A DESCRIPTION OF THE PROJECTIVE STONE ALGEBRAS

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0. Introduction. In his book [**6**] Grätzer sets a guideline for the research on Stone algebras; in view of his and Chen's triple characterization of Stone algebras [**3**], he considers a problem for these algebras solved if it can be reduced to a problem for Boolean algebras and one for distributive lattices with 1. In the same book he lists as Problem 53: describe the projective Stone algebras. For a Stone algebra L, let B_L be its centre and D_L its dense set. In order that L be projective, B_L has to be a Boolean retract of B_F for some free Stone algebra F, and D_L has to be a retract of D_F in the category of distributive lattices with 1. These conditions are, however, not sufficient, so one hopes to characterize the projective Stone algebras by adding further conditions, and in this spirit we in fact arrive at a description of these algebras. We also show, however, that every such algebra is a retract of some D_F, F a free Stone algebra, so we conclude that there is no nice structural characterization of the projective Stone algebras along the line of Grätzer's programme.

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1. Preliminaries. A Stone algebra $(L, +, \cdot, {}^*, 0, 1)$ is a pseudocomplemented distributive lattice with 0 and 1, such that $x^* + x^{**} = 1$ for all $x \in L$. Throughout the paper any structure will be identified with its underlying set. The set $B_L = \{x^* \mid x \in L\}$ is called the centre of L; it is the set of complemented elements of L, and, as a subalgebra of L, it is a Boolean algebra. The set $D_L = \{x \in L \mid x^* = 0\}$ is called the dense set of L, and it is a filter in L. Finally, L is a double Stone algebra if every $x \in L$ has a dual pseudocomplement x^+, and if $x^+ \cdot x^{++} = 0$.

The reader is referred to [**6**] for the basic facts on Stone algebras. A Stone algebra homomorphism is a lattice homomorphism which also preserves *, 0, and 1.

1.1. Lemma (Chen, Grätzer [**3**]). *Let L and M be Stone algebras, and suppose $f_1 : B_L \to B_M$ is a Boolean homomorphism, and that $f_2 : D_L \to D_M$ is a lattice homomorphism preserving* 1. *Then there is a Stone algebra homomorphism $f : L \to M$ such that $f \mid B_L = f_1$ and $f \mid D_L = f_2$ if and only if $f_1(a) \leq f_2(x)$ for all $a \in B_L$ and $x \in D_L$ which satisfy $a \leq x$. In this case, the extension f is unique. Furthermore, f is onto (one-to-one) if and only if f_1 and f_2 are onto (one-to-one).*

If L is a double Stone algebra, this easily implies

1.2. Lemma. *Let L, M, f_1, and f_2 be defined as in Lemma* 1.1., *and let L be a double Stone algebra. Then there is an extension f of f_1 and f_2 over L if and only if $f_1(x^{++}) \leq f_2(x)$ for all $x \in D_L$.*

Note that f does not necessarily preserve $^+$.

If I is an ideal of a Stone algebra L, then I determines a lattice congruence if we set $x \equiv y$ if there is a $z \in I$ such that $x+z=y+z$, but L/I does not have to be a Stone algebra, and, even when it is, the canonical epimorphism need not preserve *. The following lemma, the proof of which is left to the reader, tells us when it does. Call an ideal I *normal* if $x \in I$ implies $x^{**} \in I$; note that a normal ideal has the form $(J]$ for an ideal J of B_L.

1.3. Lemma. *Let I be an ideal of a Stone algebra L. Then the induced congruence preserves pseudocomplementation if and only if I is normal.*

Note that in particular L/I is a Stone algebra for a Stone algebra L and I a normal ideal of L.

Let α be a nonzero cardinal, and for every $i<\alpha$ let F_i be the free Stone algebra generated by some element a_{i0}.

1.4. Theorem (Balbes, Horn [**1**]). *Let F be the free product of $\{F_i \mid i<\alpha\}$ in the category of distributive lattices with 0 and 1. Then F is the free Stone algebra on the free generators $\{a_{i0} \mid i<\alpha\}$.*

For the rest of the paper F_α will be the free Stone algebra on the free generators $\{a_{i0} \mid i<\alpha\}$ with B_α its centre, and D_α its dense set. For every $i<\alpha$ set $a_{i1}=a_{i0}^{*}$, $a_{i2}=a_{i0}^{**}$, and let, for $r \in \{0,1,2\}$,

$$A^r=\{a_{ir} \mid i<\alpha\}.$$

Furthermore, set $A=A^0 \cup A^1 \cup A^2$. Then Theorem 1.4 implies that for every $x \in F_\alpha \backslash \{0,1\}$ there are nonempty finite subsets $T_1,\ldots,T_m$ and $S_1,\ldots,S_r$ of A, such that $x=\sup T_1 \cdot \ldots \cdot \sup T_m$, and $x=\inf S_1+\ldots+\inf S_r$. The sets used in the representation of an $x \in F_\alpha$ are always assumed to be nonempty finite subsets of A. If T is such a set, let

$$\bar{T}=T \cup \{a_{i0} \mid a_{i2} \in T\}.$$

Then $\sup T=\sup \bar{T}$, since $a_{i0} \leq a_{i2}$. The following two lemmas are taken from Katrinák [**7**].

1.5. Lemma. *Let T be a nonempty finite subset of A.*

(1) $\sup T=1$ *if and only if there is an $i<\alpha$, such that $a_{i1}, a_{i2} \in T$.*

(2) $\inf T=0$ *if and only if there is an $i<\alpha$, such that $a_{i0}, a_{i1} \in \bar{T}$.* (cf. also Balbes, Horn [**1**])

(3) $\sup T \in D_\alpha$ *if and only if there is an $i<\alpha$, such that $a_{i0}, a_{i1} \in \bar{T}$.*

1.6. Lemma. *Let $x, y \in F_\alpha$, $x=\sup T_1 \cdot \ldots \cdot \sup T_m$, $y=\sup W_1 \cdot \ldots \cdot \sup W_p$, such that $\sup W_j \neq 1$ for all $j \leq p$. Then $x \leq y$ if and only if for every $j \leq p$ one of the following conditions holds:*

(1) *there is an $i \leq m$, such that T_i is a subset of $\bar{W}_j$;*

(2) *for all $i \leq m$, T_i is not a subset of $\bar{W}_j$, and*

$$\sup(T_1 \backslash \bar{W}_j) \cdot \ldots \cdot \sup(T_m \backslash \bar{W}_j)=0.$$

In the same paper Katrinák has also shown that B_α is the free Boolean algebra freely generated by $\{a_{i2} \mid i<\alpha\}$.

2. The characterization. Recall that a Stone algebra L is projective if and only if it is a retract of some F_α. It follows from 1.1 that then B_L is a Boolean retract of B_α, i.e. a projective Boolean algebra by Katrinák's result, and that D_L is a retract of D_α in the category of distributive lattices with 1. We have shown in [**5**]

2.1. LEMMA. *A projective Stone algebra is a double Stone algebra.*

Proof. F_1 is isomorphic to its dual lattice, so it follows from 1.4 that any free Stone algebra is a double Stone algebra. Now let L be projective, and $f: F_\alpha \to L$ be an onto Stone homomorphism. Then there is a Stone homomorphism $g: L \to F_\alpha$, such that $f \circ g = \mathrm{id}\, L$. For $x \in L$ define $x^+ = f(g(x)^+)$; it is not hard to show that x^+ is the dual pseudocomplement of x, and that $x^+ \cdot x^{++} = 0$.

Note that neither f nor g necessarily preserves $^+$, but that for any $y \in g[L]$, $f(y^+) = f(y)^+$.

In order to arrive at our description of the projective Stone algebras, we shall first exhibit a certain class of such algebras. As a preparation we shall need one more lemma, the proof of which is an easy consequence of 1.6 and is left to the reader.

2.2. LEMMA. *Suppose $b, c \in B_\alpha$, and M_1 and M_2 are nonempty finite subsets of A^0, which are disjoint. Then* $\inf M_1 \cdot b \leq \sup M_2 + c$ *implies* $\inf M_1 . b \leq c$.

Note that, by the lemma, $a_{i0} \not\leq c$ and $j \neq i$ imply $a_{i0} \not\leq a_{j0} + c$.

2.3. THEOREM. *Let I be a normal ideal of F_α, such that $L = F_\alpha/I$ has a projective centre. Then L is a projective Stone algebra.*

Proof. Let $f: F_\alpha \to L$ be the canonical epimorphism, and let $g_1: B_L \to B_\alpha$ be a Boolean homomorphism such that $f \circ g_1 = \mathrm{id}\, B_L$. Let

$$J = \{i < \alpha \mid f(a_{ik}) \notin \{0, 1\} \text{ for some } k \in \{0, 1, 2\}\};$$

we can assume without loss of generality that J is nonempty. Then,

$$S = \{f(a_{ik}) \mid i \in J, k = 0, 1, 2\}$$

is a set of generators for L. Now we define a mapping $r: S \to F_\alpha$ by

$$r(f(a_{ik})) = \begin{cases} g_1(f(a_{ik})) & \text{if} \quad k = 1, 2, \\ g_1(f(a_{i2})) \cdot (a_{i0} + a_{i1}) & \text{if} \quad k = 0. \end{cases}$$

For all $i \in J$, a_{i0} is not an element of I, hence $a_{i0} \not\leq b$ for all $b \in I \cap B_\alpha$. It is then implied by 2.2 that for all $i, j \in J$, $i \neq j$, we have $f(a_{i0}) \not\leq f(a_{j0})$; hence, r is well-defined.

Suppose

$$T = \{f(a_{i_1 0}), \ldots, f(a_{i_k 0}), f(b_1), \ldots, f(b_r)\},$$

and

$$W=\{f(a_{j_10}),\ldots,f(a_{j_s0}),f(c_1),\ldots,f(c_t)\}$$

are subsets of S, where $b_1,\ldots,b_r$ and $c_1,\ldots,c_t$ are elements of $A^1\cup A^2$. Let $\inf T\leqslant \sup W$; it is well-known that r can be extended to a lattice homomorphism over L if $\inf r[T]\leqslant \sup r[W]$; the extension preserves 0 (1), if $\inf T=0$ ($\sup T=1$) implies $\inf r[T]=0$ ($\sup r[T]=1$).

$\inf T\leqslant \sup W$ implies that there is an $x\in I$, such that

$$a_{i_10}\cdot\ldots\cdot a_{i_k0}\cdot b_1\cdot\ldots\cdot b_r\leqslant a_{j_10}+\ldots+a_{j_s0}+c_1+\ldots+c_t+x.$$

Since I is normal, we can suppose without loss of generality that x is an element of B_α. If $a_{i_n0}=a_{j_m0}$ for some $n\leqslant k$, $m\leqslant s$, then obviously $\inf r[T]\leqslant \sup r[W]$, so suppose without loss of generality that $\{a_{i_10},\ldots,a_{i_k0}\}$ and $\{a_{j_10},\ldots,a_{j_s0}\}$ are disjoint. Lemma 2.2 tells us now that

$$a_{i_10}\cdot\ldots\cdot a_{i_k0}\cdot b_1\cdot\ldots\cdot b_r\leqslant c_1+\ldots+c_t+x.$$

Since the right side of the inequality is Boolean, this in turn implies

$$a_{i_12}\cdot\ldots\cdot a_{i_k2}\cdot b_1\cdot\ldots\cdot b_r\leqslant c_1+\ldots+c_t+x,$$

hence

$$g_1(f(a_{i_12}))\cdot\ldots\cdot g_1f(a_{i_k2}))\cdot g_1(f(b_1))\cdot\ldots\cdot g_1(f(b_r))\leqslant g_1(f(c_1))+\ldots+g_1(f(c_t)),$$

and therefore

$$g_1(f(a_{i_12}))\cdot(a_{i_10}+a_{i_11})\cdot\ldots\cdot g_1(f(a_{i_k2}))\cdot(a_{i_k0}+a_{i_k1})\cdot g_1(f(b_1))\cdot\ldots\cdot g_1(f(b_r))$$
$$\leqslant g_1(f(c_1))+\ldots+g_1(f(c_t))$$

This easily implies $\inf r[T]\leqslant \sup r[W]$. It is also not hard to show that $\inf T=0$ ($\sup T=1$) implies $\inf r[T]=0$ ($\sup r[T]=1$), so r can be extended to a lattice homomorphism $\bar{r}:L\to F_\alpha$, which preserves 0 and 1.

To show that $\bar{r}$ preserves pseudocomplementation, it suffices to show that $\bar{r}(f(a_{i0}+a_{i1}))$ is in D_α:

$$\begin{aligned}\bar{r}(f(a_{i0}+a_{i1}))^* &= (r(f(a_{i0}))+r(f(a_{i1})))^*\\ &= r(f(a_{i0}))^*\,.\,r(f(a_{i1}))^*\\ &= (g_1(f(a_{i2}))\cdot(a_{i0}+a_{i1}))^*\cdot g_1(f(a_{i1}))^*\\ &= g_1(f(a_{i2}))^*\,.\,g_1(f(a_{i1}))^*\\ &= 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}f(r(f(a_{i0}))) &= f(g_1(f(a_{i2}))\cdot(a_{i0}+a_{i1}))\\ &= f(a_{i2})\cdot(f(a_{i0})+f(a_{i1}))\\ &= f(a_{i0}).\end{aligned}$$

Thus, L is a retract of F_α, and therefore projective.

For those readers who are familiar with the topological duality of Stone algebras (see Priestley [**8**]), the following facts seem worthy of mention. If M is an arbitrary Stone algebra, the topological and order components of $S(M)$, the Stone space of M, are of the form $\{Q \in S(M) \mid P \subseteq Q\}$, where P is a minimal prime ideal of M (cf. [**5**]). The prime ideals of L are in one-to-one correspondence with those prime ideals of F_α which contain I, and I is the intersection of a family of minimal prime ideals of F_α. Thus, $S(L)$ consists of "whole" components of $S(F_\alpha)$.

The proof of the following corollary can be derived mutatis mutandis from the proof of Theorem 3.4 of [**5**], of which it is a generalization.

2.4. COROLLARY. *Let L be a projective Stone algebra and let I be a normal ideal of L. If L/I has a projective centre, then it is a projective Stone algebra.*

The fact that every countable Boolean algebra is projective provides us with

2.5. COROLLARY. *If L is a projective Stone algebra with a countable centre and I a normal ideal of L, then L/I is projective.*

Now we can describe the projective Stone algebras.

2.6. THEOREM. *A Stone algebra L is projective if and only if it is a double Stone algebra and there are a free Stone algebra F and an onto Stone homomorphism* $f: F \to L$, *such that the following two conditions hold.*

(1) *There is a Boolean homomorphism* $g_1: B_L \to B_F$ *such that* $f \circ g_1 = \mathrm{id}\, B_L$.

(2) *There is a lattice homomorphism* $g_2: D_L \to D_F$ *such that* $f \circ g_2 = \mathrm{id}\, D_L$, $g_2(1) = 1$, *and* $f(g_2(x)^+) = x^+$ *for all* $x \in D_L$.

Proof. Because of 2.1 and the remarks at the beginning of this section we need only show sufficiency.

Suppose I is the ideal of F which is generated by $f^{-1}(0) \cap B_F$. Then I is normal, and the centre of F/I is isomorphic to the centre of L which is projective by our hypothesis. Hence, F/I is a projective Stone algebra by Theorem 2.3. Let $s: F \to F/I$ be the canonical epimorphism; note that s also preserves dual pseudocomplementation. We shall show now that L is a retract of F/I.

Define $q: F/I \to L$ by $q(s(x)) = f(x)$ for all $x \in F$. Suppose $s(x) = s(y)$; then there is an $a \in I \cap B_F$, such that $x + a = y + a$. Hence, $f(x) = f(y)$, and q is well defined. Obviously, q is onto, and q_1, the restriction of q to the centre of F/I, is an isomorphism. Let $g_1: B_L \to B_F$ be a Boolean homomorphism which satisfies $f \circ g_1 = \mathrm{id}\, B_L$, and set $p_1 = s \circ g_1$. Then $p_1 = q_1^{-1}$.

Let $g_2: D_L \to D_F$ be a mapping having the properties prescribed in (2) above. Note that $f(g_2(x)^+) = x^+$ implies that $f(g_2(x)^{++}) = x^{++}$, since $g_2(x)^{++} = g_2(x)^{+*}$, and f preserves *. Let $p_2 = s \circ g_2$. Then p_2 is a lattice homomorphism from D_L into the dense set of F/I, which preserves 1. Furthermore, for all $x \in D_F$,

$$
\begin{aligned}
q(p_2(f(x))) &= q(s(g_2(f(x)))) \\
&= f(g_2(f(x))) \\
&= f(x),
\end{aligned}
$$

so all that is left to show is that p_1, p_2 can be extended over L. So, let y be an element of D_L. Then

$$\begin{aligned} q(p_2(y)^{++}) &= q(s(g_2(y))^{++}) \\ &= q(s(g_2(y)^{++})), \quad \text{since } s \text{ preserves } {}^+, \\ &= f(g_2(y)^{++}) \\ &= f(g_2(y))^{++}, \quad \text{by our hypothesis,} \\ &= y^{++}. \end{aligned}$$

Since $p_1 = q_1^{-1}$, we have $p_1(y^{++}) = p_2(y)^{++} \leq p_2(y)$, so there is an extension of p_1 and p_2 over L by 1.2, and thus L is projective.

3. The circle game

3.1. THEOREM. *Let α be infinite. Then F_α is a retract of D_α in the category of distributive lattices with* 1.

Proof. Let L be the sublattice of F_α which is generated by

$$\{a_{ik} \mid 0 < i < \alpha, k = 0, 1, 2\},$$

and let

$$M = \{a_{00} + a_{01} + x \mid x \in L\}.$$

Then L and M are isomorphic to F_α, and M is a $1-$sublattice of D_α. Define $f: D_\alpha \to M$ by

$$f(y) = a_{00} + a_{01} + y.$$

If f is well-defined, it obviously is a retraction, so suppose y is an element of D_α. We have to show that $f(y)$ is an element of M, i.e. we are looking for an element x of L which satisfies

$$a_{00} + a_{01} + y = a_{00} + a_{01} + x.$$

Since L is a lattice, and f is a lattice homomorphism, we can suppose $y = \sup T$ for some nonempty finite subset of A (cf. 1.4). If $a_{02} \leq y$, then $f(y) = 1$, and if $y = a_{00} + a_{01}$, then $f(y) \in M$. So, suppose $a_{02} \notin T$, and $T \neq \{a_{00}, a_{01}\}$. Now set $W = T \setminus \{a_{00}, a_{01}\}$; then

$$W \cup \{a_{00}, a_{01}\} = T \cup \{a_{00}, a_{01}\},$$

and $\sup W$ is an element of L. We then have

$$\begin{aligned} a_{00} + a_{01} + y &= a_{00} + a_{01} + \sup T \\ &= \sup(T \cup \{a_{00}, a_{01}\}) \\ &= \sup(W \cup \{a_{00}, a_{01}\}) \\ &= a_{00} + a_{01} + \sup W. \end{aligned}$$

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Hence, M—and therefore F_α—is a retract of D_α in the category of distributive lattices with 1.

Incidentally, we have shown that $[a_{i0}+a_{i1})$ is isomorphic to F_α for any $i<\alpha$.

3.2. COROLLARY. *If L is a projective Stone algebra such that $|L|\leq\alpha$, then L is a retract of D_α in the category of distributive lattices with* 1.

A structural characterization of the projective Stone algebras in the sense of Grätzer's programme has to make use of the retracts of D_α, but we have just seen that among those retracts are, for infinite α, all the projective Stone algebras with cardinality less than or equal to α. Thus, we have come full circle.

4. Odds and ends. Let us take a brief look at the free objects in the category of double Stone algebras (and double Stone homomorphisms). It was observed in **[4]** by Davey and Goldberg that, by analogy with Theorem 1.4, the free double Stone algebra on α free generators is the coproduct in the category of distributive lattices with 0 and 1 of α copies of the free double Stone algebra F on one generator. The ordered set of prime ideals of F looks like this:

$S(F)$

It follows, e.g. from Theorem 3.9 of **[5]**, that F is projective as a Stone algebra. Since the coproduct of projective Stone algebras is projective, and every morphism of double Stone algebras is a morphism of Stone algebras, we arrive at

4.1. THEOREM. *Every projective double Stone algebra is projective as a Stone algebra.*

It is worth noticing that by results of Priestley **[8]**, a free double Stone algebra cannot be the image of a free Stone algebra by a double Stone homomorphism, so indeed Theorem 4.1 makes sense. On the other hand, one can easily show that the free Stone algebra on one generator is projective in the category of double Stone algebras. Hence, every free Stone algebra has this property. The following example shows that this cannot be generalized to arbitrary projective Stone algebras: Let P be a minimal prime ideal of a free Stone algebra F, such that F/P is not the two-element Boolean algebra. Then F/P is a projective Stone algebra, whose dense set and dually dense set have a nonempty intersection. This implies that F/P cannot be embedded in a free double Stone algebra by a double Stone homomorphism.

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