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ALMOST-P-SPACES

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A *P-space* is a topological space in which every G_δ-set is open. P-spaces are fairly rare. For example, the only compact (or even pseudocompact) P-spaces are finite. A larger class of spaces, the *almost-P-spaces*, consists of those spaces in which G_δ-sets have dense interiors. The almost-P-spaces are far less restricted than the P-spaces—for example, there are infinite, compact, connected almost-P-spaces. In this paper, we study almost-P-spaces and raise a number of questions relating to them.

1. Preliminaries. All given spaces are assumed to be completely regular. If X is a space, βX denotes the Stone-Cech compactification of X. $\mathbf{R}$ denotes the space of reals and N denotes the countable discrete space. If A is a set, $|A|$ denotes the cardinal of A. By "set theory" we mean Zermelo-Frankel set theory with the axiom of choice, that is, ZFC. *Lusin's hypothesis*, denoted LH, is the set theoretic assumption that $2^{\aleph_1} = 2^{\aleph_0}$. The continuum hypothesis, denoted CH, is the statement that $\aleph_1 = 2^{\aleph_0}$. If ZF is consistent, so are $ZFC + LH$ and $ZFC + CH$ (see [**1**])

The proof of the following proposition is easy.

PROPOSITION 1.1. *For a topological space the following are equivalent:*

(i) *Every non-empty zero set has non-empty interior.*

(ii) *Every non-empty G_δ-set has non-empty interior.*

(iii) *Every zero-set is a regular-closed set.*

(iv) *If G is a G_δ-set,* $\text{Int}_X G$ *is dense in G.*

A space which satisfies the conditions of Proposition 1.1 is called an *almost-P-space.*

Examples. 1) Any P-space is an almost-P-space.

2) The one-point compactification of an uncountable discrete space is an almost-P-space since any non-empty G_δ of such a space contains an isolated point of the space.

3) W. Rudin proved in [**9**] that $\beta N - N$ is an almost-P-space.

4) A generalization of Example 3 due to Fine and Gillman [**2**] is that if X is locally compact and realcompact, $\beta X - X$ is an almost-P-space. Thus, for example, $\beta\mathbf{R} - \mathbf{R}$ is an almost-P-space.

5) In [**6**] it is proved that $\bar{R}$ which is the Dedekind completion of the η_1-set $\bar{Q}$ is an almost-P-space (See [**3**, Chapter 13] for definitions).

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2. Properties of almost-P-spaces.

PROPOSITION 2.1. *A dense subset or an open subset of a (Baire) almost-P-space is a (Baire) almost-P-space.*

Proof. The statements regarding open subsets are trivial. Suppose X is an almost-P-space and D is a dense subset of X. Suppose A is a non-empty G_δ-set of D; then $A = B \cap D$ where B is a G_δ-set of X. $\text{Int}_X B \neq \emptyset$, so $(\text{Int}_X B) \cap D \neq \emptyset$. Therefore $\text{Int}_D A \neq \emptyset$. Thus, D is an almost-P-space. Now suppose that X is a Baire space. If U_i is a dense open subset of D for each i in N, there are sets V_i such that each V_i is open in X and $U_i = V_i \cap D$. $\bigcap_{i=1}^\infty U_i = (\bigcap_{i=1}^\infty V_i) \cap D$. Since X is Baire, $\bigcap_{i=1}^\infty V_i$ is dense in X and so, by (iv) of Proposition 1.1, $\text{Int}_X \bigcap_{i=1}^\infty V_i$ is dense in X. Therefore, since $\bigcap_{i=1}^\infty U_i \supseteq (\text{Int}_X \bigcap_{i=1}^\infty V_i) \cap D$, $\bigcap_{i=1}^\infty U_i$ is dense in D. This shows that D is a Baire space.

Remark. Not every subspace of an almost-P-space is necessarily an almost-P-space. In [**5**] an example is given of an almost-P-space which contains a closed copy of the space of rationals.

It is not in general the case that if X is an almost-P-space, βX is an almost-P-space. For example, βN is not an almost-P-space since the non-empty G_δ-set $\beta N - N$ has empty interior. However, we have the following:

PROPOSITION 2.2. *βX is an almost-P-space if and only if X is a pseudocompact almost-P-space.*

Proof. (Necessity) If βX is an almost-P-space, so is X by Proposition 2.1. Furthermore, if X were not pseudocompact, some non-empty zero-set of βX would be contained in $\beta X - X$ and hence would have empty interior.

(Sufficiency) Suppose X is a pseudocompact almost-P-space. If Z is a non-empty zero-set of βX, $Z \cap X \neq \emptyset$ (since X is pseudocompact). Therefore, $\text{Int}_{\beta X} Z \supseteq \text{Int}_X(Z \cap X) \neq \emptyset$.

Remark. It is not hard to prove that X is an almost-P-space if and only if the Hewitt realcompactification of X is.

The space $\bar{R}$ which was Example 5 of § 1 will be of particular interest to us. We therefore summarize in the following proposition the properties of $\bar{R}$ which we will need.

PROPOSITION 2.3. *$\bar{R}$ is a connected, totally ordered (and hence locally compact) almost-P-space such that $|\bar{R}| = 2^{\aleph_1}$. $\bar{R}$ has no first or last element and $\beta\bar{R}$ is the two-point compactification of $\bar{R}$. $\beta\bar{R}$ is a compact, connected, totally ordered almost-P-space such that $|\beta\bar{R}| = 2^{\aleph_1}$. $\bar{R}$ and $\beta\bar{R}$ each have $2^{\aleph_0}$ points which fail to be P-points.*

Proof. See [**3**] and [**4**].

3. Compactness and cardinal questions. According to Proposition 2.3, if $2^{\aleph_1} = 2^{\aleph_0}$, that is LH, there is a compact dense-in-itself almost-P-space of cardinal $2^{\aleph_0}$, namely, $\beta\bar{R}$. In fact, LH is the only condition under which such a space exists.

PROPOSITION 3.1. (See [**8**] or [**12**]). *If X is a compact, dense-in-itself almost-P-space, then $|X| \geqq 2^{\aleph_1}$.*

COROLLARY 3.2. *LH is equivalent to the existence of a compact dense-in-itself almost-P-space of cardinal* $2^{\aleph_0}$.

Remarks. 1) The one-point compactification of the discrete space of cardinal $2^{\aleph_0}$ is a compact almost-P-space of cardinal $2^{\aleph_0}$ even without set theoretic assumptions, so the "dense-in-itself" is essential to the corollary.

2) The set X_0 of non-P-points of $\bar{R}$ can be easily seen to be a countably compact almost-P-space of cardinal $2^{\aleph_0}$ and hence compactness is also essential in the corollary.

Corollary 3.2 suggests the following question.

Question 1. Can a dense-in-itself almost-P-space X have a compactification of cardinal $2^{\aleph_0}$? What if X is Baire?

Of course, by Corollary 3.2, Question 1 is interesting only if LH is not assumed.

One consequence of the fact that $\bar{R}$ has cardinal $2^{\aleph_1}$ involves the following:

THEOREM 3.3 (Mrowka [**7**]). *Every compact space of cardinal less than $2^{\aleph_1}$ has a point of first countability. Thus if LH fails, every compact space of cardinal $2^{\aleph_0}$ has a point of first countability.*

In his proof of the above theorem, Mrowka explicitly assumes the denial of LH. We ask if the denial of LH is needed to prove Mrowka's theorem. The answer is that it is indeed required—under LH, $\beta\bar{R}$ is compact, has cardinal $2^{\aleph_0}$, and yet has no point of first countability since it is a dense-in-itself almost-P-space. Thus, we have:

COROLLARY 3.4. *LH fails if and only if every compact space of cardinal $2^{\aleph_0}$ has a point of first countability.*

Question 2. If LH, does every compact space of cardinal $2^{\aleph_0}$ have a non-trivial convergent sequence?

Remark. A famous problem attributed to Efimov asks whether every compact space contains either a non-trivial convergent sequence or a copy of βN. A negative answer to Question 2 clearly implies a negative answer to Efimov's problem under LH.

4. P-points of almost-P-spaces. Formally the definition of an almost-P-space is close to that of a P-space—in a P-space zero sets are open whereas in an almost-P-space zero sets have dense interiors. We have seen examples of almost-P-spaces which are not P-spaces (the one-point compactification of an uncountable discrete space, for example). Furthermore, it is possible for an almost-P-space to have no P-points. In fact, $X_0 = \{x \in \bar{R} | x \text{ is not a } P\text{-point of } \bar{R}\}$ is a countably compact almost-P-space with no P-points. The question of P-points in compact almost-P-spaces seems to be difficult.

PROPOSITION 4.1. *Any compact, totally ordered, zero-dimensional almost-P-space* $[a, b]$ *has a dense set of P-points.*

Proof. It suffices to prove that if $(c, d) = \emptyset$, then c is a P-point. If $c = a$ we are done. Therefore, we may assume $c > a$. Any G_δ-set which contains c contains a set of the form $\bigcap_{i=1}^{\infty} (y_i, c]$ where $y_i < c$ for each c. This is an interval, and, since $[a, b]$ is an almost-P-space and c has immediate successor, it contains an open set containing c. Thus c is a P-point.

The next theorem, although not insuring the existence of P-points, does say that there are points which act like P-points with respect to certain families of functions.

THEOREM 4.2. *Suppose X is a compact almost-P-space and* $\mathfrak{F}$ *is a family of continuous functions on X such that the density of* $\mathfrak{F}$ *in the uniform norm topology is at most* $\aleph_1$. *Then there is a dense subset D of X such that if* $f \in \mathfrak{F}$ *and* $x \in D$, *there is a neighborhood V of x such that f is constant on V.*

LEMMA (See [**8**; **12**; or **13**]). *If X is a compact almost-P-space and* $\{B_\lambda\}_{\lambda<\omega_1}$ *is a family of open dense subsets of X, then* $\bigcap_{\lambda<\omega_1} B_\lambda$ *is dense in X.*

Proof of Theorem 4.2. Let $\{g_\lambda\}_{\lambda<\omega_1}$ be a dense subset of $\mathfrak{F}$. For each $\lambda < \omega_1$, let

$$\hat{B}_\lambda = \bigcap_{\delta<\lambda} \left(\bigcup_{r\in\mathbf{R}} \text{Int}_X g_\delta^{-1}(r) \right).$$

Since X is an almost-P-space, each $\bigcup_{r\in R} \text{Int}_X g_\delta^{-1}(r)$ is dense and open. Therefore, by the Baire category theorem each $\hat{B}_\lambda$ is dense in X. Since X is an almost-P-space and $\hat{B}_\lambda$ is a G_δ-set, $B_\lambda = \text{Int}_X \hat{B}_\lambda$ is also dense in X. Therefore, by the lemma, $D = \bigcap_{\lambda<\omega_1} B_\lambda$ is dense in X. Now suppose $f \in \mathfrak{F}$. There are indices $\lambda_1, \lambda_2, \ldots$ such that $\{g_{\lambda_k}\}$ converges to f. Now if $x \in D$, let $V = \text{Int}_X \bigcap_{k=1}^{\infty} g_{\lambda_k}^{-1}(g_{\lambda_k}(x))$. Then since $x \in D$, V is a neighborhood of x. Furthermore, each g_{λ_k} is constant on V. Therefore, if $y \in V$, $f(y) = \lim_k g_{\lambda_k}(y) = \lim_k g_{\lambda_k}(x) = f(x)$. Hence, f is constant on V.

The following corollary which generalizes a theorem of Plank [**8**] also follows from a theorem of Vekslar [**11**] which states that if the weight of a compact almost-P-space is $\aleph_1$, then the space has a dense set of P-points, and a theorem of Smirnov [**10**] which says that for compact X, the weight of X is the same as the density of $C(X)$.

COROLLARY 4.3. *If X is a compact almost-P-space such that the density of the space of continuous functions is $\aleph_1$, then X has a dense set of P-points. In particular, under CH if there are only $2^{\aleph_0}$ continuous functions, X has a dense set of P-points.*

Remark. It is possible to show that if in Theorem 4.2, X is assumed to be dense-in-itself, then D may be taken so that $|D| \geqq 2^{\aleph_1}$. Thus, in Corollary 4.3 if X is dense-in-itself, there are at least $2^{\aleph_1}$ P-points.

Question 3. Does every compact almost-P-space have a P-point? If LH, does every compact almost-P-space of cardinal $2^{\aleph_0}$ have a P-point?

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