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THE EXCEPTIONAL SETS IN THE DEFINITION OF THE P^n-INTEGRAL

BY

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ABSTRACT. It has been recently observed by S. N. Mukhopadhyay that the various definitions of the P^n-integral are not complete unless it is shown that the exceptional scattered set allowed in the definition is not important. Utilizing the fact that on the real line a scattered set is countable, and adapting known methods for coping with exceptional countable sets, it is proved that the definitions of the P^n-integral are complete. It is then clear that the concept of scattered set is not essential to the definition of the P^n-integral.

1. **Introduction.** It has recently been observed by Mukhopadhyay [11] that the definitions of the P^n-integral as given in [7], [9] and [4], and the definition of the P_n^*-integral in [5], are not complete without some argument to show that the exceptional scattered set allowed in the definition of major and minor functions is not important.

For if one defines major and minor functions with respect to a fixed scattered set S_1 and obtains an integral of f, one may define major and minor functions with respect to another fixed scattered set S_2 and obtain a new integral of f. An argument to show independence of the scattered set would be required.

On the other hand if the scattered set varies from one pair of major and minor functions to another a difficulty arises in proving that $Q_1(x)-q_2(x)$ is convex, where Q_1 is a major function with respect to a scattered set S_1 and q_2 is a minor function with respect to a different scattered set S_2. One way around the difficulty is to impose n-smoothness everywhere on the major and minor functions, and then, since the union of two scattered sets is scattered, the usual arguments go through, and the integral is well defined. This is essentially what was done in the case of the original P^2-integral [10] and the SCP-integral [3]. Insofar as application to trigometric series is concerned, this approach results in no loss of generality, since the sums of the series involved in the construction of major and minor functions are n-smooth everywhere.

The methods of Bosanquet [1] and Grimshaw [6] for removing exceptional points are of course well known, as is the more recent approach of Taylor [13] and Mukhopadhyay [11]. The difficulty in James' definitions arose when

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making the change from the definition in [10] to the definition in [7], the condition of smoothness on the major and minor functions was imposed only on the exceptional countable set. This same incompleteness in the definition was continued in [9] and by Cross in [4] and [5].

It is shown in the following how the exceptional sets in the various definitions may be removed. An argument in the context of an exceptional countable set is sufficient since in a second countable space a scattered set is countable [12].

It is then clear that the concept of scattered set is not an essential part of the definition of the P^n-integral.

2. **Preliminaries.** We adopt the notations and conventions of [4]. The (upper, lower) Peano derivative of order $2n$ of F at x_0 is denoted by $(\bar{F}_{2n}(x_0), \underline{F}_{2n}(x_0))$ $F_{2n}(x_0)$ and the (upper, lower) de la Vallée Poussin derivative of order $2n$ of F at x_0 by $(\Delta^{2n}F(x_0), \delta^{2n}F(x_0))$ $D_{2n}F(x_0)$. The function F is said to be $2n$-smooth at x_0 if $\lim_{h\to 0} h\theta_{2n}(F; x_0, h)=0$, where

$$\theta_{2n}(F; x_0, h)=\left(\frac{(2n)!}{h^{2n}}\right)\left(\frac{F(x_0+h)+F(x_0-h)}{2}-\sum_{k=0}^{n-1}\frac{h^{2k}}{(2k)!}D_{2k}F(x_0)\right).$$

The function F is said to satisfy condition A^*_{2n} $(n\geq 1)$ in $[a, b]$ if it is continuous in $[a, b]$, if, for $1\leq k\leq 2n-2$, each $F_{(k)}(x)$ exists and is finite in (a, b) and if F is $2n$-smooth for $x\in(a, b)-E$ where E is countable.

There are corresponding definitions for odd indices. In the following, discussion will be restricted to the even case, and for simplicity in notation we shall use n to denote an even integer. The arguments for the odd case are similar.

3. **The P^n-integral.** Detail for the argument in the case of the P^n-integral of [4] is provided in this section.

Definition 3.1. Let $f(x)$ be a function defined in $[a, b]$ and let a_i, $i=1, 2, \ldots, n$, be fixed points such that $a=a_1<a_2<\cdots<a_n=b$. The functions $Q(x)$ and $q(x)$ are called P^n-major and minor functions respectively of $f(x)$ over $(a_i)=(a_1, a_2, \ldots, a_n)$ if

(3.1) $Q(x)$ and $q(x)$ satisfy condition A^*_n in $[a, b]$;

(3.2) $Q(a_i)=q(a_i)=0$, $i=1, 2, \ldots, n$;

(3.3) $\partial^n Q(x)\geq f(x)\geq \Delta^n q(x)$, $x\in(a, b)-E$, $|E|=0$;

(3.4) $\partial^n Q(n)\neq -\infty$, $\Delta^n q(x)\neq +\infty$, $x\in(a, b)-S$, S a scattered set;

(3.5) Q and q are n-smooth in S.

(Condition (3.5) is stronger than the corresponding condition in [9] and [4] but seems more natural.)

Lemma 3.1. *Given $x_0\in(a, b)$ and $\varepsilon>0$, there exists a P^n-major function $Q(x)$*

for the function $J(x)\equiv 0$ *over* $(a_i)=(a_1, a_2, \ldots, a_n)$ *such that:*

(3.6) $\partial^n Q(x)\geq 0$, *for all* $x\in(a, b)$;

(3.7) $\theta_n(Q; x, h)\geq 0$, *for* x, $x\pm h\in(a, b)$;

(3.8) $\lim_{h\to 0} h\theta_n(Q; x_0, h)>0$:

(3.9) $|D_{2k}Q(x)|<\varepsilon$, $x\in(a, b)$, $0\leq k\leq(n/2)-1$.

Proof. Let

$$g(x)=\begin{cases} x, & x\leq x_0 \\ 2x-x_0, & x\geq x_0\end{cases}$$

and define $G(x)$ as the $(n-2)$th indefinite integral of $g(x)$ on $[a, b]$:

$$G(x)=\frac{1}{(n-3)!}\int_a^x (x-t)^{n-3}g(t)\,dt.$$

Then the function Q defined by

$$Q(x)=c\left[G(x)-\sum_{i=1}^{n}\lambda(x; a_i)G(a_i)\right],$$

where

$$\lambda(x; a_i)=\prod_{j\neq i}(x-a_j)/(a_i-a_j),$$

satisfies the conditions of the lemma if $c>0$ is chosen so that $|D_{2k}Q(x)|<\varepsilon$, $0\leq k\leq(n/2)-1$. Indeed, it is clear that $\partial^n Q(x)\geq 0$ if $x\neq x_0$, and

$$\partial^n Q(x_0)=c\partial^n G(x_0)\geq c\partial^2(D_{n-2}G(x_0))=c\partial^2 g(x_0)>0.$$

It follows that $Q(x)$ is n-convex in $[a, b]$ and (3.7) follows from Lemma 4.1 [8]. Moreover

$$\lim_{h\to 0} h\theta_n(Q; x_0, h)$$

$$=\lim_{h\to 0} c\left\{\frac{n!\left[\frac{Q(x_0+h)+Q(x_0-h)}{2}-\sum_{k=0}^{(n/2)-1}\frac{h^{2k}}{(2k)!}D_{2k}Q(x_0)\right]^{(n-2)}}{(n-1)!\,h}\right\}$$

$$=\lim_{h\to 0} nc\left[\frac{g(x_0+h)+g(x_0-h)-2g(x_0)}{2h}\right]$$

$$=\lim_{h\to 0} nc\left[\frac{2(x_0+h)-x_0+x_0-h-2x_0}{2h}\right]=\frac{nc}{2}>0$$

LEMMA 3.2. *Suppose* $M(x)$ *is a* P^n*-major function of* $f(x)$ *over* $(a_i)=(a_1, a_2, \ldots, a_n)$ *and let* $\varepsilon>0$. *Then there exists a* P^n*-major function* $F(x)$ *such that*

$$|F(x)-M(x)|<\varepsilon, \qquad x\in[a, b].$$

and $\partial^n F(x)>-\infty$, *for all* x *in* (a, b).

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Proof. Let E be the set of points where $\partial^n M(x) = -\infty$. Then E is scattered and hence, as noted in the introduction, E is countable. Moreover, by definition, $M(x)$ is n-smooth for $x \in E$.

Let the points of E be enumerated

$$x_1, x_2, \ldots, x_n, \ldots$$

and let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence such that $\sum_{i=1}^{\infty} \varepsilon_1 < \varepsilon$. Let $M_i(x)$ be the major function defined by Lemma 3.1 with ε, x_0 replaced by ε_i, x_i. Let G be defined by

$$G(x) = \sum_{i=1}^{\infty} M_i(x), \qquad x \in [a, b].$$

Then $G(x)$ is the sum of a uniformly convergent series of continuous functions and so $G(x)$ is continuous. Also $|G(x)| < \varepsilon$, $G(a_i) = 0$, $i = 1, 2, \ldots, n$. For $x \in (a, b)$, we have,

$$\theta_n(G; x, h) = \sum_{i=1}^{\infty} \theta_n(M_i; x, h),$$

where for each i, $\theta_n(M_i; x, h) \geq 0$. Therefore for each N and x, $x \pm h \in (a, b)$, we have

$$\theta_n(G; x, h) \geq \sum_{i=1}^{N} \theta_n(M_i; x, h)$$

and

$$\partial^n G(x) = \varliminf_{h \to 0} \theta_n(G; x, h) \geq \sum_{i=1}^{N} \varliminf_{h \to 0} \theta_n(M_i; x, h) \geq 0,$$

for $x \in (a, b)$.

For $x_{i_0} \in E$, $h > 0$.

$$h\theta_n(G; x_{i_0}, h) = \sum_{i=1}^{\infty} h\theta_n(M_i; x_{i_0}, h) \geq \sum_{i=1}^{N} h\theta_n(M_i; x_{i_0}, h).$$

It follows that

$$\varliminf_{h \to 0+} h\theta_n(G; x_{i_0}, h) \geq \sum_{i=1}^{N} \varliminf_{h \to 0+} h\theta_n(M_i; x_{i_0}, h) > 0,$$

since N can be chosen so that $i_0 = N$.

Now define $F(x) = M(x) + G(x)$. Then

$$\partial^n F(x) \geq \partial^n M(x) + \partial^n G(x) \geq f(x) \quad \text{a.e. in } [a, b],$$

and

$$\partial^n F(x) > -\infty \quad \text{if} \quad x \notin E.$$

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At points $x_{i_0} \in E$.

$$\underline{\lim}_{h\to 0+} [h\theta_n(F; x_{i_0}, h)] \geq \underline{\lim}_{h\to 0+} [h\theta_n(M; x_{i_0}, h)] + \underline{\lim}_{h\to 0+} [h\theta_n(G; x_{i_0}, h)]$$

$$\geq \underline{\lim}_{h\to 0+} [h\theta_n(G; x_{i_0}, h)] > 0,$$

since M is n-smooth in E.

Therefore $\underline{\lim}_{h\to 0+} [\theta_n(F; x_{i_0}, h)] = +\infty$.

But from the format of $\theta_n(F; x_{i_0}, h)$, with n even, this is the same as $\underline{\lim}_{h\to 0-} [\theta_n(F; x_{i_0}, h)]$. Therefore $\partial^n F(x_{i_0}) = +\infty$.

LEMMA 3.3. *Suppose E is a set of measure zero contained in (a, b) and $\varepsilon > 0$. Then there exists a P^n-major function $F_1(x)$ for the function $J(x) \equiv 0$ over $(a_i) = (a_1, a_2, \ldots, a_n)$ such that*

(3.10) $\partial^n F_1(x) \geq 0$, *for all* $x \in (a, b)$;

(3.11) $\partial^n F_1(x) = +\infty$, *for* $x \in E$;

(3.12) $0 \leq F_1(x) < \varepsilon$, *for* $x \in [a, b]$.

Proof. This follows using the same technique as in the proof of Theorem 16 of [2].

Given $\varepsilon > 0$ let χ be a function on $[a, b]$ such that

(i) χ is absolutely continuous;

(ii) χ is differentiable;

(iii) $\chi'(x) = \infty$, $x \in E$;

(iv) $0 \leq \chi'(x) < \infty$, $x \notin E$,

(v) $\chi(a) = 0$, $0 \leq \chi(b) \leq \varepsilon/(b-a)^{n-1}$,

Define $\Psi(x)$ as the $(n-1)$th integral of $\chi(t)$:

$$\Psi(x) = \frac{1}{(n-2)!} \int_a^x (x-t)^{n-2} \chi(t)\, dt,$$

and

$$F_1(x) = c\left(\Psi(x) - \sum_{i=1}^{n} \lambda(x_i, a_i)\Psi(a_i)\right)$$

with c chosen that (3.12) is satisfied.

LEMMA 3.4. *Suppose $M(x)$ is a P^n-major function and $m(x)$ a P^n-minor function of $f(x)$ over $(a_i) = (a_1, a_2, \ldots, a_n)$ in $[a, b]$. Then there exists a P^n-major function $Q(x)$ and a P^n-minor function $q(x)$ such that for all $x \in (a, b)$:*

$$\partial^n Q(x) \geq f(x) \geq \Delta^n q(x)$$

$$\partial^n Q(x) > -\infty, \qquad \Delta^n q(x) < +\infty$$

$$|Q(x) - M(x)| < \varepsilon, \quad \text{and} \quad |q(x) - m(x)| < \varepsilon.$$

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Proof. The result for major functions follows directly by setting

$$Q(x)=F(x)+F_1(x).$$

No new construction is needed in the case of a minor function. Merely use the major function case applied to $-m(x)$ and $-f(x)$.

THEOREM 3.1. *If $M(x)$ and $m(x)$ are P^n-major and minor functions of $f(x)$ over $(a_i)=(a_1, a_2, \ldots, a_n)$ then the difference $M(x)-m(x)$ is n-convex in $[a, b]$.*

Proof. Let $M(x)$ and $m(x)$ be given major and minor functions. By Lemma 3.4 there is a sequence $M_k(x)$ of major functions and a sequence $m_k(x)$ of minor functions such that $M_k(x)-m_k(x)$ converges (uniformly) to $M(x)-m(x)$ on $[a, b]$ and such that, everywhere in (a, b).

$$\partial^n M_k(x)\geq f(x)\geq \Delta^n m_k(x)$$

$$\partial^n M_k(x)>-\infty, \qquad \Delta^n m_k(x)<+\infty, \quad \text{for each } k.$$

It follows that the difference $M_k(x)-m_k(x)$ is n-convex for each k, and hence so is $M(x)-m(x)$.

4. **The P_n^*-integral.** The exceptional sets in the definition of the P_n^*-integral may be disposed of similarly as is shown below.

DEFINITION 4.1. The functions $Q(x)$ and $q(x)$ are called P_n^*-major and minor functions, respectively, of $f(x)$ on $[a, b]$ if

(4.1) $Q(x)$ and $q(x)$ satisfy condition A_n^* on $[a, b]$;

(4.2) $Q_{(k)}(a+)=q_k(a+)=0$; $0\leq k\leq n-1$;

(4.3) $\partial^n Q(x)\geq f(x)\geq \Delta^n q(x)$, in $[a, b-E$, $|E|=0$;

(4.4) $\partial^n Q(x)>-\infty$, $\Delta^n q(x)<+\infty$, $x\in[a, b]-S$, *S a scattered set*;

(4.5) *$Q(x)$ and $q(x)$ are n-smooth in S.*

LEMMA 4.1. *Given $x_0\in(a, b)$ and $\varepsilon>0$, there exists a P_n^*-major function $Q(x)$ for the function $J(x)\equiv 0$ on $[a, b]$ such that*

(4.6) $\partial^n Q(x)\geq 0$, *for* $x\in(a, b)$;

(4.7) $\theta_n(Q; x, h)\geq 0$, *for* x, $x\pm h\in(a, b)$;

(4.8) $\lim_{h\to 0} h\theta_n(Q; x_0, h)>0$;

(4.9) $|Q_{(k)}(x)|<\varepsilon$, $x\in[a, b]$, $1\leq k\leq n-1$.

Proof. Let h be defined on $[a, b]$ by

$$h(x)=\begin{cases}(x-a)^2, & x\leq x_0,\\ 2(x-a)^2-(x_0-a)^2, & x\geq x_0.\end{cases}$$

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Then if Q is defined by

$$Q(x)=\frac{c}{(n-3)!}\int_a^x (x-t)^{n-3}h(t)\,dt,$$

where $c>0$ is chosen so that (4.9) holds, it is clear that Q satisfies the conditions required.

Now using the same approach as in the preceding section we obtain the following:

THEOREM 4.1. *If $M(x)$ and $m(x)$ are P_n^*-major and minor functions of $f(x)$ on $[a, b]$, then the difference $M(x)-m(x)$ is n-convex in $[a, b]$.*

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