Three discs for the Mittenpunkt

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1. *Introduction*

The aim of this paper is to give a solution to three conjectures from Euclidean geometry concerning the location of the Mittenpunkt. The first two are solved without dependence on computer technology and with only a moderate amount of calculations. They were initially tackled by heavy calculations using computer algebra systems.

For unknown reasons, the remarkable *Mittenpunkt* M is a somewhat neglected triangle centre, in spite of its intrinsic importance and usefulness in triangle geometry [1]. In the comprehensive Kimberling's encyclopedia of triangle centers (ETC) [2], it is denoted by X_9. The centre, discovered by the German mathematician Christian Heinrich von Nagel (1803-1882) in 1836, is the point of concurrency of the lines passing through the centres of the excircles and the corresponding midpoints of the sides of the triangle. Hence its name Mittenpunkt. By definition, it is the symmedian point of the excentral triangle of ABC. For many more properties of the Mittenpunkt the interested reader is referred to [3].

The incentre I of a triangle with orthocentre H and centroid G lies in the disc with diameter HG, the well-known orthocentroidal disc $\mathbb{D}_{HG}$ [4].

The *Brocard disc* of a triangle ABC is the disc $\mathbb{D}_{OK}$ with diameter OK where O and K are the circumcentre and the symmedian point, i.e. the point of concurrence of the symmedians of ABC, respectively, see Figure 1. It is named after the French army officer Henri Brocard (1845-1922). Along with Lemoine and Neuberg, he is widely considered as one of the founders of the modern geometry of the triangle. A large part of it is consequently named ‘Brocard geometry’ and includes the two points Ω_1 and Ω_2 which bear his name. These are the points in ABC with the following equal angle property:

$$\angle\Omega_1 AB = \angle\Omega_1 BC = \angle\Omega_1 CA \text{ and } \angle\Omega_2 BA = \angle\Omega_2 CB = \angle\Omega_2 AC.$$

The two angles formed by Ω_1 and Ω_2 are equal and this common angle is the *Brocard angle* ω of ABC, see [3], [5], [6].

We call the disc on diameter OG the *circumcentroidal disc* and denote it by $\mathbb{D}_{OG}$. The *Parry circle* [3] is the circumcircle of triangle J_1J_2G, where J_1 and J_2 are the isodynamic points, denoted in ETC by X_{15} and X_{16}, respectively. We denote the corresponding Parry disc by $\mathbb{D}_P$.

In [7], Graeme Taylor poses three conjectures about the location of the Mittenpunkt.

Conjectures:

1. The Mittenpunkt is constrained to the Brocard disc $\mathbb{D}_{OK}$.
2. The Mittenpunkt is constrained to the circumcentroidal disc $\mathbb{D}_{OG}$.
3. The Mittenpunkt cannot lie in the Parry disc $\mathbb{D}_P$.

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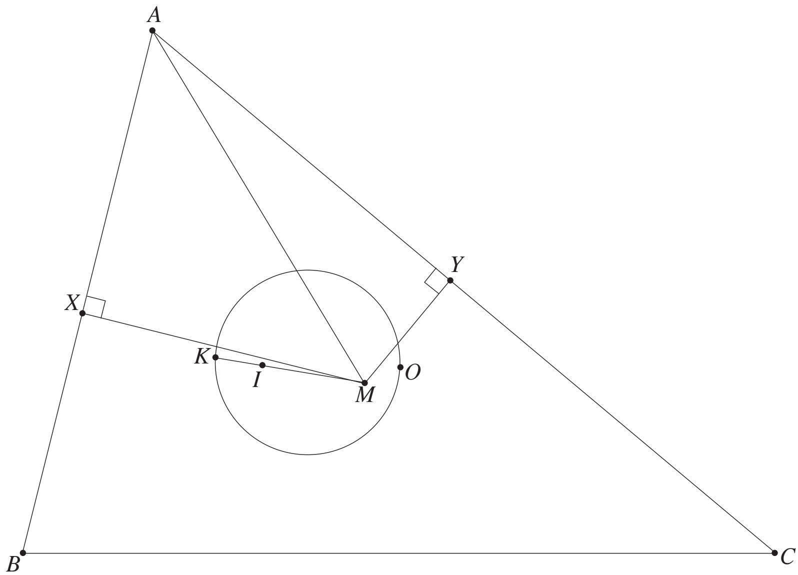


FIGURE 1: Brocard disc $\mathbb{D}_{OK}$, Mittenpunkt M and the Incentre I

The computer proof of the second conjecture given in [7] uses *Maple* to deal with the unwieldy expressions which end up in a nasty inequality. The result for the Mittenpunkt is very nice but the geometer is left unsatisfied when he himself cannot confirm the calculations and must leave them to be confirmed by *Mathematica*.

In this paper we will present proof of the properties concerning the Mittenpunkt M, the Brocard disc $\mathbb{D}_{OK}$ and the circumcentroidal disc $\mathbb{D}_{OG}$. The first two geometric objects have been known for more than a century, but the relationship between them is new, showing that the subject of Euclidean geometry is still vibrant. *A posteriori*, from the first conjecture, the incentre I also lies in the Brocard disc [8], since the centres K, I and M are collinear (in that order) [9]. Recently, in [10], the term 'symmedicentroidal disc' is coined for the disc $\mathbb{D}_{KG}$ with diameter KG and it was shown that the incentre is additionally constrained to this disc.

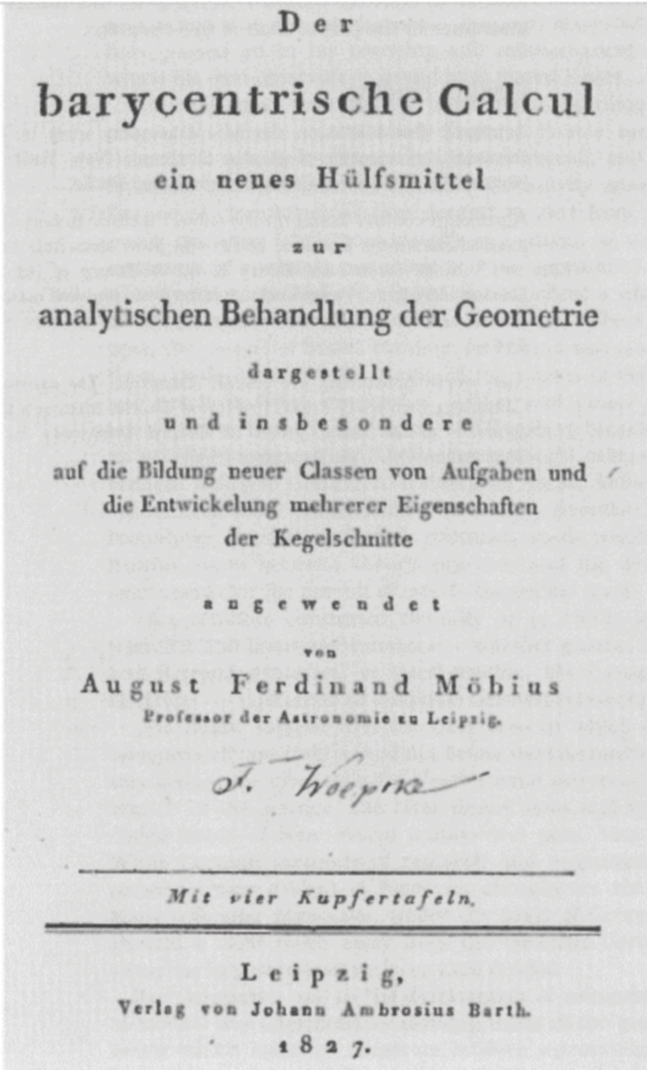
2. *Möbius' 'Der barycentrische Calcul'*

To prove that M lies in the Brocard disc $\mathbb{D}_{OK}$ and the circumcentroidal disc $\mathbb{D}_{OG}$, we will show that the angles $\angle OMK$ and $\angle OMG$ are obtuse. Hence it will be sufficient to prove the inequalities

$$OK^2 \geqslant OM^2 + MK^2 \tag{1}$$

and

$$OG^2 \geqslant OM^2 + MG^2. \tag{2}$$



Der

barycentrische Calcul

ein neues Hülfsmittel

zur

analytischen Behandlung der Geometrie

dargestellt

und insbesondere

auf die Bildung neuer Classen von Aufgaben und
die Entwickelung mehrerer Eigenschaften
der Kegelschnitte

angewendet

von

August Ferdinand Möbius

Professor der Astronomie zu Leipzig.

F. Woepcke

Mit vier Kupfertafeln.

Leipzig,

Verlag von Johann Ambrosius Barth.

1827.

FIGURE 2: Der barycentrische Calcul

Since the inequalities are hard to access by Euclidean geometry, in order to make them more tractable, we will make use of the barycentric calculus developed by the German mathematician and astronomer August Ferdinand Möbius (1790-1868) in his 1827 book *Der barycentrische Calcul*. Möbius is famous for his surface with just one side, called the Möbius strip, but he also made other major contributions to mathematics. Named after him are Möbius transformations in analysis, the Möbius function and the Möbius inversion formula in number theory. However, of all Möbius' works, we

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read in Felix Klein's influential 1926 book [11, p. 117], that the most important is the first-mentioned:

'Unter den Werken von Moebius steht zeitlich und inhaltlich als sein Fundamentalwerk Der Barycentrische Calcul von 1827 voran, eine wahre Fundgrube neuer Ideen in wunderbar abgeklärter Darstellung.'

Carl B. Boyer, another historian of prominence, thirty years later praises the work even more [12, p. 242]:

'The year 1827 is of considerable importance in the history of analytic geometry in Germany for reasons far removed from Jacobi's work. It is sometimes said that Descartes arithmetized geometry, but this is not strictly correct. For almost two hundred years after his time coordinates were in essence geometric. Cartesian coordinates were line segments, and polar coordinates were vectorial radii and circular arcs. Even the areal coordinates of Carnot were largely geometric. The arithmetization of coordinates took place not in 1637 but in the crucial years 1827-1829. Bobillier should be remembered as anticipating the new point of view to a certain extent, but otherwise the change came with a certain suddenness in 1827 with the Barycentrische Calcul of A. F. Mobius.'

The interested reader can find much more about Möbius' life, his legacy, and mathematics and astronomy of the period in [13].

We return now to the treatment of the problem using barycentric calculus. Barycentric coordinates of the centres will enable us to calculate the distances between them in terms of the circumradius R, inradius r and semiperimeter s of the triangle ABC with sides a, b and c.

Let the point $P = (x : y : z)$ be given in areal coordinates, that is, with normalised barycentric coordinates, $x + y + z = 1$. If the distances from an arbitrary point Q to the vertices of ABC are known, then its distance from P can be calculated by the following important formula:

$$QP^2 = xQA^2 + yQB^2 + zQC^2 - yza^2 - zxb^2 - xyc^2. \tag{3}$$

In conjunction with the areal coordinates of the Mittenpunkt [9],

$$M = \left(\frac{a(s-a)}{D} : \frac{b(s-b)}{D} : \frac{c(s-c)}{D}\right), \quad D = \sum a(s-a) = 2r(4R+r),$$

the formula (3) is used in [1] to find the distance OM where

$$OM^2 = R^2 - \frac{2R(2R - r)s^2}{(4R + r)^2}. \tag{4}$$

The symmedian point has barycentric coordinates [3], [9] $K\left(a^2 : b^2 : c^2\right)$. Hence by (3) it is immediate that

$$OK^2 = R^2 - \frac{3a^2b^2c^2}{(a^2 + b^2 + c^2)^2}. \tag{5}$$

We are left with the task of finding MG^2 and MK^2, which are the key difficulties in the proof. In general, the distance of $P = (x : y : z)$ from BC is

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$2\Delta|x|/a$ where Δ is the area of ABC. Thus the perpendicular distances of the Mittenpunkt M from the sides AB and AC are

$$MX = \frac{s(s-c)}{4R+r}, \qquad MY = \frac{s(s-b)}{4R+r}. \tag{6}$$

The central place in the proof of the two conjectures is the following lemma which is of independent interest.

Lemma 1: The distance between M and the vertex A is given by

$$MA^2 = \frac{s^2}{(4R+r)^2}\left[4R^2 - \frac{bc(s-b)(s-c)}{s(s-a)}\right].$$

Proof: From the cyclic quadrilateral $AXMY$ and triangle XMY with circumradius $\frac{1}{2}MA$ (see Figure 1) we find, without going into the details of the simple calculations, using (6) that

$$MA^2 = \frac{XY^2}{\sin^2 A} = \frac{1}{\sin^2 A}\left(MX^2 + MY^2 + 2MX \times MY \cos A\right)$$
$$= \frac{s^2}{(4R+r)^2}\left[4R^2 - \frac{bc(s-b)(s-c)}{s(s-a)}\right].$$

3. *The proof of the second conjecture*

We begin by proving the second conjecture. The crucial step in the proof of (1) is the following theorem

Theorem 1: The distance between M and the centroid G is given by

$$MG^2 = \frac{(12R^2 + 8Rr - r^2)s^2 - r(4R+r)^3}{3(4R+r)^2} - \frac{a^2+b^2+c^2}{9}. \tag{7}$$

Proof: We will make use of the triangle identity

$$\sum bc(s-b)^2(s-c)^2 = r^3\left[(4R+r)^3 - (8R-r)s^2\right]. \tag{8}$$

It follows easily from

$$(s-b)^2(s-c)^2 = s^4 - 2as^3 + \left(a^2 - 2bc\right)s^2 + 2abcs + b^2c^2$$

and the well-known triangle identities

$$\sum ab = s^2 + r(4R+r), \sum a^2 = 2\left(s^2 - r(4R+r)\right), \sum a^3 = 2s\left(s^2 - 6Rr - 3r^2\right).$$

To calculate MG^2 we use the Leibniz formula

$$GP^2 = \frac{1}{3}\sum AP^2 - \frac{a^2+b^2+c^2}{9},$$

where P is an arbitrary point in the plane of triangle ABC.

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Hence from Lemma 1 and (8), we obtain

$$MG^2 = \frac{1}{3}\sum AM^2 - \frac{a^2 + b^2 + c^2}{9}$$

$$= \frac{4R^2s^2}{(4R+r)^2} - \frac{1}{3r^2(4R+r)^2}\sum bc(s-b)^2(s-c)^2 - \frac{a^2+b^2+c^2}{9}$$

$$= \frac{(12R^2 + 8Rr - r^2)s^2 - r(4R+r)^3}{3(4R+r)^2} - \frac{a^2+b^2+c^2}{9}.$$

In the calculations, Heron's formula for the area, $\Delta = rs$ and $abc = 4Rrs$ are also used.

Next we prove Conjecture 2 after it is reduced to an interesting triangle inequality. The inequality is given two totally different proofs, one direct and the other less straightforward but shorter.

Theorem 2: The Mittenpunkt M lies within the circumcentroidal disc $\mathbb{D}_{OG}$.

Proof: The identity

$$OG^2 = R^2 - \frac{a^2 + b^2 + c^2}{9}$$

is well known. By (4) and (7), the inequality (2) can be rewritten as

$$6R(2R - r)s^2 \geqslant (12R^2 + 8Rr - r^2)s^2 + r(4R + r)^3,$$

which is equivalent to

$$s^2 \leqslant \frac{(4R + r)^3}{14R - r}. \tag{9}$$

This inequality follows from the fundamental Kooi's inequality [14, 1]

$$s^2 \leqslant \frac{R(4R + r)^2}{2(2R - r)},$$

since $\frac{R(4R + r)^2}{2(2R - r)} \leqslant \frac{(4R + r)^3}{14R - r}$ is equivalent to $(2R + r)(R - 2r) \geqslant 0$, which is true by Euler's inequality $R \geqslant 2r$.

A more fanciful proof of (9) can be given using the mixtilinear radii ρ_A, ρ_B, ρ_C [16]. From the relations

$$\rho_A + \rho_B + \rho_C = r\left(1 + \left(\frac{4R + r}{s}\right)^2\right)$$

and

$$\frac{1}{\rho_A} + \frac{1}{\rho_B} + \frac{1}{\rho_C} = \frac{4R + r}{2Rr},$$

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we observe that the Cauchy-Schwarz inequality

$$\left(\rho_A + \rho_B + \rho_C\right)\left(\frac{1}{\rho_A} + \frac{1}{\rho_B} + \frac{1}{\rho_C}\right) \geqslant 9$$

is equivalent to (9). The proof is complete.

4. *The proof of the first conjecture and consequences*

Theorem 3: The distance between the Mittenpunkt M and the symmedian point K is given by

$$MK^2 = \frac{4R^2s^2}{(4R+r)^2} - \frac{8Rr\left(8R^2+2Rr-s^2\right)s^2}{\left(a^2+b^2+c^2\right)(4R+r)^2} - \frac{3a^2b^2c^2}{\left(a^2+b^2+c^2\right)^2}. \tag{10}$$

Proof: The triangle identity needed for the proof is

$$\sum a(s-b)^2(s-c)^2 = 2r^2s\left(8R^2+2Rr-s^2\right). \tag{11}$$

It is proved similarly as the identity (8). Employing (3), Lemma 1 and the triangle identity (11), we get

$$\begin{aligned} MK^2 &= \sum \frac{a^2}{a^2+b^2+c^2}MA^2 - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2} \\ &= \frac{4R^2s^2}{(4R+r)^2} - \frac{abc\sum a(s-b)^2(s-c)^2}{\left(a^2+b^2+c^2\right)(4R+r)^2r^2} - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2} \\ &= \frac{4R^2s^2}{(4R+r)^2} - \frac{8Rr\left(8R^2+2Rr-s^2\right)s^2}{\left(a^2+b^2+c^2\right)(4R+r)^2} - \frac{3a^2b^2c^2}{(a^2+b^2+c^2)^2}. \end{aligned}$$

We now have all the necessary ingredients for the proving the first conjecture

Theorem 4: The Mittenpunkt M is constrained to the Brocard disc $\mathbb{D}_{OK}$.

Proof: The statement is equivalent to the inequality (1). Having found all the distances between the triangle centres O, K and M, by (5), (4) and (10), the inequality (1) reads

$$\frac{8Rr\left(8R^2+2Rr-s^2\right)s^2}{\left(a^2+b^2+c^2\right)(4R+r)^2} \geqslant \frac{4R^2s^2}{(4R+r)^2} - \frac{2R(2R-r)s^2}{(4R+r)^2}.$$

Sorting out, by $\sum a^2 = 2\left(s^2 - r(4R+r)\right)$, the inequality can be rewritten as

$$3s^2 \leqslant (4R+r)^2. \tag{12}$$

This last inequality follows from (9) and Euler's inequality $R \geqslant 2r$. We obtain

$$3s^2 \leqslant \frac{3(4R+r)^3}{14R-r} \leqslant (4R+r)^2.$$

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It is worth remarking that (12) is an equivalent form of the famous Finsler-Hadwiger inequality from 1937 (see [17, 18, 19]):

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\Delta + (a - b)^2 + (b - c)^2 + (c - a)^2 .$$

The proof that the Mittenpunkt M is in the Brocard disc is complete.

As a direct consequence of the theorem, in addition to the given restrictions for the location of the incentre I from [4], we now have one more. Since I is collinear with K and M, and lies between them [9], we deduce:

Theorem 5: The incentre I lies inside the Brocard disc $\mathbb{D}_{OK}$.

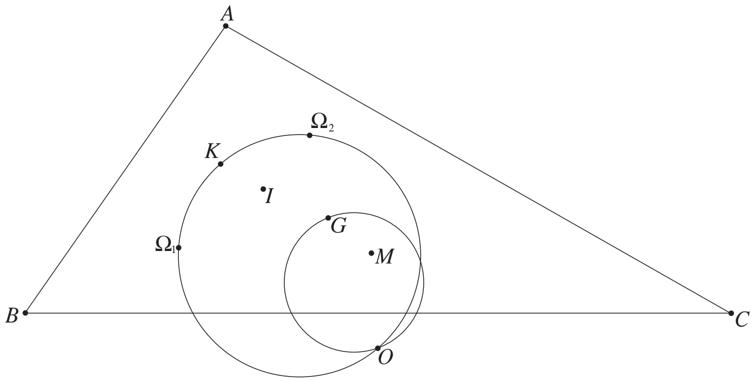

FIGURE 3: Mittenpunkt M, constrained to circumcentroidal disc $\mathbb{D}_{OG}$ and to Brocard disc $\mathbb{D}_{OK}$

Another, direct and simpler proof of the last theorem can be found in [8].

We now have a pretty good understanding of the location of the Mittenpunkt M; see Figure 3 with the two discs $\mathbb{D}_{OK}$ and $\mathbb{D}_{OG}$.

It is worthwhile summarising the known facts about various points and their relationship to the Brocard circle and the Brocard disc; see Figure 3. We have four points on the Brocard circle

- the circumcentre O;
- the symmedian point K;
- the Brocard points Ω_1 and Ω_2.

In the Brocard disc $\mathbb{D}_{OK}$ we have

- the incentre I;
- the Mittenpunkt M.

The Brocard disc $\mathbb{D}_{OK}$ certainly contains many more interesting centres that remain to be discovered.

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5. *The proof of the third conjecture*

This time the conjecture is about the place where the Mittenpunkt should not be looked for – within the Parry circle. Unfortunately we have not been able to give a simple proof like the proofs of the two previous conjectures.

Let P be the centre of the Parry circle, r its radius and M the Mittenpunkt. The centre P of the Parry circle is known to be X(351) in ETC where its barycentric coordinates are given by:

$$\left(a^2\left(b^2-c^2\right)\left(-2a^2+b^2+c^2\right), b^2\left(c^2-a^2\right)\left(a^2-2b^2+c^2\right), c^2\left(a^2-b^2\right)\left(a^2+b^2-2c^2\right)\right).$$

The squared radius of the Parry circle is

$$r^2 = \frac{a^2b^2c^2\left(a^4 - a^2b^2 + b^4 - a^2c^2 - b^2c^2 + c^4\right)^2}{9\left(a^2 - b^2\right)^2\left(a^2 - c^2\right)^2\left(b^2 - c^2\right)^2}.$$

Conjecture 3 that the Mittenpunkt M lies outside $\mathbb{D}_P$ is actually the inequality

$$MP^2 - r^2 > 0.$$

The distance MP^2 is calculated to be something really nasty by *Mathematica* and then

$$MP^2 - r^2 > 0$$

reduces to showing that the term

$$\frac{2Rs^2\left(r^3 - 10r^2R + 5r\left(s^2 - 4R^2\right) + 2Rs^2\right)}{3\left(r + 4R\right)^2\left(r^2 + 2rR + s^2\right)}$$

is always positive, which is true and not difficult to show. The interested readers can find the algorithm used in [20].

We would like to see a proof of this conjecture in the spirit of the previous two, but for now it seems out of reach.

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