# AMENABILITY AND INVARIANT SUBSPACES

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## 1. Introduction

Let $E$ be a topological vector space (over the real or complex field). A well-known geometric form of the Hahn-Banach theorem asserts that if $U$ is an open convex subset of $E$ and $M$ is a subspace of $E$ which does not meet $U$, then there exists a *closed* hyperplane $H$ containing $M$ and not meeting $U$. In this paper we prove, among other things, that if $S$ is a left amenable semigroup (which is the case, for example, when $S$ is abelian or when $S$ is a solvable group, see [3, p.11]), then for any right linear action of $S$ on $E$, if $U$ is an invariant open convex subset of $E$ containing an invariant element and $M$ is an invariant subspace not meeting $U$, then there exists a closed invariant hyperplane $H$ of $E$ containing $M$ and not meeting $U$. Furthermore, this geometric property characterizes the class of left amenable semigroups.

In section 4 of this paper, we also characterize amenability of a semigroup $S$ by similar geometric properties when $S$ acts on partially ordered topological vector spaces with a topological order unit.

Since our method of proof carries over quite easily to topological semigroups, it is in this more general setting that we shall state our results. However, it is to our best knowledge that both Theorem 1 and 2 are new even for discrete semigroups except (a)$\Leftrightarrow$(b) in Theorem 1 which is due to Silverman [10].

## 2. Preliminaries

For the rest of this paper $S$ will be fixed *topological semigroup* (i.e., $S$ is a semigroup with a Hausdorff topology such that the mapping $(a, b) \to ab$ from $S \times S$ into $S$ is separately continuous).

Let $C(S)$ be the space of bounded continuous real-valued functions on $S$

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with the sup norm topology. For each $f$ in $C(S)$ and $a$ in $S$, define the *left translate* of $f$ by: $l_a f(t) = f(at)$ for all $t$ in $S$. Then a function $f$ in $C(S)$ is *left uniformly continuous* if the mapping $a \to l_a f$ from $S$ into $C(S)$ is continuous. Then as is known (see [9, p. 64] and [8]), LUC($S$), the space of left uniformly continuous functions on $S$, is translation invariant, sup norm closed subspace of $C(S)$ containing 1, the constant one function on $S$. Following Namioka [9, p. 67], we call $S$ *left amenable* when LUC($S$) has a LIM (*left invariant mean*) $\phi$, i.e. $\phi$ is a linear functional on LUC($S$) such that $\|\phi\| = \phi(1) = 1$ and $\phi(l_a f) = \phi(f)$ for all $a \in S$, $f \in$ LUC($S$). Note that this definition of amenability agrees with that of Day [2] for discrete semigroups. However, there exists many topological semigroups $S$ such that LUC($S$) has a LIM but $S$ is not left amenable *as a discrete semigroup* (see e.g. [7], p.72]). For excellent expositions of the subject on amenable semigroups, we refer the readers to Day [2], [3] and Greenleaf [5].

Let $E$ be a topological vector space. Then a *right linear action* of $S$ on $E$ is a separately continuous map from $S \times E \to E$, denoted by $(s, x) \to s \cdot x$ satisfying

(1) $(ab) \cdot x = b \cdot (a \cdot x)$ for all $a, b \in E$

(2) for each $s \in S$, the map $x \to s \cdot x$ is a linear from $E$ into $E$.

## 3. Invariant subspaces

In this section we shall be concerned with linear actions of $S$ on topological vector spaces. Theorem 1 (a)$\Leftrightarrow$(b) is due to Silverman [10] for the case when $S$ has the discrete topology. (see also [3, p. 4.] and [11, p. 576]).

A real-valued function on a vector space $E$ is *sublinear* if $p(x + y) \leqq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for all $x, y \in E$, $\lambda \geqq 0$.

THEOREM 1. *The following conditions on $S$ are equivalent*:

(a) *$S$ is left amenable*

(b) *For any right linear action of $S$ on a real topological vector space $E$, if $p$ is a continuous sublinear map on $E$ such that $p(s \cdot x) \leqq p(x)$ for all $s \in S$, $x \in E$, and if $\phi$ is an invariant linear functional on an invariant subspace $F$ of $E$ such that $\phi \leqq p$, then there exists an invariant extension $\tilde{\phi}$ of $\phi$ to $E$ such that $\tilde{\phi} \leqq p$.*

(c) *For any right linear action of $S$ on a topological vector space $E$, if $U$ is an invariant open convex subset of $E$ containing an invariant element, and $M$ is an invariant subspace of $E$ which does not meet $U$, then there exists a closed invariant hyperplane $H$ of $E$ such that $H$ contains $M$ and $H$ does not meet $U$.*

(d) *For any right linear action of $S$ on a Haudsroff topological vector space $E$ with a base of the neighbourhoods of the origin consisting of invariant open convex sets, then any two points in $E_f = \{x \in E; sx = x \text{ for all } s \in S\}$ can be separated by a continuous invariant linear functional on $E$.*

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PROOF. $(a) \Rightarrow (b)$: Let $m$ be a LIM on LUC($S$). By Hahn-Banach extension theorem, $\phi$ has an extension $\gamma$ to $E$ such that $\gamma \leqq p$. For each $x \in E$ define $(T_x\gamma)(s) = \gamma(s \cdot x)$. Then $T_x\gamma \in C(S)$. Furthermore, $T_x\gamma$ is even left uniformly continuous since if $\{a_\alpha\}$ is a net in $S$ and $a_\alpha \to a$, let $y_\alpha = a_\alpha \cdot x - a \cdot x$, then

$$\|l_{a_\alpha}(T_x\gamma) - l_a(T_x\gamma)\| = \sup\{|\gamma(s \cdot y_\alpha)|;\ s \in S\} \leqq |p(y_\alpha)| + |p(-y_\alpha)| \to 0$$

since $-p(-y) \leqq \gamma(s \cdot y) \leqq p(y)$ for each $s \in S$, $y \in E$, and $y_\alpha \to 0$. For any $x \in E$, define $\tilde{\phi}(x) = m(T_x\gamma)$, then for any $x \in F$, $T_x\gamma(s) = \phi(x)$ for all $s \in S$, and hence $\tilde{\phi}(x) = \phi(x)$. Furthermore, if $x \in E$, then $\tilde{\phi}(x) \leqq \sup\{\gamma(s \cdot x);\ s \in S\} \leqq p(x)$ and $\tilde{\phi}(s \cdot x) = m(T_{s \cdot x}\gamma) = m(l_s(T_x\gamma)) = m(T_x\gamma) = \tilde{\phi}(x)$ for all $s \in S$.

$(b) \Rightarrow (c)$. We first assume that $E$ is a real topological vector space. Let $e$ be an invariant element in $U$, $W = U - e$ and $p$ be the Minkowski functional on $E$ for $W$ (i.e. $p(x) = \inf\{\lambda > 0;\ x \in \lambda W\}$). Then $p$ is sublinear, non-negative and continuous on $E$. Furthermore, $p(s \cdot x) \leqq p(x)$ for all $s \in S$, $x \in E$. Let $F$ be the linear span of $M$ and $e$. Then $F$ is invariant. Define on $F$ an invariant linear functional $\phi$ by: $\phi(x) = \lambda$ if $x = h - \lambda e$, $h \in M$. Then $\phi \leqq p$. Indeed, if $p(x) < \lambda$ and $x = h - \lambda e$, $h \in M$, (we may assume that $\lambda > 0$), then $p(h/\lambda - e) < 1$ and hence $h/\lambda - e \in W$ which is impossible since $M$ does not meet $U$. Hence using (b) we may obtain an extension $\tilde{\phi}$ of $\phi$ to $E$ such that $\tilde{\phi} \leqq p$. Then $\tilde{\phi}$ is continuous, and $H = \ker \tilde{\phi}$ is a closed invariant hyperplane of $E$ containing $M$. Furthermore, if $x \in H \cap U$, then $x - e \in W$ and hence $p(x - e) < 1$. Consequently $\tilde{\phi}(x) \neq 0$, which is impossible. Hence $H$ does not meet $U$.

If $E$ is a topological vector space over the complex field, as usual, we can regard $E$ as a vector space over the real field and then obtain a real closed invariant hyperplane $K$ containing $M$ and not meeting $U$. Let $H = K \cap iK$, then $H$ is a closed invariant *complex* hyperplane containing $M$ and not meeting $U$.

$(c) \Rightarrow (d)$. If $x, y$ in $E_f$ are distinct, using $(c)$ we can obtain an invariant closed hyperpalane $H$ such that $x - y \notin H$. Define $\phi(z) = \lambda$ if $z = h + \lambda(x - y)$ with $h \in H$. Then $\phi$ is continuous and invariant. Furthermore $\phi(x) \neq \phi(y)$.

$(d) \Rightarrow (a)$. Consider the right linear action of $S$ on LUC($S$) defined by the map $(s, f) \to l_{as}f$. Then $\{\lambda W;\ \lambda > 0\}$ where $W = \{f \in \text{LUC}(S);\ \|f\| < 1\}$ is a base of open convex invariant neighbourhoods of the origin. Hence there exists a continuous invariant linear functional $\phi$ on LUC($S$) such that $\phi(1) \neq 0$. It follows from [9, prop. 3.2] that $\phi^+$ is also invariant and $\mu = \phi^+/\phi^+(1)$ is a LIM on LUC($S$)

COROLLARY. *If $S$ is abelian, a solvable group, or a compact semigroup with finite intersection property for right ideals, then $S$ has properties (b), (c) and (d) of Theorem* 1.

PROOF. It is known that in each of the cases, $S$ is left amenable (see [3, p.11] and [4, p.70]).

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## 4. Invariant ideals

Let $E$ be a partially ordered topological vector space over the real field. An element $e \in E$ is a *topological order unit* if $e$ is an *order unit* (i.e., for each $x \in E$, there exists $\lambda > 0$ such that $-\lambda e \leqq x \leqq \lambda e$) and the absolutely convex set $[-e, e]$ is a neighbourhood of $E$, where $[a, b] = \{x \in E;\ a \leqq x \leqq b\}$ for any $a, b \in E$. A subspace $I$ of $E$ is a *proper ideal* if $I \neq E$ and $x \in E$ implies $[0, x] \subseteq I$. An action of $S$ on $E$ is *positive* if $s \cdot x \geqq 0$ for all $s \in S$ and $x \geqq 0$. The action is *normalized* (*with respect a topological order unit* $e$) if $s \cdot e = e$ for all $s \in S$.

Note that if $E$ is a partially order vector space (no topology) and $e$ is an order unit of $E$. Then $p$, the Minkowski functional of $[-e, e]$ on $E$ i.e. $p(x) = \inf\{\lambda > 0;\ -\lambda e \leqq x \leqq \lambda e\}$ is a semi-norm on $E$, and $e$ will become a topological order unit of the locally convex space $E$ equipped with the topology determined by $p$.

An important example of partially ordered topological vector space that we shall be concerned with is LUC($S$) with the natural ordering $f \leqq g$ if and only if $f(s) \leqq g(s)$ for all $s \in S$. In this case, 1, the constant one function on $S$, is a topological order unit of LUC($S$).

It is known [1, p.124] that if $\mathscr{F}$ is a commuting family of positive normalised linear endomorphisms of a partially ordered vector space $E$ with a unit, then $E$ contains a proper maximal ideal which is invariant under each map in $\mathscr{F}$. Our next result shows that a much stronger result also holds.

THEOREM 2. *The following conditions on $S$ are equivalent*:

(a) *$S$ is left amenanble.*

(b) *For any positive normalised right linear action of $S$ on a partially ordered topological vector space $E$ with a topological order unit $e$, if $F$ is an invariant subspace of $E$ containing $e$, and $\phi$ is an invariant monotonic linear functional on $F$, then there exists an invariant monotonic linear function $\tilde{\phi}$ on $E$ extending $\phi$.*

(c) *For any positive normalised right linear action on a partially ordered topological vector space $E$ with a topological order unit $e$, $E$ contains a maximal proper ideal which is invariant under $S$.*

PROOF. (a) $\Rightarrow$ (b). Without loss of generality, we may assume $\phi(e) = 1$. Let $m$ be a LIM on LUC($S$) and $p$ be the Minkowski functional on the set $[-e, e]$ Then $p$ is a continuous semi-norm on $E$. Furthermore, $p(s \cdot x) \leqq p(x)$ for all $s \in S$. By a theorem of Ruth and Krein (see [6, p. 24, proof of Theorem 1.6.1]) $\phi$ has a monotonic extension $\gamma$ to $E$ and $\gamma \leqq q$ where $q$ is the sublinear functional on $E$ defined by

$$q(x) = \inf\{\phi(y);\ y \in F \text{ and } y \geqq x\}$$

Then for any $x \in E$, if $-\lambda e \leqq x \leqq \lambda e$, $\lambda > 0$, then $q(x) \leqq \phi(\lambda e) = \lambda$. Hence

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$\gamma \leqq q \leqq p$. Now as in the proof of Theorem 1 (a) $\Rightarrow$ (b), define $\tilde{\phi}(x) = m(T_x\gamma)$, where $(T_x\gamma)(s) = \gamma(s \cdot x)$. Then $\tilde{\phi}$ is an invariant extension of $\phi$, and $\tilde{\phi}$ is monotonic since if $x \geqq 0$, then $T_x\gamma \geqq 0$ and hence $m(T_x\gamma) \geqq 0$.

(b) $\Rightarrow$ (c). If $E$ has dimension 1, then $\{0\}$ is the maximal proper ideal of $E$ which is invariant. If $E$ has dimension greater than 1, define a monotonic invariant function $\phi$ on $F = \{\lambda e;\ \lambda \in R\}$ by $\phi(\lambda \cdot e) = \lambda$ for all $\lambda \in R$. Now if $\tilde{\phi}$ is a monotonic invariant extension of $\phi$ to $E$ and $I = \ker \tilde{\phi}$, then $I$ is an invariant hyperplane of $E$. Furthermore, $I$ is an ideal, since if $x \in I$ and $0 \leqq y \leqq x$, then $\tilde{\phi}(y) = 0$ i.e. $[0, y] \subseteq I$.

(c) $\Rightarrow$ (a). Consider a normalised positive right linear action of $S$ on LUC($S$) defined by the map $(s,f) \to l_s f$. If LUC($S$) consists of only constant fucntions, then any mean on LUC($S$) is a LIM. Otherwise, LUC($S$) has a non-trivial maximal proper ideal $M$ which is invariant under the action of $S$. Then there exist a linear function $\phi$ on LUC($S$) such that $\phi(1) = 1$, $\phi \geqq 0$ and $\phi(f) = 0$ for all $f \in M$ (see [1, p. 121]). It is easy to see that $\phi$ is even a LIM.

COROLLARY. *If S is abelian, a solvable group or a compact semigroup with finite intersection property for right ideals, then S has properties* (*b*) *and* (*c*) *of Theorem* 2.

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