceptional sequence.

Remark. In fact, we may have a stronger assertion than Corollary 5.2. Let $\lambda \in \mathcal{P}$ be an exceptional module, our algorithm provides a universal formula to express u_λ as a combination of monomials of u_i, $i \in I$, that is, the formula is unchanged whenever we chose different k for the realization of Λ corresponding to a given Δ. So the element u_λ belongs to the generic composition algebra, that is, to the quantum group.

Let $\mathcal{S}$ be a R-subalgebra of $\mathcal{C}(\Lambda)$. If for any $x \in \mathcal{S}$ we have $\phi_1(x)(\mathcal{S}) \subset \mathcal{S}$ and $\phi_2(x)(\mathcal{S}) \subset \mathcal{S}$, where ϕ_1 and ϕ_2 are given in Proposition 3.4, then $\mathcal{S}$ is called a derivation subalgebra of $\mathcal{C}(\Lambda)$.

COROLLARY 5.3. *Let $A = (V_{\alpha_1}, \ldots, V_{\alpha_n})$ be any complete exceptional sequence of Λ-mod. Then the derivation subalgebra generated by $\{u_{\alpha_1}, \ldots, u_{\alpha_n}\}$ coincides with the whole $\mathcal{C}(\Lambda)$.*

Proof. Since we can get the complete exceptional sequence consisting of simple modules by the action of the braid group on A.

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