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THE LEAST α FOR WHICH $E(\alpha)$ IS INADMISSIBLE

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Abstract

This paper attempts to classify the least ordinal α_0 for which $E(\alpha_0)$ (the E closure of $\alpha_0 \cup \{\alpha_0\}$) is inadmissible. Among the results proved are (i) $L_{\alpha_0} = \mathrm{ZFC}^-$; (ii) α_0 is very large in comparison with the least ordinal satisfying (i); (iii) $(\alpha_0, \alpha]$ marks precisely an ω-Gap, where $\bar{\alpha} = E(\alpha_0) \cap \mathrm{ON}$; (iv) the K_r-sequence of α_0 has length ω.

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E-recursion, which is a generalisation of recursion on objects of higher type to arbitrary sets, is obtained by adding to the rudimentary operations of Jensen [8] ((i)–(v) below) a reflection scheme ((vi) below). See Normann [3] or Fenstad [2].

(i) $f(x_1, \ldots, x_n) = x_i$, $e\langle 1, n, i\rangle$.

(ii) $f(x_1, \ldots, x_n) = x_i - x_j$, $e = \langle 2, m, i, j\rangle$.

(iii) $f(x_1, \ldots, x_n) = \{x_i, x_j\}$, $e = \langle 3, m, i, j\rangle$.

(iv) $f(x_1, \ldots, x_n) = \bigcup_{y \in x_1} h(y, x_1, \ldots, x_n)$, $e = \langle 4, n, e'\rangle$, where e' is the index for h,

(v) $f(x_1, \ldots, x_n) \simeq h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$, $e = \langle 5, n, me', e_1, \ldots, e_m\rangle$, where e' is the index for h and $e_1, \ldots, e_m$ indices for $g_1, \ldots, g_m$,

(vi) $f(e_1, x_1, \ldots, x_n, y_1, \ldots, y_m) \simeq \{e_1\}^R(x_1, \ldots, x_n)$, $e = \langle 7, n, m\rangle$.

Condition (v) is the substitution scheme. On the right of each scheme, we give the indices which are carried along in the induction. For any set x, $E(x)$ is defined to be the closure of $x \cup \{x\}$ under the above operations. If x is transitive, then so is $E(x)$, and the strength of $E(x)$ is in general weaker than admissibility. Lemma 1 below proves the existence of ordinals α for which $E(\alpha)$ is inadmissible. The problem naturally arises to characterise the first ordinal α_0 for which $E(\alpha_0)$ is inadmissible. This is what this paper attempts to do. Note that $E(\omega)$ is admissible by Gandy selection.

MW below will abbreviate 'Moschovakis Witness' to the divergence of a computation in E-recursion. The notation $\langle e, \beta\rangle\downarrow$ means the function with index $e \in \omega$ converges for β, and $\langle e, \beta\rangle\uparrow$ will abbreviate divergence. When $\langle e, \beta\rangle\downarrow$ there is a well-founded computation tree which lists the inductive stages involved. The height of the tree will denote the length of the computation $|\langle e, \beta\rangle|$. The ordering of the computation tree is by the relation of subcomputation. When $\langle e, \beta\rangle\uparrow$, the computation tree is not well-founded and there is an infinitely descending sequence of computations witnessing the divergence of $\{e\}(\beta)$. Such a sequence is called a Moschovakis Witness (MW). (See Fenstad [2], and Slaman [7].)

Many of the results flow out of the paper on ω-gaps by Marek and Srebrny [5]. The notation $a \leqslant_E b$ means a is E-recursive (or E computable) in b.

Let α_0 be at least such that $E(\alpha_0) \vDash \neg KP$. The first two lemmata are intended to prove the existence of α_0 and to fix upper bounds on its value.

LEMMA 1. *Let α_1 be the least ordinal for which there exists $\beta > \alpha_1$ such that L_β is Σ_2-admissible (equivalently π_1-admissible) and $L_\beta \vDash$ "α_1 is the least uncountable cardinal." Then $E(\alpha_1)$ is inadmissible.*

PROOF. The set $\{\langle e, \bar\beta\rangle \mid \bar\beta \in \alpha_1 \wedge \{e\}(\bar\beta, \alpha_1)\downarrow\} = S$ is Σ_1 and a subset of α_1. By Δ_2-separation, $S \in L_\beta$. Hence $E(\alpha_1) \in L_\beta$. Let $E(\alpha_1) = L_\alpha$. Thus $\alpha \in \beta$. Over L_α we could define MW's for those divergent computations $\langle e, \bar\beta\rangle\uparrow$. These will be elements of ${}^\omega\alpha_1$ and will belong to $L_{\alpha+1} \subseteq L_\beta$. If m is such a witness, then $m \subseteq \gamma < \alpha_1$ for some γ, since $L_\beta \vDash cf(\alpha_1) > \omega$. It follows by a standard collapsing argument that $m \in L_{\alpha_1}$. Since all computation trees for $\langle e, \bar\beta\rangle\downarrow$ belong to $E(\alpha_1)$, in the usual manner we could express $S \subseteq \alpha_1$ as a Δ_1 predicate over $E(\alpha_1)$. Since $S \notin E(\alpha_1)$, Δ_1-separation fails and $E(\alpha_1)$ is inadmissible. (We could E-recursively compute $E(\alpha_1)$ from S. Whence if $S \in E(\alpha_1)$, then $E(\alpha_1) \in E(\alpha_1)$.)

LEMMA 2. $\alpha_0 < \delta_0$, *δ_0 being the least stable ordinal.*

PROOF. Let $\phi(\beta, \gamma)$ say that $L_\beta \vDash \Sigma_2$-admissibility $\wedge L_\beta \vDash \gamma$ is the least uncountable cardinal. $\phi(\beta, \gamma)$ is Δ_1 and $L \vDash \exists\langle\beta, \gamma\rangle\phi(\beta, \gamma)$ (i.e. $\beta = \aleph_2^L$, $\gamma = \aleph_1^L$). Whence $L_{\delta_0} \vDash \exists\langle\beta, \gamma\rangle\phi(\beta, \gamma)$. The result follows.

LEMMA 3 (T. Slaman [7]). *Let α be the least ordinal such that $E(\alpha) \vDash$ "α is an uncountable cardinal." Then $E(\alpha)$ is admissible.*

In what follows we shall further describe α_0. We shall first run through some results in ω-gaps in the constructible universe. See Marek and Srebrny [5] α is said to be a gap ordinal if and only if $(L_{\alpha+1} - L_\alpha) \cap P(\omega) = \varnothing$.

LEMMA 4 (Boolos). *If α is not a gap ordinal, then there is an arithmetical copy of E_α of L_α such that $E_\alpha \in L_{\alpha+1} - L_\alpha$.*

(For our purposes it is sufficient that E_α is a (well-founded) ω-diagram of L_α that could be unravelled by E-recursion.)

LEMMA 5. *α starts a gap if and only if $L_\alpha \vDash ZFC^- + V = HC$. ($V = HC$ means that all sets are hereditarily countable, i.e. $TC(x)$ is countable for all x.)*

THEOREM 6 (D. Guaspari). *If α starts a gap, $\beta > \alpha$ and $L_\alpha \cap P(\omega) = L_\beta \cap P(\omega)$ (i.e. if β is still in the gap), then α is β-stable, i.e. $L_\alpha <_1 L_\beta$.*

PROOF. Let $\phi(x, b)$ be a Σ_0 formula with parameter $b \in L_\alpha$. Suppose $L_\beta \vDash \exists x \phi(x, b)$. Let $b \in L_\gamma \in L_\alpha$. Let X_γ be the Σ_∞-Hull of $L_\gamma \cup \{b\}$ in $L_{\bar\beta}$, for $\bar\beta < \beta$ such that there exists $x \in L_{\bar\beta+1} \phi(x, b)$, with x first order definable in $L_{\bar\beta}$ from d. Let $\bar X_\gamma = L_\delta$ be the Mostowski collapses of X_γ. Letting $x = \{y | L_{\bar\beta} \vDash \psi(y, d)\}$, since ϕ is Σ_0 we may obtain a first order formula Ψ from $\phi(x, b)$ by replacing $y \in x$ wherever it occurs by $\psi(y, d)$.

Thus $L_{\bar\beta+1} \vDash \phi(x, b)$ if and only if $L_{\bar\beta} \vDash \Psi(b, d)$. Let $\bar d$ be the collapse of d an $\bar x = \{y | L_\delta \vDash \psi(y, \bar d)\}$. Thus $L_\delta = \Psi(b, \bar d)$ and by reversing the process of obtaining Ψ, we have $L_{\delta+1} \vDash \phi(\bar x, b)$ since $b \in L_\gamma \subseteq L_\delta$. We also have $L_{\delta+1} \vDash \exists x \phi(x, b)$.

We must show that $\delta < \alpha$. Since L_γ is countable in L_α, X_γ is countable in L_β, whence L_β contains an ω-code θ of $X_\gamma \cong L_\delta$. Then $\theta \in L_\alpha$ as $L_\alpha \cap P(\omega) = L_\beta \cap P(\omega)$. Since $L_\alpha \vDash$ ZFC$^-$, θ can be unravelled inside L_α, i.e. $L_\delta \in L_\alpha$. Let $E(\alpha_0) = L_{\bar\alpha}$.

THEOREM 7.
(i) *α_0 is a gap ordinal,*
(ii) *α_0 is the first uncountable cardinal in $E(\alpha_0) = L_{\bar\alpha}$,*
(iii) *$[\alpha_0, \bar\alpha)$ lies in an ω-gap and the gap commences at α_0 and extends precisely to all ordinals less than α.*

PROOF.
(i) If not there is an ω-code of α_0 in $L_{\alpha_0+1} \in E(\alpha_0)$, which makes α_0 countable in $E(\alpha_0)$, i.e. $E(\alpha_0)$ is admissible by Gandy selection.

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(ii) α_0 is obviously uncountable in $L_{\bar{\alpha}}$. Suppose there exists $\beta \in \alpha_0$ such that $L_{\bar{\alpha}} \models |\alpha_0| = \beta$. Since $L_{\bar{\alpha}}$ is E closed inadmissible it contains MW's for computations in $E(\beta)$. If m is such an MW, $m \in {}^{\omega}\beta$. Since $cf(\beta) > \omega$ in $L_{\bar{\alpha}}$, $m \in L_{\beta}$. Arguing as in Lemma 1 we conclude that $E(\beta)$ is inadmissible, a contradiction by the minimality of α_0.

If X is an uncountable cardinal $< \alpha_0$ in $E(\alpha_0)$, we could argue likewise for $E(\gamma)$ and conclude that it is inadmissible, again not possible.

(iii) Suppose the gap commences at $\beta < \alpha_0$. Then $L_{\bar{\alpha}} \models$ "β is countable by a function f" (since α_0 is the least uncountable in $L_{\bar{\alpha}}$). By a collapsing argument we see that the $<_L$ least such f belongs to L_{α_0}. Hence there is an ω-code m of β in L_{α_0} and hence in L_β since $P(\omega) \cap L_{\alpha_0} = P(\omega) \cap L_\beta$. This is a contradiction since $L_\beta \models \text{ZFC}^-$.

Suppose the gap stops at $\gamma < \bar{\alpha}$. Then γ and hence α_0 are countable in $L_{\bar{\alpha}}$ by Lemma 4, a contradiction. Suppose the gap proceeds to γ beyond $\bar{\alpha}$, so $L_\gamma \models \exists x[\exists y \in X(x = E(y) \wedge X \models \sim KP)]$. Since L_γ contains MW's for divergent computations from $\alpha \cup \{\alpha\}$, we replace "$x = E(y)$" by a Σ_1 formula with the same effect. The formula in square brackets is also Σ_1; call it ϕ. Since $L_{\alpha_0} <_{\Sigma_1} L_\gamma$ by Theorem 6, $L_{\alpha_0} \models \phi$, a contradiction.

COROLLARY 8. $L_{\alpha_0} \models ZFC^- + V = HC$.

But α_0 is far from being the least α such that $L_\alpha \models \text{ZFC}^- + V = HC$. This could be seen from the following two lemmas from Marek and Srebrny [5].

LEMMA 9. *If α starts a gap of length greater than* 1, *then α is the limit of the sequence of beginnings of gaps of length* 1.

LEMMA 10. *If α starts a gap of length p and $p \in \alpha$, then for each $\sigma \in p$,* $\sup\{\beta < \alpha \mid \beta$ *starts a gap of length* $\sigma\} = \alpha$.

We shall now say something about α_0^+, the least admissible ordinal greater than α_0.

THEOREM 11.
(i) *There are no gaps between $\bar{\alpha}$ and α_0^+.*
(ii) *$L_{\alpha_0}^+$ is locally countable.*

PROOF.
(i) If γ (where $\bar{\alpha} < \gamma < \alpha_0^+$) begins a gap, then $L_\gamma \models \text{ZFC}^-$, so by definition of α_0^+, $\alpha_0^+ = \gamma$. But α_0^+ is a successor admissible, and hence is not even Σ_2-admissible, and this is a contradiction α_0^+ does not begin a gap.

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(ii) This follows from (i) and Lemma 4.

Let μ_0 be the ordinal considered in Lemma 3 by Slaman. The arguments above could be used to show

THEOREM 12.
(i) μ_0 *begins a gap which lasts precisely up to* μ_0^+.
(ii) $L_{\mu_0} \models ZFC^- + HC$.
(iii) μ_0 *is countable in* $L_{\mu_0^+ + 1}$.

Let $E(\alpha)$ be inadmissible and α the greatest cardinal in $E(\alpha)$. Slaman [7] defines the K_r sequence for $E(\alpha)$ as follows:

$$K_r(0) = 0, \quad a_0 = 0;$$
$$K_r(\beta + 1) = K_r^{a_\beta, \alpha}, \quad a_{\beta+1} = \min(\delta)\big(K_0^{\delta,\alpha} > K_r(\beta + 1)\big).$$

For $\lim(\lambda)$, $K_r(\lambda) = \sup_{\beta<\lambda} K_r(\beta)$, $a_\lambda = \min \delta(K_0^{\delta,\alpha} > K_r(\lambda))$ if $\sup_{\beta<\lambda} K_r(\beta) < E(\alpha) \cap \mathrm{ON}$.

Let θ_α be the order type of the K_r sequence for α.

LEMMA 13 (Slaman [7]). *Let* $\lim(\delta) \wedge \delta \leqslant \theta_\alpha$. *And let* α *be a successor cardinal* β^+ *in* $E(\alpha)$. *Then there exists* $\alpha' \leqslant \alpha$ *such that* $E(\alpha')$ *is inadmissible,* $\theta_{\alpha'} = \delta$ *and* $E(\alpha')$ *has the same cardinal structure as* $E(\alpha)$.

PROOF. Let $x = \{\bar{\beta} < \alpha \mid K_r^{\bar{\beta},\alpha} < K_r(\delta)\} \cup \{\alpha\}$. Let M be the E-Hull of x.

M is closed under pairing, MW's for divergent computations and computation trees for convergent computations. Let $\overline{M}$ be the collapse of M and let α collapse to α'. Then $\beta \subseteq \alpha'$ since $\beta \subseteq M$ and $\overline{M}$ is closed under MW's and hence inadmissible. Thus $E(\alpha')$ has the same cardinal structure as $E(\alpha)$, i.e. $E(\alpha') \models \beta^+ = \alpha'$ and $\theta_{\alpha'} = \delta$.

THEOREM 14. $\theta_{\alpha_0} = \omega$.

PROOF. For if $\theta_{\alpha_0} > \omega$ we could as in Lemma 13 obtain $\alpha' < \alpha_0$ with $\theta_{\alpha'} = \omega$. This would contradict the minimality of α_0. Let $|\alpha$-recursive$|$ be the supremum of order types of all α-recursive well-orderings of α. Gostanian [4] calls an admissible α 'bad' when $\alpha^+ > |\alpha$-recursive$|$ and proves that the least bad ordinal b_0 is the least Σ_1^1-reflecting ordinal. Since α-recursive well-orderings of α belong to $E(\alpha)$, we may be tempted to conjecture that $\alpha_0 = b_0$. We shall show that this is false. It is obvious that α is good (i.e. $\alpha^+ = |\alpha$-recursive$|$) implies that $E(\alpha) = L_{\alpha^+}$ and is admissible.

We need the following result from Barwise, Gandy and Moschovakis [1].

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LEMMA 15. *If $\phi(\bar{V})$ is a Σ_1^1 formula in L_{ZF}, there is a π_1 formula ϕ^* such that given a nonempty countable transitive set A, an admissible set B with $A \in B$ and an element $\bar{a} \in A$, then $A \vDash \phi(\bar{a})$ if $B \vDash \phi^*(A, \bar{a})$.*

PROOF. The lemma could be extracted from the following. Suppose ϕ is π_1^1, say ϕ is $\forall R \psi(R, \bar{x})$. If $A \vDash \phi(\bar{a})$, let Dia(A) be the diagram of A. Dia(A) is Δ_0-recursive in A for any admissible B containing A. The following is a valid statement in the language of B (we may take B countable):

$$\mathrm{Dia}(A) \wedge \forall y \Big(\bigvee_{b \in A} b = y \Big) \to \psi(R, \bar{a}).$$

Hence it has a proof in B by the Barwise completeness theorem. The predicate "p is a proof of σ" is Δ_1 in p, σ.

LEMMA 16. *b_0 is Σ_1-projectible into ω in L_{b_0}.*

PROOF. We show that working Σ_1-recursively inside L_{b_0}, we could assign a unique integer code to each $\alpha < b_0$. Note that b_0 is recursively inaccessible by Σ_3^0-reflection. Hence $f(\alpha) = \alpha^+$ is Σ_1-recursive.

Assume each ordinal $< \alpha$ has been assigned a unique integer.

Case 1. Succ(α). Say $\alpha = \beta + 1$ and Code(β) $= n$. Then Code(α) $= \langle 1, n \rangle$.

Case 2. Lim(α), α is not admissible. Then there exist $\gamma_1, \gamma_2 < \alpha$, and $\phi(\Delta_0)$ such that $L_\alpha \vDash \forall x \in \gamma_1 \exists_y \phi(y, \gamma_2) \wedge \sim L_\alpha \vDash \exists z \forall x \in \gamma_1 \exists_y \in z \phi(y, \gamma_2)$. Let Code($\alpha$) $= \langle 2, \ulcorner\phi\urcorner, \mathrm{Code}(\gamma_1), \mathrm{Code}(\gamma_2) \rangle$ for the least such triple $\langle \ulcorner\phi\urcorner, \gamma_1, \gamma_2 \rangle$ in terms of some canonical ordering of triples of ordinals.

Case 3. α is admissible and is the least admissible ordinal greater than γ for some $\gamma < \alpha$. Let Code(α) $= \langle 3, \mathrm{Code}(\gamma) \rangle$ for the least such γ.

Case 4. α is admissible and recursively inaccessible. Then L_α does not reflect some Σ_1^1 formula $\phi(\bar{\beta})$. Let ϕ^* be as in Lemma 15. Then $L_{\alpha^+} \vDash \phi^*(L_\beta, \bar{\beta})$ and $\forall \beta < \alpha$ [Admissible(β) $\to \sim L_{\beta^+} \vDash \phi^*(L_\beta \bar{\beta})$]. Let Code($\alpha$) $= \langle 4, \ulcorner\phi\urcorner, \mathrm{Code}(\bar{\beta}) \rangle$ for the least such $\langle \ulcorner\phi\urcorner, \bar{\beta} \rangle$ and we are done.

COROLLARY 17. *$E(b_0)$ is admissible.*

PROOF. Now b_0 has a counting Σ_1 in L_{b_0} which is hence E-recursive in b_0. Hence the result follows by Gandy selection.

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Indeed, we could see that every incaccessible $\alpha \leqslant b_0$ has a counting Σ_1 in α and hence in $L_{\alpha+1}$. Thus b_0 is less than the first gap ordinal and $b_0 \ll \mu_0 < \alpha_0$.

Let α be a limit ordinal and α^+ the least admissible ordinal greater than α. It follows essentially from the results of Grilliot that

$$\alpha^+ = |\Sigma_2\text{-hyperelementary}|_\alpha,$$

$$E(\alpha) \cap \mathrm{ON} = |\pi_1\text{-hyperelementary}|_\alpha,$$

where $|\Phi\text{-hyperelementary}|_\alpha$ is the supremum of order of types of well-founded hyperelementary subsets of α obtained from formulae $\phi(xy, S) \in \Sigma$. Also α_0 is the first ordinal 'bad' in the following sense:

$$\alpha^+ > |\pi_1\text{-hyperelementary}|_\alpha.$$

To see how α_0 compares with the first gap ordinal for ${}^3E$ in L (call this γ_0^L), let λ, μ, γ be ordinals such that $\mu, \gamma \in \lambda \leqslant \aleph_2^L$, $L_\gamma = E(\mu)$ and

$$\lim(\lambda) \wedge L_\lambda \vDash \exists y\big(\mu = \aleph_1^{L_\lambda} \wedge y = E(\mu)\big).$$

We could define over L_γ MW's to divergent computations in $\mu \cup \{\mu\}$. These will be members of $L_{\gamma+1}$ $(\in L_\lambda)$. These MW's being countable subsets of $\mu = \aleph_1^{L_\lambda}$ will (by a collapsing argument) be members of L_μ $(\in L_\gamma)$. Whence (by the existence of MW's) $E(\mu)$ is inadmissible.

The same collapsing argument which shows $L_{\aleph_1^L} \leqslant_{\Sigma_1} L_{\aleph_2^L}$ gives us $L_\mu \leqslant_{\Sigma_1} L_\lambda$. Thus

$$(*) \qquad L_\mu \vDash \exists\theta\exists p\big(p = \aleph_1^{L_\theta} \wedge L_\theta = E(p) \wedge \sim L_\theta \vDash KP\big).$$

Now consider E-recursive computations with μ. For δ such that $\delta = \omega\delta$ (i.e. $J_\delta = L_\delta$), the computation of L_δ from δ has length ω (close under the rudimentary schemata). The evaluation of a first order predicate over L_μ is a computation of finite length with L_μ (and the parameters). The evaluation of the least p satisfying $(*)$ (where it is plain that the least such $p = \theta_0$) is first order definable over L_μ and hence of computational length less than $\omega + \omega$ in μ. The $<_L$ least counting of α_0 is first order definable over L_μ. Hence this counting is $\leqslant_E \mu$ with length less than $\omega + \omega$. It easily follows that $\alpha_0 < \gamma_0^\mu$, the first gap ordinal for computations with μ.

We must thus state

THEOREM 18. *Let μ be as in the above discussion. Then $\alpha_0 < \gamma_0^\mu \leqslant \gamma_0^L$, the first gap ordinal for ${}^3E$ in L.*

For more about γ_0 the reader may consult the article by Normann in Moldestad [6].

We shall now endeavour to say more about μ_0 in terms of entities related to α_0.

For an ordinal δ such that $E(\delta)$ inadmissible $\wedge \aleph_1^{E(\delta)} = \delta$, let the 1-section of δ be defined as $S_\delta = \{x \mid x \in L_\delta \vDash x \leqslant_E \delta\}$. When we take $\delta = \aleph_1^L$, S_δ would be the classical 1-section for ${}^3E$ in L.

Let

$$\mathrm{Hull}(\delta) = \left\{x \mid x \in L_{\kappa_0^\delta} \wedge \exists\phi \in \Sigma_1 L_{\kappa_0^\delta} \vDash \exists! y\phi(y,\delta) \wedge \phi(x,\delta)\right\},$$

$$\mathrm{Hull}^-(\delta) = \mathrm{Hull}(\delta) \cap L_\delta,$$

$$\eta(\delta) = \mathrm{Hull}^-(\delta) \cap \mathrm{ON}.$$

It is not difficult to see that $S_\delta = \mathrm{Hull}^-(\delta) = L_{\eta(\delta)}$. See Fenstad [2] for Normann's results on Spectra. Furthermore Hull(δ) is 'Admissible with gaps' (Sacks [9]).

LEMMA 19. $\mu_0 < \gamma_0^{\alpha_0} < \eta(\alpha_0) < \alpha_0$.

PROOF. This is seen by a repetition of the argument for Theorem 18.

MORAL. Let μ be such that $E(\mu) \vDash \mu = (\aleph_1)_{E(\mu)}$. Then any ordinal phenomenon which occurs before μ and is first order describable in L_μ without parameters, occurs before γ_0^μ; for example (Moldestad [6]), $V = L \vDash \exists x$ (x is transitive $\wedge\, x \vDash$ ZFC) $\to \exists\alpha < \gamma_0^{\aleph_1}(L_\alpha \vDash$ ZFC).

We now compare α_1 of Lemma 1 with α_0. Call an ordinal α *wonderful* if $E(\alpha)$ is inadmissible.

LEMMA 20. $\alpha_0 < \alpha_1$ *and indeed* α_1 *is a limit of wonderfuls, a limit of limits of wonderfuls, etcetera.*

PROOF. Let β be as in Lemma 1. Choose $\gamma < \beta$ such that $E(\alpha_1) \in L_\gamma$. We can prove as in Theorem 7 that α_1 begins a gap. But unlike in the case of α_0 this gap extends through $E(\alpha_1)$ and L_γ upto β. For if the gap stops short of β, α_1 will be countable in L_β and if it goes beyond β, the reflection of Theorem 6 will contradict the minimality of α_1. Hence $\alpha_0 < \alpha_1$.

Let δ be any ordinal less than α_1. Let $\theta_1(\eta)$ be the following Σ_1 sentence: $\exists y(E(\eta) = y \wedge \delta < \eta \wedge y \vDash \neg KP)$. Then $\theta_1(\eta)$ says $\eta > \delta$ and η is wonderful.

Now $L_\gamma \vDash \exists\eta\theta_1(\eta)$. By reflection $L_{\alpha_1} \vDash \exists\eta\theta(\eta)$. It follows that α_1 is a limit of wonderfuls. That 'η is a limit of wonderfuls' can be expressed as a Δ_0 sentence with parameter L_η. By reflecting again we prove that α_1 is a limit of limits of wonderfuls. The lemma follows by simple induction.

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