Canad. Math. Bull. Vol. 15 (2), 1972

PROBLEMS AND SOLUTIONS

This department welcomes problems believed to be new. Solutions should accompany proposed problems.

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PROBLÈMES ET SOLUTIONS

Cette section a pour but de présenter des problèmes inédits. Les problèmes proposés doivent être accompagnés de leurs solutions.

Veuillez adresser les communications concernant cette section à

E. C. Milner, Problem Editor
Canadian Mathematical Bulletin
Department of Mathematics
University of Calgary
Calgary 44, Alberta

PROBLEMS FOR SOLUTION

P.198. A Hausdorff space X has the fixed point set property (fpsp) iff any non-empty closed subset of X is the fixed point set $\{y \in X : f(y)=y\}$ of some continuous function $f: X \to X$. Prove that: (1) a convex subset of a metric linear space with the induced topology has the fpsp; (2) a compact, strongly convex space (i.e. a compact space with metric d in which $\{z \in X : d(x, z)+d(z, y)=d(x, y)\}$ is isometric with a real closed interval of length $d(x, y)$) has the fpsp.

Give an example of a metric space which lacks the fpsp.

Simeon Reich,
The Technion, Haifa, Israel

P.199. Let $X \subset P=\{0, 1, \ldots, p-1\}$ be such that whenever $x, y \in X$ then there is $z \in X$ such that $x+y \equiv 2z \pmod p$. Show that, if p is prime and $|X|>1$, then $X=P$.

E. C. Milner,
University of Calgary

P.200. Let $a_1, a_2, a_3, \ldots$ be any sequence of numbers with $0<a_i<1$. Show that every number α, $0<\alpha<1$, has a representation in the form $\alpha=\sum_{j=1}^{\infty} a_{i_j} . 2^{-j}$ with

$(i_1, i_2, \ldots)$ a permutation of $(1, 2, \ldots)$ if and only if 0 and 1 are both limit points of the set $\{a_1, a_2, a_3, \ldots\}$.

P. ERDÖS,
HUNGARIAN ACADEMY OF SCIENCES

P.201. Let H be a subgroup of the (non-abelian) torsion-free group G. It is known (B. H. Neumann and J. A. H. Shepperd, *Finite extensions of fully ordered groups*, Proc. Roy. Soc. London, Ser. A, **239** (1957), 320–327, Corollary 4.1) that the following three conditions are sufficient to ensure that a partial order for H may be extended to some full order for G: (1) the partial order for H is a full order, (2) H is normal in G and the inner automorphisms of G induce order automorphisms of H, (3) G/H is locally finite. Which of these conditions are necessary?

DONALD P. MINASSIAN,
BUTLER UNIVERSITY

P.202. Show that if $y \in C^\infty[x]$, then for all integers m, n $(1 \le m \le n)$

$$\sum_{j=1}^{m} (-1)^{m-j}\binom{m}{j} y^{m-j}\left(\frac{d}{dx}\right)^n (y^j) = m!n! \sum_{p=1}^{n} \sum_{i \in S(m,n,p)} \prod_{j=1}^{p} \left(\frac{y^{(j)}}{i_j! j!}\right)^{i_j},$$

where

$$S(m, n, p) = \left\{(i_1, \ldots, i_p) : i_p \neq 0,\ \sum_{j=1}^{p} i_j = m,\ \sum_{j=1}^{p} j i_j = n\right\}.$$

D. K. COHOON,
UNIVERSITY OF MINNESOTA

SOLUTIONS

P.170. In the n-dimensional tic-tac-toe board let $f(n)$ be the maximal number of "squares" which can be entered without getting three in a line. Prove that $f(n) \ge c\, 3^n/\sqrt{n}$. (Whether $f(n) = o(3^n)$ is unknown.)

L. MOSER,
UNIVERSITY OF ALBERTA

Solution by J. Komlos, Hungarian Academy of Sciences. Let the 3^n points be $(x_1, x_2, \ldots, x_n)$, $x_i = 1$, 2 or 3. Three of these points, say $A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n)$, $C = (c_1, \ldots, c_n)$, with B between A and C, are collinear iff $b_k = \frac{1}{2}(a_k + c_k)$, $k = 1, \ldots, n$, i.e., for each k, EITHER $a_k = b_k = c_k$ OR $a_k = 1$, $b_k = 2$, $c_k = 3$ OR $a_k = 3$, $b_k = 2$, $c_k = 1$. Hence, for a given integer m, the set of points each having precisely m of its coordinates equal to 2 has no three points collinear. This set contains $\binom{n}{m} 2^{n-m}$ points. Therefore

$$f(n) \ge \binom{n}{m} 2^{n-m}.$$

https://doi.org/10.1017/S0008439500031556 Published online by Cambridge University Press

Taking $m=\left[\frac{n}{3}\right]$, we have

$$\binom{n}{m} 2^{n-m} \sim \frac{3^{n+1}}{\sqrt{2\pi n}},$$

and hence for some $c>0$,

$$f(n) \geq c3^n/\sqrt{n}.$$

P.171. Evaluate

$$\sum_{(a,b)=1} \frac{1}{a^{2m}b^{2n}}$$

for m and n positive integers. For which values of m and n is the sum rational?

L. MOSER,
UNIVERSITY OF ALBERTA

Solution by M. S. Klamkin, Ford Motor Company, Dearborn, Michigan. If $\sum A_i$, $\sum B_i$ are both absolutely convergent series, then the resulting double series of their product can be summed in any order to give the same sum. In particular,

$$\sum_{i=1}^{\infty} A_i \sum_{i=1}^{\infty} B_i = \sum_{i,j=1}^{\infty} A_i B_i = \sum_{(a,b)=1}^{\infty} \sum_{k=1}^{\infty} A_{ak} B_{bk}.$$

Putting $A_a = a^r \zeta(r+s)$, $B_b = b^s$, we obtain

$$\sum_{(a,b)=1} \frac{1}{a^r b^s} = \frac{\zeta(r)\zeta(s)}{\zeta(r+s)},$$

where $\zeta(r)$ is the Riemann zeta function. Since $\zeta(2n)=2^{2n-1}\pi^{2n}B_n/(2n)!$, where the B_n are the Bernoulli numbers (all rational), the proposed sum is equal to $\binom{2m+2n}{2m} B_m B_n/2B_{m+n}$ which is always a rational number.

Also solved by W. J. Blundon, O. P. Lossers, and F. G. Schmitt, Jr.

(Klamkin remarks that different versions of this problem appear as Problems E1550, E1762 (Amer. Math. Monthly, October 1963 and April 1966) and in Problem 63–17 (SIAM Rev., July 1965). Also, one has the more general relation

$$\sum_{(i_1, i_2, \ldots, i_n)=1} \prod i_j^{-r_j} = \frac{\zeta(r_1)\zeta(r_2)\ldots\zeta(r_n)}{\zeta(r_1+r_2+\cdots+r_n)}.)$$

P.174. Prove that any measurable subset S of $[0, 1]$ having the property that $x, y \in S \Rightarrow (x+y)/2 \notin S$ has measure 0.

L. MOSER,
UNIVERSITY OF ALBERTA

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands. (Evidently the condition should read: if x and y are *distinct* points in S, then $\frac{1}{2}(x+y) \notin S$.)

Suppose $S \subset [0, 1]$ and $\mu(S) > 0$. Choose an open set $U \supset S$ such that $\mu(U) < \frac{4}{3}\mu(S)$. Since U is a countable union of disjoint open intervals, there is an open interval $I \subset [0, 1]$ such that $\mu(I) < \frac{4}{3}\mu(I \cap S)$. Let $T = I \cap S$ and $x \in T$. If $T_1 = \frac{1}{2}(x+T)$, then $\mu(T \cap T_1) > 0$. For if not, then $\mu(T \cup T_1) = \mu(T) + \mu(T_1) = \frac{3}{2}\mu(T) > \frac{9}{8}\mu(I)$. But $T \cup T_1 \subset I$, and therefore $\mu(T \cup T_1) \leq \mu(I)$: contradiction. We conclude that there exists a point $y \in T$, $y \neq x$, such that $\frac{1}{2}(x+y) \in S$.

Also solved by D. Borwein, P. Erdös and J. Komlos.

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