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LUSTERNIK-SCHNIRELMANN CATEGORY AND ALGEBRAIC *R*-LOCAL HOMOTOPY THEORY

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ABSTRACT. In this paper, we define the notion of R_*-LS category associated to an increasing system of subrings of $\mathbb{Q}$ and we relate it to the usual LS-category. We also relate it to the invariant introduced by Félix and Lemaire in tame homotopy theory, in which case we give a description in terms of Lie algebras and of cocommutative coalgebras, extending results of Lemaire-Sigrist and Félix-Halperin.

Introduction. Let $r \geq 3$ be a natural number. Let R be a subring of $\mathbb{Q}$ and $R_* = (R_i)_{i\geq 0}$ an increasing system of subrings of $\mathbb{Q}$ such that $R_i \supseteq R$ for $i \geq 0$. We call "(R_*, r)-homotopy theory" the homotopy category of spaces of the homotopy type of r-reduced CW-complexes X which are R_*-local, *i.e.* $\pi_{r+i}(X)$ is an R_i-module for $i \geq 0$. The most interesting of these theories is tame homotopy theory [5] where the rings R_i have to satisfy certain divisibility conditions. In fact, tame homotopy theory is equivalent to the homotopy theory of a closed model category Lie_s of s-reduced ($s = r - 1$) differential Lie algebras over R by [5].

We begin the present investigation by defining a notion of R_*-Lusternik-Schnirelmann category (R_*-cat($\_$) for short) for any r-reduced CW-complex. Our first main result then states that R_*-cat(Y) = cat(Y) (ordinary LS-category) provided Y is an r-reduced R-local CW-complex of R-dimension m and $R_i = R$ for $i \leq m - r$. In passing we establish a mapping theorem for cat for maps between such complexes. We also show that R_*-cat(Y) equals an invariant defined by Y. Félix and J. M. Lemaire [8], [9]. But it is the invariant R_*-cat(Y) we need to work with.

Our next objective is to consider tame homotopy theory and to establish an algorithm for computing R_*-cat(Y) from the Lie algebra model of Y. To this end we transfer the notion of "fibrations à la Ganea" developed in [21] into the tame setting. For the case of R-local CW-complexes of R-dimension m as above we obtain a particularly simple method of calculation which will be illustrated by examples.

The third main point is to demonstrate that the description of rational LS-category as given by Y. Félix and S. Halperin [7] can also be obtained for R_*-cat in tame theory. We use the description of tame homotopy theory via differential cocommutative coalgebras over R [20]. Let C be the coalgebra representing a space Y, let C_i be the i-th term in the primitive filtration of C, then $C_i \longrightarrow C$ is a model of an i-th fibration à la Ganea.

The last result may open up a way to extend the proof of the rational Ganea conjecture [11], [14] to obtain the following: Given an r-reduced CW-complex X and $n \geq r$, then

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one has R_*-cat$(X \times S^n) = R_*$-cat$(X) + 1$ provided R_* is tame. In particular, if $R_i = R$ for $i \le m - r$, X is R-local and R-dim$(X) + n \le m$, then cat$(X \times S^n) =$ cat$(X) + 1$. The idea is to dualize the proof of [11], [14] to the category of cocommutative differential coalgebras. In fact, we know of such attempts presently undertaken.

1. LS-**category and** R_*-**homotopy.** We first recall some facts about classical LS-category. Then we shall discuss R_*-homotopy theory (with tame homotopy theory as a particular case) and consider the fibre-cofibre construction in model categories. In 1.4 we define R_*-category. In 1.5 we recall an invariant of Félix-Lemaire and prove our first main result. As an application we prove in 1.6 a mapping theorem for LS-category of r-reduced R-local CW-complexes of R-dimension $\le m$. In Section 1.7 we transfer, in the tame situation, the definition of R_*-cat to the homotopy category of Lie$_s$.

In all what follows "space" will mean a pointed space of the pointed homotopy type of a pointed CW-complex.

1.1. *Lusternik-Schnirelmann category.* We refer to the survey article [13] for a discussion of all the statements in Section 1.1.

DEFINITION 1.1. The Lusternik-Schnirelmann category, cat(X), of a space X is the smallest integer k, $k \ge 0$, such that X can be covered by $(k+1)$ open subsets which are contractible in X, or it is infinity, if no such k exists.

NOTE. The original definition [16] worked with $k \ge 1$ and coverings by k open contractible sets.

Recall that the "fat wedge", $T^k(X)$, of a space X is the subspace of X^{k+1} of points having at least one component equal to the base point. Then one has:

PROPOSITION 1.2. *For any X cat(X) is the smallest integer k (or infinity) such that the diagonal $\Delta\colon X \longrightarrow X^{k+1}$ factors up to homotopy through the inclusion $j\colon T^k(X) \longrightarrow X^{k+1}$; i.e. there exists $\sigma\colon X \longrightarrow T^k(X)$ with $\Delta \sim j \cdot \sigma$.*

We also have to recall the "fibre-cofibre construction":

Given a map $Y \longrightarrow X$. Factorize it as $Y \xrightarrow{\sim} Y' \overset{p'}{\twoheadrightarrow} X$, a homotopy equivalence followed by a fibration p' (We call p' the associated fibration of $Y \longrightarrow X$). Let F' be the fibre of p' and form $Y_1 := Y' \cup C(F')$ where $C(F')$ is the reduced cone on F' and define $p_1\colon Y_1 \longrightarrow X$ by $p_1|Y' = p'$, $p_1|C(F') = *$. The map p_1 is called the *fibre-cofibre construction* of $Y \longrightarrow X$.

The sequence $p_i\colon G_i(X) \longrightarrow X$ of Ganea maps is inductively defined as follows: p_0 is $* \longrightarrow X$ and p_i is the fibre-cofibre construction of p_{i-1} for $i \ge 1$. The associated fibrations are called *Ganea fibrations*.

Note that $G_1(X) \longrightarrow X$ is equivalent to the evaluation map $\Sigma\Omega(X) \longrightarrow X$ (where Ω resp. Σ denotes loop space resp. reduced suspension).

PROPOSITION 1.3. *The value* cat(X) *is equal to the smallest integer k (resp. infinity) such that $p_k\colon G_k(X) \longrightarrow X$ admits a section up to homotopy.*

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REMARK. Gilbert proved this [10] by showing that the k-th Ganea fibration is equivalent over X to the pullback by $\Delta\colon X \longrightarrow X^{k+1}$ of the fibration associated to $j\colon T^k(X) \longrightarrow X^{k+1}$.

For a proof in a more general setting see [3].

1.2. *R_*-homotopy.* Let the subring $R \subseteq \mathbb{Q}$ and the integer $r \geq 3$ be fixed. An R-system of rings is a sequence $R_* = (R_i)_{i\geq 0}$ of increasing subrings of $\mathbb{Q}$ such that $R \subseteq R_0$.

The R-system R_* is called "tame", if each $k \geq 0$ with $2k-3 \leq i$ is invertible in R_i.

Denote by $\mathcal{S}$ the category of simplicial sets and by $\mathcal{S}_r$ the category of r-reduced simplicial sets. The category $\mathcal{S}_r$ carries the following closed model category structure, to be denoted by R_*-$\mathcal{S}_r$, [5]: The cofibrations are the injective maps; the weak equivalences are the maps f such that $\pi_{r+i}(f) \otimes R_i$ is an isomorphism for all $i \geq 0$; the fibrations are implicitly defined.

We will need a partial direct characterization of fibrations in R_*-$\mathcal{S}_r$ given in [5]: A morphism f in R_*-$\mathcal{S}_r$ is a fibration in R_*-$\mathcal{S}_r$ and $\pi_0(f) \otimes R_0$ is surjective, if and only if f is a Kan fibration in $\mathcal{S}$ and for all $k \geq 0$ (a) $\pi_{r+k}(F)$ is an R_k-module (F the fibre of f) and (b) $\operatorname{cokernel}\bigl(\pi_{r+k+1}(f)\bigr)$ is without p-torsion, p invertible in R_{k+1}.

In particular, an object $X \in R_*$-$\mathcal{S}_r$ is fibrant, if it is a Kan complex and $\pi_{r+i}(X)$ is an R_i-module for $i \geq 0$.

If $R_i = R$, $i \geq 0$, we denote R_*-$\mathcal{S}_r$ by R-$\mathcal{S}_r$; the fibrant objects are then called "R-local", and the corresponding homotopy theory is the usual R-local homotopy theory (From $\mathbb{Q}$-$\mathcal{S}_r$ we obtain rational homotopy theory).

Note that $\mathbb{Z}$-$\mathcal{S}_r$ defines "ordinary" homotopy theory.

We will also have to consider a particular subcategory of the homotopy category Ho-R_*-$\mathcal{S}_r$ of R_*-$\mathcal{S}_r$.

An r-reduced R-local CW-complex of R-dimension m is a cellular complex constructed from $*$ by successively attaching cones on R-local spheres, S^n_R, $r-1 \leq n < m$.

Let $R - \mathrm{CW}^m_r$ be the category of such spaces. Then (see [17]) the ordinary homotopy category of R-CW^m_r embeds as a full subcategory into Ho-R_*-$\mathcal{S}_r$ provided $R_i = R$ for $i = 0, \ldots, m-r$.

NOTATION. We use "$\underset{R_*}{\sim}$" (resp. "$\sim$") to denote weak equivalences in $R_* - \mathcal{S}_r$ (resp. $\mathbb{Z} - \mathcal{S}_r$). The ornamented arrows "$\rightarrowtail$", "$\twoheadrightarrow$" indicate cofibrations (resp. fibrations) in various model categories.

1.3. *The fibre-cofibre construction in a model category.* Let $\mathcal{M}$ be a pointed model category. We want to give a simple-minded fibre-cofibre construction in $\mathcal{M}$. In fact it is the exact analogue of the ordinary construction recalled in 1.1. On the other hand it is a particular case of the more general "join" construction of J. P. Doeraene [3].

Let $X \in \mathcal{M}$ be fibrant. Let $Y \longrightarrow X$ be a morphism with Y cofibrant. Factor $Y \longrightarrow X$ into $Y \overset{\sim}{\rightarrowtail} X' \twoheadrightarrow X$, a cofibration and weak equivalence followed by a fibration. Let F' be the fibre of $X' \twoheadrightarrow X$ and factor $F' \longrightarrow *$ in $F' \rightarrowtail C(F') \overset{\sim}{\longrightarrow} *$, a cofibration followed by

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a weak equivalence. Define $Y_1 \longrightarrow X$ by the following diagram:

$$\begin{tikzcd} F' \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & X' \arrow[d] \arrow[dr] & \\ C(F') \arrow[r] & Y_1 \arrow[r] & X \end{tikzcd}$$

We now assume that $\mathcal{M}$ satisfies the following property:

($*$) Given a fibration $Z' \longrightarrow Z$ in $\mathcal{M}$ with Z' cofibrant, then the fibre is cofibrant.

One can then show that up to weak equivalence over X the morphism $Y_1 \longrightarrow X$ does not depend on the choices made, nor does it depend on the weak equivalence class of Y over X (all objects over X taken cofibrant!). The assumptions have been arranged such that the gluing lemmas (comp. [2]) can be applied. No assumption about "properness" of $\mathcal{M}$ is needed.

1.4. R_*-LS-*category.* Let $\mathcal{M}$ be a model category as above. For X fibrant we can then define a sequence of Ganea maps by starting at $* \longrightarrow X$ giving rise to a notion of $\mathcal{M}$-LS-category in analogy to 1.1. For details we refer to [3] and [4].

Applying this procedure to R_*-$\mathcal{S}_r$ leads to the following phenomenon which—from a geometrical viewpoint—is undesirable:

Let $S^3_{\mathbb{Q}}$ be the $\mathbb{Q}$-local sphere (and a Kan complex) in $\mathbb{Q} - \mathcal{S}_3$. Then the map $* \longrightarrow S^3_{\mathbb{Q}}$ is a fibration. With $\mathcal{M} = \mathbb{Q} - \mathcal{S}_3$ this implies that all $\mathcal{M}$-Ganea fibrations of $S^3_{\mathbb{Q}}$ are equal to $* \longrightarrow S^3_{\mathbb{Q}}$ and $\mathcal{M}$-cat$(S^3_{\mathbb{Q}}) = \infty$ (see [4]). Of course, the usual category of the "space" $S^3_{\mathbb{Q}}$ is 1.

On the other hand we wish to have a good definition of cat in the model categories R_*-$\mathcal{S}_r$. For, if R_* is tame, we can then read the definition in the category Lie$_s$ of Lie algebras.

The solution is to change the beginning of the construction of the Ganea fibrations.

We need the following convention: Let f be a map between $(r-1)$-connected spaces. Then a morphism $f'\colon K \longrightarrow L$ in $\mathcal{S}_r$ is called a *model* of f, if there exists a homotopy commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & {|K|} \arrow[l, "\sim"'] \arrow[d, "{|f'|}"] \\ Y & {|L|} \arrow[l, "\sim"'] \end{tikzcd}$$

where $|\_|$ means geometric realization and "$\sim$" homotopy equivalence.

Such a model always exists. One may take for K the subcomplex ${}_rS(X)$ of the singular complex of X consisting of those simplices whose i-th faces are at $*$ for $i < r$, and similarly $L := {}_rS(Y)$. Then the diagram

$$\begin{tikzcd} X \arrow[d] & {|{}_rS(X)|} \arrow[l, "\sim"'] \arrow[d] \\ Y & {|{}_rS(X)|} \arrow[l, "\sim"'] \end{tikzcd}$$

even commutes. The above definition, however, will enable particular choices.

DEFINITION 1.4. Let $X \in R_*\text{-}\mathcal{S}_r$ be fibrant.

Define $q_1: R_*\text{-}G_1(X) \longrightarrow X$ as a model of $G_1(|X|) \longrightarrow |X|$ and $q_i: R_*\text{-}G_i(X) \longrightarrow X$ as the fibre-cofibre construction of q_{i-1} for $i \geq 2$. (Note that $R_*\text{-}\mathcal{S}_r$ satisfies condition 1.3 $(*)$, because all objects are cofibrant).

The spaces $R_*\text{-}G_i(X)$, $i \geq 1$, will be called *Ganea spaces* in $R_*\text{-}\mathcal{S}_r$, the fibrations associated to the q_i are the Ganea fibrations.

DEFINITION 1.5. Let $X \in R_*\text{-}\mathcal{S}_r$.

If X is homotopy equivalent to $*$ in the homotopy category of $R_*\text{-}\mathcal{S}_r$, then we set $R_*\text{-cat}(X) = 0$.

Otherwise we define $R_*\text{-cat}(X) := \inf\{n \mid q_n: R_*\text{-}G_n(X_f) \longrightarrow X_f$ admits a section in the homotopy category$\}$. (Here X_f is a fibrant model of X. Therefore, by definition $R_*\text{-cat}(X) = R_*\text{-cat}(X_f)$).

(Note that—by the discussion above—the definition does not depend on the choice of the fibrant model of X).

CONVENTIONS. Since for $X \in \mathbb{Z}\text{-}\mathcal{S}_r$ we have $\text{cat}(|X|) = \mathbb{Z}\text{-cat}(X)$, we will in the following simply write $\text{cat}(X)$ for $\mathbb{Z}\text{-cat}(X)$.

If X is an $(r-1)$-connected space, we will write $R_*\text{-cat}(X)$ for $R_*\text{-cat}(K)$, where K is a model of X in $\mathcal{S}_r$.

1.5. *Comparing* cat, R_*-cat *and an invariant of Félix-Lemaire.*

DEFINITION 1.6. Let $X \in R_*\text{-}\mathcal{S}_r$, let $T^k(X) \subset X^{k+1}$ be the fat wedge, let $T^k(X)_f$ and X_f^{k+1} be fibrant models. Denote by $\tilde{\Delta}_k: X \longrightarrow X_f^{k+1}$ the composition of $\Delta_k: X \longrightarrow X^{k+1}$ with $X^{k+1} \longrightarrow X_f^{k+1}$. Then one sets [8], [9]

$$\text{fw-}R_*\text{-cat}(X) := \inf\{k \mid k \geq 0 \text{ and } \tilde{\Delta}_k \text{ factors through } T^k(X)_f \longrightarrow X_f^{k+1} \text{ in the homotopy category of } R_*\text{-}\mathcal{S}_r\}.$$

(Of course, "fw" should remind us of "fat wedge").

We are now able to formulate the first main result:

THEOREM 1. *Let X be an $(r-1)$-connected* CW*-complex.*

(i) Then $\text{fw-}R_*\text{-cat}(X) = R_*\text{-cat}(X) \leq \text{cat}(X)$.

(ii) If R_* is an R-system such that $R_i = R$ for $i = 0, \ldots, m-r$ and X is an R-local CW*-complex of R-dim$(X) \leq m$, then* $R_*\text{-cat}(X) = \text{cat}(X)$.

The proof will follow easily from two lemmas. To simplify the notation we will notationally not distinguish between the Ganea maps and the associated Ganea fibrations. (Recall that $G_i(X)$ denotes Ganea space with respect to $\mathbb{Z}\text{-}\mathcal{S}_r$).

LEMMA 1.7. *Let $X \in R_*\text{-}\mathcal{S}_r$ be fibrant. Then $G_i(X)_f$ is equivalent in $R_*\text{-}\mathcal{S}_r$ over X to $R_*\text{-}G_i(X)$ for $i \geq 1$.*

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PROOF. The existence of a commutative diagram

$$\begin{tikzcd} G_1(X) \arrow[rr, "\widetilde{R}_*"'] \arrow[dr, two heads] & & R_*\text{-}G_1(X) \arrow[dl, two heads] \\ & X & \end{tikzcd}$$

follows from the definitions. Both fibrations are Kan fibrations and surjective on homotopy groups. Hence the long exact homotopy sequences decompose into short ones what implies that the map induced on the fibres is a weak equivalence in $R_*\text{-}\mathcal{S}_r$. By induction we now suppose that, for $i \geq 2$, a weak equivalence $G_{i-1}(X) \longrightarrow R_*\text{-}G_{i-1}(X)$ over X exists; it induces a weak equivalence between the respective fibres F_{i-1} and $R_*\text{-}F_{i-1}$. In the following diagram

$$\begin{tikzcd} G_{i-1}(X) \cup C(F_{i-1}) \arrow[rr, "\widetilde{R}_*"'] \arrow[d, "\sim"'] \arrow[dr] & & R_*\text{-}G_{i-1}(X) \cup C(R_*\text{-}F_{i-1}) \arrow[d, "\widetilde{R}_*"] \arrow[dl] \\ G_i(X) \arrow[r, two heads] \arrow[rr, dashed, bend left, "\alpha"] & X & R_*\text{-}G_i(X) \arrow[l, two heads] \end{tikzcd}$$

the weak equivalence α in $R_*\text{-}\mathcal{S}_r$ making the diagram commute exists by the properties of a closed model category.

LEMMA 1.8. *For each $X \in R_*\text{-}\mathcal{S}_r$ one has*

$$\mathrm{fw}\text{-}R_*\text{-}\mathrm{cat}(X) = R_*\text{-}\mathrm{cat}(X).$$

PROOF. We assume X fibrant and regard the following diagram.

$$\begin{tikzcd} G_k(X) \arrow[r, "\sim"] \arrow[dr] & P \arrow[r] \arrow[d, two heads] \arrow[rrrr, dashed, bend left, "\beta"] & E \arrow[d, two heads] \arrow[rr, dashed, bend left, "\alpha"] & T^k(X) \arrow[l, hook', "\sim"'] \arrow[r, "\widetilde{R}_*"'] \arrow[d] & T^k(X)_f \arrow[d, two heads] & Q \arrow[l] \arrow[d, two heads] \\ & X \arrow[r, "\Delta"] & X^{k+1} \arrow[r, equal] & X^{k+1} \arrow[r, equal] & X^{k+1} & X \arrow[l, "\Delta"'] \end{tikzcd}$$

The map $T^k(X) \longrightarrow X^{k+1}$ is factored into a product of a trivial cofibration $T^k(X) \overset{\sim}{\rightarrowtail} E$ and a fibration $E \twoheadrightarrow X^{k+1}$ in $\mathcal{S}_*$ on one side; on the other side it is factored into a product of a weak equivalence $T^k(X) \longrightarrow T^k(X)_f$ and a fibration $T^k(X)_f \twoheadrightarrow X^{k+1}$ in $R_*\text{-}\mathcal{S}_r$. The morphisms $P \longrightarrow X$ and $Q \longrightarrow X$ are the pullbacks by Δ. The morphism α making the diagram commute exists by the properties of the model category $R_*\text{-}\mathcal{S}_r$; hence β exists by the pullback property.

Observe that $E \longrightarrow X^{k+1}$ and $T^k(X)_f \longrightarrow X^{k+1}$ induce surjective homomorphisms of homotopy groups. It follows in particular that $T^k(X)_f \longrightarrow X^{k+1}$ is also a Kan fibration (by the criterion recalled above). Therefore the exact homotopy sequences of $E \longrightarrow X^{k+1}$ and $T^k(X)_f \longrightarrow X^{k+1}$ decompose into short exact sequences. Since α is a weak equivalence in

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R_*-$\mathcal{S}_r$, this implies that the map induced by α on the fibres of $E \longrightarrow X^{k+1}$ and $T^k(X)_f \longrightarrow X^{k+1}$ is a weak equivalence in R_*-$\mathcal{S}_r$. Then it follows from the exact homotopy sequences of $P \longrightarrow X$ and $Q \longrightarrow X$ that β is a weak equivalence in R_*-$\mathcal{S}_r$.

By [10] there is a weak homotopy equivalence (in $\mathcal{S}_r$) $G_k(X) \longrightarrow P$ over X; by Lemma 1.7 we deduce R_*-$G_k(X) \overset{\widetilde{R}_*}{\longrightarrow} P_f$. Thus fw-$R_*$-cat$(X)$ = R_*-cat(X).

PROOF OF THEOREM 1. Part (1) is given by Lemma 1.8 because, obviously, cat$(X) \geq$ fw-R_*-cat(X).

To prove part (2) it suffices to show that cat$(X) \leq$ fw-R_*-cat(X). Since X is R-local, so is the fat wedge $T^k(X)$. Hence $T^k(X) \longrightarrow T^k(X)_f$ is an m-equivalence (in the R-local sense), and the result follows.

1.6. *A mapping theorem for cat in* CW_r^m. Suppose $f: X \longrightarrow Y$ is a morphism in R_*-$\mathcal{S}_r$. Assume that $\Omega(X_f)$ and $\Omega(Y_f)$ are homotopy equivalent to weak products of Eilenberg-MacLane complexes and that $f_*: \pi_i(X_f) \longrightarrow \pi_i(Y_f)$ is split injective for $i \geq r$. By [8] we then have R_*-cat$(X) \leq R_*$-cat(Y). In particular, if R_* is an R-system and $X, Y \in \mathrm{CW}_r^m$, then cat$(X) \leq$ cat(Y). But in that case the following is the appropriate formulation:

PROPOSITION 1.9. *Let $X, Y \in \mathrm{CW}_r^m$ and $f: X \longrightarrow Y$ be a map. Suppose R-dim$(X) \leq k \leq m$; assume that $f_*: \pi_i(X) \longrightarrow \pi_i(Y)$ is split injective for $i \leq k$ and that there is a k-equivalence $\Omega Y \longrightarrow \prod_{i=r}^k K\big(\pi_i(Y), i-1\big)$.*

Then we have cat$(X) \leq$ cat(Y).

PROOF. Consider the fibre sequence

$$\longrightarrow \Omega X \longrightarrow \Omega Y \xrightarrow{\ h\ } F \longrightarrow X \xrightarrow{\ f\ } Y$$

of the map f. Denote by A_i the cokernel of $\pi_i(f)$, $r \leq i \leq k$. Then there is a k-equivalence

$$\Omega Y \xrightarrow{(g,h)} \Big(\prod_{i=r}^{k} K(A_i, i-1)\Big) \times F.$$

Let $F^k \longrightarrow F$ be a k-equivalence with F^k R-local of R-dim $\leq k$. Then $F^k \longrightarrow F$ factors (up to homotopy) through $h: \Omega Y \longrightarrow F$ and hence the composite $F^k \longrightarrow F \longrightarrow X$ is nullhomotopic. Assume cat$(Y) \leq q$ and let $Y_0 \cup \cdots \cup Y_q$ be a covering of Y by in Y contractible subcomplexes. We may assume that f is cellular. Let $X_i = f^{-1}(Y_i)$, then $X_i \longrightarrow X$ factors through $F \longrightarrow X$ and $(X_i)_R \longrightarrow X$ through $F^k \longrightarrow X$, because R-dim$(X_i)_R \leq k$. Hence cat$(X) \leq q$.

REMARK. If R_* is tame, a k-equivalence $\Omega Y \longrightarrow \prod_{i=r}^k K\big(\pi_i(Y), i-1\big)$ exists [19].

1.7. *Translating the definition of R_*-cat into* Lie_s. Let R_* be an R-system, $r \geq 3$ and $s = r - 1$.

Denote by Ch_s the category of s-reduced chain complexes over R. It carries the following closed model category structure: The cofibrations are the injective morphisms with degreewise projective cokernel; the weak equivalences are the morphisms f such that $H_{s+i}(f;R_i)$ is an isomorphism for all $i \geq 0$; a morphism f is a fibration, if it is surjective in degrees $> s$, if H_{s+i} (kernel (f)) is an R_i-module and if cokernel $(H_{s+i}(f))$ is without q-torsion for q invertible in R_i, $i \geq 0$. This closed model category structure is denoted by R_*-Ch_s.

Assume now that R_* is mild, *i.e.* for $i \geq 0$ the positive integers k with $sk \leq s+i$ are invertible in R_i. Note that "tame" implies "mild".

Then [5] the category Lie_s of s-reduced differential Lie algebras over R has the following closed model category structure denoted by R_*-Lie_s: A morphism in Lie_s is a weak equivalence (resp. a fibration) if it is a weak equivalence (resp. fibration) as map in R_*-Ch_s; the cofibrations are implicitly defined.

By [5] there is a sequence of pairs of adjoint functors between R_*-Lie_s and R_*-$\mathcal{S}_r$ inducing adjoint functors on the corresponding homotopy theories (comp. [20]). If R_* is tame, these induced functors are equivalences. If $L \in R_*$-Lie_s and $X \in R_*$-$\mathcal{S}_r$ correspond to each other via these functors, L is called a *model* of X.

REMARK. To avoid the presence of 2- and 3-torsion in free Lie algebras over R we suppose that the Lie bracket always satisfies the following conditions:

(1) For all x of pair degree $[x,x]=0$,

(2) For all homogeneous x one has $\big[x,[x,x]\big]=0$.

As it was remarked in [20] this has no effect on Ho-R_*-Lie_s for R_* mild.

DEFINITION 1.10. Let $L \in R_*$-Lie_s be fibrant.

The first Ganea map R_*-$G_1(L) \longrightarrow L$ is a model of R_*-$G_1(X) \longrightarrow X$ where L is a model of the fibrant object X.

For $i \geq 2$, the Ganea maps R_*-$G_i(L) \longrightarrow L$ are given by the fibre-cofibre construction on R_*-$G_{i-1}(L) \longrightarrow L$.

Property 1.3($*$) is true for R_*-Lie_s, because the cofibrant objects are the free Lie algebras and sub-Lie algebras of free ones are free.

DEFINITION 1.11. In analogy to Definition 1.5 we define R_*-$\mathrm{cat}(L)$ for L fibrant. (Details may be omitted).

For arbitrary $K \in R_*$-Lie_s we set R_*-$\mathrm{cat}(K) := R_*$-$\mathrm{cat}(K_f)$ where K_f is a fibrant model of K.

PROPOSITION 1.12. *Let R_* be tame. Let $X \in R_*$-$\mathcal{S}_r$ and L be a model of X in R_*-*Lie_s. *Then* R_*-$\mathrm{cat}(X) = R_*$-$\mathrm{cat}(L)$.

PROOF. We may assume X, L fibrant. Then, by definition, R_*-$G_1(X) \longrightarrow X$ and R_*-$G_1(L) \longrightarrow L$ correspond to each other under the equivalence of homotopy theories. By [4] also the following fibre-cofibre constructions R_*-$G_i(X) \longrightarrow X$ and R_*-$G_i(L) \longrightarrow L$ correspond.

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2. R_*-cat **and** LS-**fibrations.**

2.1. LS-*fibrations in* R_*-$\mathcal{S}_r$.

DEFINITION 2.1. Let $X \in R_*$-$\mathcal{S}_r$ with fibrant model X_f.

A map $f\colon Y \longrightarrow X'$ is called an "n-LS-morphism", if two commutative rectangles exist in Ho-R_*-$\mathcal{S}_r$ as follows:

$$\begin{tikzcd} R_*\text{-}G_n(X_f) \arrow[d] \arrow[r, shift left] & Y \arrow[l, shift left] \arrow[d, "f"] \\ X_f \arrow[r, phantom, "\widetilde{R}_*"] & X' \end{tikzcd}$$

If f is a fibration, we call it an "n-LS-fibration".

REMARK. If $Y_n \longrightarrow X'_n$, $n \geq 1$, is a sequence of n-LS-morphisms, then R_*-cat$(X) = 0$, or R_*-cat$(X) = \inf\{n \mid Y_n \longrightarrow X'_n$ admits a section in Ho-R_*-$\mathcal{S}_r\}$.

Following [21] we will construct sequences of n-LS-morphisms in R_*-$\mathcal{S}_r$ (and in R_*-Lie$_s$ in 2.2).

Let $\bar{\Omega}(\ )$ denote a suitable loop space functor $\bar{\Omega}\colon \mathcal{S}_r \longrightarrow \mathcal{S}_{r-1}$.

THEOREM 2. *Let $X \in R_*$-$\mathcal{S}_r$ and $Y \longrightarrow X$ a morphism in $\mathcal{S}_r$ such that*

(i) $\bar{\Omega}(Y_f) \longrightarrow \bar{\Omega}(X_f)$ *admits a section up to homotopy,*

(ii) R_*-cat$(Y) \leq 1$,

*then $Y \longrightarrow X$ is a 1-*LS-*morphism; the homotopy fibre F of $Y \longrightarrow X_f$ has* R_*-cat$(F) \leq 1$.

PROOF. The proofs of [21], Proposition 2.2 and Proposition 4.5 apply here as well. (In fact, the proof can also be left as an exercise).

DEFINITION 2.2. Let a pointed model category $\mathcal{M}$ be given. Let $F \xrightarrow{i} E \xrightarrow{f} X$ be a fibration in $\mathcal{M}$ with fibre F. Let $j\colon A \longrightarrow F$; we factorize $A \longrightarrow *$ as $A \rightarrowtail C(A) \xrightarrow{\sim} *$ and define $E' \longrightarrow X$ by the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & F \arrow[r, "i"] & E \arrow[d] \arrow[dl] \\ C(A) \arrow[r] & E' \arrow[r] & X. \end{tikzcd}$$

The construction will be called "modified fibre-cofibre construction with respect to j".

THEOREM 3. *Let $X \in R_*$-$\mathcal{S}_r$ be fibrant, let $F \longrightarrow E \longrightarrow X$ be an n-*LS-*fibration. Given $j\colon A \longrightarrow F$ let $E' \longrightarrow X$ be the modified fibre-cofibre construction with respect to j.*

If $F \longrightarrow E'$ is trivial in Ho-R_*-$\mathcal{S}_r$, then $E' \longrightarrow X$ is an $(n+1)$-LS-morphism.

PROOF. The proof of Théorème 1 in [21] applies here as well.

We also want to transcribe the way the holonomy was used in [21] into the present situation.

Let $X \in R_*$-$\mathcal{S}_r$ be fibrant and $f\colon E \longrightarrow X$ a fibration such that $\pi_r \otimes R_0$ is surjective. Then f is also a Kan fibration and we may consider its holonomy (calculated in $\mathcal{S}_{r-1}$)

$$m\colon \bar{\Omega}(X) \times F \longrightarrow F$$

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where F is the fibre of f.

Given $A \xrightarrow{j} F$ we denote by

$$m' \colon \bar{\Omega}(X) \times A \longrightarrow F$$

the composition of m with $(\mathrm{id} \times j) \colon \bar{\Omega}(X) \times A \longrightarrow \bar{\Omega}(X) \times F$.

If the connecting map (of the Kan fibration) $\partial \colon \bar{\Omega}(X) \longrightarrow F$ is homotopically trivial, we obtain a map

$$\bar{m}' \colon \bar{\Omega}(X) \times A / \bar{\Omega}(X) \longrightarrow F.$$

Note that $\partial \sim *$, if f is an LS-fibration.

Let $E' \longrightarrow X$ be obtained by the modified fibre-cofibre construction with respect to $\bar{m}'$.

PROPOSITION 2.3. *Let f be an n-LS-fibration and $\bar{m}'$ a homotopical epimorphism in R_*-$\mathcal{S}_r$ (i.e. the homotopy class of $\bar{m}'$ is an epimorphism in* Ho-R_*-$\mathcal{S}_r$ *in the categorical sense). Then $E' \longrightarrow X$ is an $(n+1)$-LS-morphism whose homotopy fibre (with respect to R_*-$\mathcal{S}_r$) F' has R_*-$\mathrm{cat}(F') \leq 1$.*

PROOF. Let R_*-$E' \longrightarrow X$ be the fibration associated to $E' \longrightarrow X$ in R_*-$\mathcal{S}_r$ (whose fibre is F' by definition), let $\tilde{E}' \longrightarrow X$ be the fibration associated to $E' \longrightarrow X$ in $\mathbb{Z}$-$\mathcal{S}_r$ with fibre $\tilde{F}$. From the diagram

$$\begin{tikzcd}
\tilde{F} \arrow[d] & & F' \arrow[d] \\
\tilde{E}' \arrow[rr, dashed, bend left] \arrow[dr, two heads] & E' \arrow[l, "\sim"] \arrow[r, "\widetilde{R_*}"'] \arrow[d] & R_*\text{-}E' \arrow[dl, two heads] \\
& X &
\end{tikzcd}$$

follows the existence of a map $\tilde{E}' \longrightarrow R_*$-$E'$ inducing a weak equivalence $\tilde{F} \longrightarrow F'$ in R_*-$\mathcal{S}_r$. Note that $\pi_r(E') \longrightarrow \pi_r(X)$ is surjective, hence $\tilde{E}' \twoheadrightarrow X$ is a Kan fibration and $\tilde{F}$ is the homotopy fibre of $E' \longrightarrow X$ in the category of pointed simplicial sets.

Hence, by [21] there is a cofibration sequence

$$\bar{\Omega}(X) \times A / \bar{\Omega}(X) \xrightarrow{\bar{m}'} F \longrightarrow \tilde{F}.$$

Therefore $F \longrightarrow \tilde{F}$ and, by the above, $F \longrightarrow F'$ is homotopically trivial in R_*-$\mathcal{S}_r$. Moreover, $\mathrm{cat}(\tilde{F}) \leq 1$ by [21], Lemma 4.7, thus R_*-$\mathrm{cat}(F') \leq 1$.

REMARK. Theorem 2 and Proposition 2.3 allow the construction of a sequence of n-LS-fibrations, $n \geq 1$. In this context a criterion for "homotopical epimorphism" is provided by the following result.

LEMMA 2.4. *Let R_* be tame. Let $g \colon Y \longrightarrow Z$ be a morphism in R_*-$\mathcal{S}_r$, let R_*-$\mathrm{cat}(Y)$, R_*-$\mathrm{cat}(Z) \leq 1$. Then g is a homotopical epimorphism in R_*-$\mathcal{S}_r$ provided the induced homomorphisms $H_{r+i}(Y; R_i) \longrightarrow H_{r+i}(Z; R_i)$ are split surjective, $i \geq 0$.*

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PROOF. For any R-module A let $M(A,k)$, $k \geq 3$, be a Moore space with reduced homology isomorphic to A concentrated in degree k. It follows from the assumptions on R_*-cat that Y and Z are equivalent to $\bigvee_{i\geq 0} M(H_{r+i}(Y;R_i), r+i)$ resp. $\bigvee_{i\geq 0} M(H_{r+i}(Z;R_i), r+i)$ in Ho-R_*-$\mathcal{S}_r$. Therefore, in Ho-R_*-$\mathcal{S}_r$ there exists $h\colon Z \longrightarrow Y$ such that $\bar{g}\circ h = \mathrm{id}_Z$ (where $\bar{g}$ is the image of g in Ho-R_*-$\mathcal{S}_r$).

NOTE. Let us assume that R_*-cat$(A) \leq 1$. Then we have also R_*-cat$\big(\bar{\Omega}(X) \times A/\bar{\Omega}(X)\big) \leq 1$. For, if A is equivalent in R_*-$\mathcal{S}_r$ to a suspension $\Sigma A'$, then $\bar{\Omega}(X) \times \Sigma A'/\bar{\Omega}(X)$ being homotopy equivalent to $(\bar{\Omega}(X) \wedge \Sigma A') \vee \Sigma A'$ is a suspension.

Hence, if also R_*-cat$(F) \leq 1$, the criterion of Lemma 2.4 can be applied to $\bar{m}'\colon \bar{\Omega}(X) \times A/\bar{\Omega}(X) \longrightarrow F$.

2.2. *Sequences of* LS-*applications in* R_*-Lie$_s$. We assume again that R_* is a tame R-system.

The results of Section 2.1 have to be translated into the language of R_*-Lie$_s$.

PROPOSITION 2.5. *Let $L \in R_*$-Lie$_s$ be fibrant. Let (V,d) be a free chain complex over R, $\mathbb{L}(V,d)$ the free R-Lie algebra over (V,d) and assume that $\mathbb{L}(V,d) \longrightarrow L$ is given such that $H_{s+i}\big(\mathbb{L}(V,d)\otimes R_i\big) \longrightarrow H_{s+i}(L\otimes R_i)$ is split surjective for $i \geq 0$. Then $\mathbb{L}(V,d) \longrightarrow L$ is a* 1-LS-*morphism whose homotopy fibre F has* R_*-cat$(F) \leq 1$.

PROOF. Let $Y \longrightarrow X$ be a map between fibrant objects of R_*-$\mathcal{S}_r$ which corresponds to $\mathbb{L}(V,d) \longrightarrow L$. Then [19], $\bar{\Omega}(Y)$ and $\bar{\Omega}(X)$ are homotopy equivalent to the weak products of the Eilenberg-MacLane-spaces $K\Big(H_{s+i}\big(\mathbb{L}(V,d)\otimes R_i\big), s+i\Big)$ resp. $K\big(H_{s+i}(L\otimes R_i), s+i\big)$. Therefore, up to a homotopy equivalence of $\bar{\Omega}(X)$ a section up to homotopy of $\bar{\Omega}(Y) \longrightarrow \bar{\Omega}(X)$ can be constructed; hence $\bar{\Omega}(Y) \longrightarrow \bar{\Omega}(X)$ has a section up to homotopy.

Moreover, $\mathbb{L}(V,d)$ models a suspension by [6], hence R_*-cat$(Y) \leq 1$. The result follows from Theorem 2.

Let L be fibrant. Let $E_n \longrightarrow L$ be an n-LS-fibration, $n \geq 1$, such that E_n is cofibrant and such that the fibre F_n has R_*-cat$(F_n) \leq 1$.

As in [21] we now want to use the holonomy of the fibration $E_n \longrightarrow L$ to simplify the construction of the next $(n+1)$-LS-morphism. We need some more conventions:

For any complex $D \in \mathrm{Ch}_s$ we set $H_*(D;R_*) := \bigoplus_{i\geq 0} H_{s+i}(D;R_i)$.

If $L \in \mathrm{Lie}_s$, we denote by ab(L) the abelianization of L.

Recall that, if L is cofibrant, $H_*\big(\mathrm{ab}(L);R_*\big)$ is up to a degree shift by 1 the homology of the space corresponding to L. By [6] there exists a free chain complex (W,d) over R and a weak equivalence $\mathbb{L}(W,d) \longrightarrow F_n$ in R_*-Lie$_s$; in particular, we have $H_*\big(\mathrm{ab}(F_n);R_*\big) \cong H_*(W;R_*)$.

Recall that, if L is cofibrant, then there is an algebra isomorphism $H_*\big(U(L);R_*\big) \longrightarrow H_*(\bar{\Omega}X;R_*)$ (where $U(L)$ denotes the universal enveloping algebra of L) [18].

Let $\tau\colon H_*(L;R_*) \longrightarrow H_*(E_n;R_*)$ be a section. We define an operation of $U\big(H_*(L;R_*)\big)$ on $H_*(W;R_*) \cong H_*\big(\mathrm{ab}(F_n);R_*\big)$ by defining it on the generators $\langle u\rangle \in H_{s+\ell}(L;R_\ell)$ by

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the formula

$$\begin{gathered} H_{s+\ell}(L;R_\ell)\otimes H_{s+k}\big(\mathrm{ab}(F_n);R_k\big)\longrightarrow H_{2s+\ell+k}\big(\mathrm{ab}(F_n);R_{s+\ell+k}\big)\\ \langle u\rangle\otimes\langle w\rangle\longmapsto\langle[\tau'(u),w]\rangle \end{gathered}$$

where the symbol "$\langle\_\rangle$ denotes homology class, where $\tau'(u)\in\tau(\langle u\rangle)$ and $[\_\,,\_]$ is the Lie bracket.

One deduces from [21], Theorem 2, that this operation coincides with the one induced by the holonomy map. Thus we finally obtain:

PROPOSITION 2.6. *Let (V,d) be a subcomplex of (W,d), set j equal to the composition $\mathbb{L}(V,d)\longrightarrow\mathbb{L}(W,d)\longrightarrow F_n$; assume that*

$$\bigoplus_{k\geq 0}\bigoplus_{i+j=k}U_{s+i}\big(H_*(L;R_*)\big)\otimes H_{s+j}(A;R_j)\otimes R_k\longrightarrow H_*(W;R_*)$$

*is split surjective. Then the modified fibre-cofibre construction on $E_n\longrightarrow L$ with respect to j is an $(n+1)$-*LS*-morphism.*

2.3. *Computation of* cat *in algebraic R-local homotopy theory.* Let R_* be a tame R-system with $R_i=R$ for $i\leq m-r$. Recall ([17], or [1]) that the homotopy category of CW_r^m (see Proposition 1.2 for definitions) is equivalent to the full subcategory of Ho-R_*-Lie$_s$ given by the free differential Lie algebras L over R with only generators x such that $s\leq\mathrm{degree}(x)\leq m-1$.

Given such a Lie algebra let us inspect what we really need to calculate its LS-category.

(i) To construct $L\rightarrowtail L_f$ involves only adding generators in degrees $\geq m$. Thus, if $E_n\longrightarrow L_f$ is an LS-fibration, the existence of a section is already detected in degrees $\leq m-1$ (*i.e.* on L).

Moreover, if $E_n'\longrightarrow L_f$ is an LS-morphism which is surjective in degrees $\leq m-1$, the construction of an LS-fibration $E_n'\overset{\sim}{\rightarrowtail}E_n\twoheadrightarrow L_f$ involves again only attaching generators in degrees $\geq m$.

(ii) In the first step (Proposition 2.5) we need to calculate $H_{s+i}(L;R_i)$ for $s+i\leq m-1$. We then can choose (V_1,d_1) (concentrated in degrees between s and m) such that $\mathbb{L}(V_1,d_1)\longrightarrow L_f$ is surjective in degrees $\leq m-1$ and $H_{s+i}(\mathbb{L}(V_1,d_1);R)\longrightarrow H_{s+i}(L_f;R)$ is split surjective for $s+i\leq m-1$. Then we may choose $\mathbb{L}(V_2,d_2)\longrightarrow L_f$, V_2 m-reduced, such that $\mathbb{L}(V_1,d_1)\sqcup\mathbb{L}(V_2,d_2)\longrightarrow L_f$ satisfies the conditions of Proposition 2.5. (But (V_2,d_2) is not needed for the interesting part in the next construction).

(iii) Suppose the n-LS-fibration $E_n\longrightarrow L_f$ has been constructed. Let F_n be the fibre. Then we need only to know $H_{s+i}(F_n;R)$ for $s+i<m-1$ to construct an $(n+1)$-LS-fibration up to degrees $\leq m-1$.

Let us give two examples.

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EXAMPLE 2.8. Let $R = \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$. Let X_R be the R-local space corresponding to the following Lie algebra where $p > 5$ is prime.

$$\begin{aligned}(L, \partial) = \big(\mathbb{L}(x_1, x_2, y, w), \partial\big); \quad |x_1| = |y| = 3,\ |x_2| = 4,\ |w| = 10,\\ \partial x_1 = \partial y = 0, \quad \partial x_2 = px_1, \quad \partial w = \big[x_1, [y, y]\big]\end{aligned}$$

In fact, X_R is the localization of $X = \big((S^4 \cup e^5) \vee S^4\big) \cup e^{11}$ with suitable attaching maps.

Inside the first step we choose $\mathbb{L}(V_1, d_1) \longrightarrow L$ as follows: $\mathbb{L}(V_1, d_1) = \mathbb{L}(\hat{x}_1, \hat{x}_2, \hat{y}, \alpha_2, \alpha_1)$, $d_1\hat{x}_2 = p\hat{x}_1$, $d_1\hat{y} = 0$, $d_1\alpha_2 = \alpha_1$; $\hat{x}_1 \longmapsto x_1$, $\hat{x}_2 \longmapsto x_2$, $\hat{y} \longmapsto y$, $\alpha_2 \longmapsto w$, $\alpha_1 \longmapsto dw = \big[x_1, [y, y]\big]$.

$$\begin{array}{r|ll|l}
10 & w & & \alpha_2 \\
9 & & \searrow\ \big[x_1, [y, y]\big] & \alpha_1 \\
8 & & & \\
7 & & & \\
6 & & & \\
5 & & & \\
4 & x_2 & & \hat{x}_2 \\
3 & x_1, y & & \hat{x}_1\hat{y} \\
\hline
\end{array}$$

Next we have to determine kernel$(\mathbb{L}(V_1) \longrightarrow L)$ in degrees ≤ 10. It is generated as R-module by $\alpha_1 - \big[\hat{x}_1, [\hat{y}, \hat{y}]\big]$. Obviously, $\mathbb{L}(V_1) \longrightarrow L$ does not yet admit a section. Looking at the next step, $\mathbb{L}(V_1) \sqcup \mathbb{L}(u)$, $du = \alpha_1 - \big[\hat{x}_1, [\hat{y}, \hat{y}]\big]$, we see the section $\mathbb{L} \longrightarrow \mathbb{L}(V_1) \sqcup \mathbb{L}(u)$, $x_1 \longmapsto \hat{x}_1$, $x_2 \longmapsto \hat{x}_2$, $y \longmapsto \hat{y}$, $w \longmapsto \alpha_2 - u$.

Therefore $\mathrm{cat}(X_R) = 2$. (We knew already at the beginning $\mathrm{cat}(X_R) \leq \mathrm{cat}(X) \leq 2$, because X is a 2-cone).

EXAMPLE 2.9. Let now $R = \mathbb{Z}[1/2, 1/3, 1/5, 1/7]$, $p > 7$ prime and $(L, \partial) = \mathbb{L}(x_1, x_2, y, z, w)$; $|x_1| = |y| = 3$; $|x_2| = 4$, $|z| = 7$, $|w| = 14$ and $\partial x_1 = \partial y = 0$, $\partial x_2 = px_1$, $\partial z = [y, y]$, $\partial w = \big[x_1, [y, z]\big]$. The corresponding space is the R-localization of

$$Y = \Big(\big((S^4 \cup e^5) \vee S^4\big) \cup e^8\Big) \cup e^{15}$$

with suitable attaching maps; in fact, Y is the analogue with torsion of the Lemaire-Sigrist example [15].

Since Y is a 3-cone, we know $\mathrm{cat}(Y_R) \leq \mathrm{cat}(Y) \leq 3$.

To prove $\mathrm{cat}(Y_R) = 3$ we even do not need to complete the first step.

Inside the first step define $\mathbb{L}(W, \partial) \longrightarrow L$, $\mathbb{L}(W) = \mathbb{L}(\hat{x}_1, \hat{x}_2, \hat{y}, \hat{z}, \alpha)$, $\partial\hat{x}_2 = p\hat{x}_1$, $\partial\hat{y} = \partial\alpha = 0$, $\partial\hat{z} = \alpha$ and $\hat{x}_1 \longmapsto x_1$, $\hat{x}_2 \longmapsto x_2$, $\hat{y} \longmapsto y$, $\alpha \longmapsto [y, y]$, $\hat{z} \longmapsto z$.

The morphism $\mathbb{L}(W) \longrightarrow L$ is surjective and split surjective in homology up to degree 9. In degrees ≤ 9 the kernel of $\mathbb{L}(W) \longrightarrow L$ is generated as R-module by the cycles $\alpha - [\hat{y}, \hat{y}]$ and $\big[\hat{y}, \alpha - [\hat{y}, \hat{y}]\big]$; $\big[\hat{y}, \alpha - [\hat{y}, \hat{y}]\big]$ is given by the map induced by the holonomy. Hence,

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in the next step one adds a generator u with $du = \alpha - [\hat{y}, \hat{y}]$ (in degrees ≤ 9).

		Step 1	Step 2
9			
8			
7	z	$\hat{z}$	u
6	$[y, y]$	α	
5			
4	x_2	$\hat{x}_2$	
3	$x_1 y$	$\hat{x}_1 \hat{y}$	

The first (completed) step does not have a section. In the second step an eventual section σ is uniquely determined by $\sigma(x_1) = \hat{x}_1$, $\sigma(x_2) = \hat{x}_2$, $\sigma(y) = \hat{y}$, $\sigma(z) = \hat{z} - u$. Whatever the completed second step may be, $\sigma(dw) = \sigma\big[x_1, [y, z]\big] = \big[\hat{x}_1, [\hat{y}, \hat{z}]\big] - \big[\hat{x}_1, [\hat{y}, u]\big]$. But, whatever element $a = \sigma(w)$ in degree 14 one chooses, the expression $\big[\hat{x}_1, [\hat{y}, u]\big]$ cannot appear in $d(a)$, only multiples of $p\big[\hat{x}_1, [\hat{y}, u]\big]$ can. Thus the second LS-fibration does not admit a section and $\operatorname{cat}(Y_R) = 3$.

We remark that the Toomer invariant of Y is 2, the cuplength of Y is 2 and $\operatorname{cat}(Y_{\mathbb{Q}}) = 2$.

3. The model $\mathcal{L}\mathcal{C}(L, \partial) \longrightarrow (L, \partial)$.

3.1. *The analogue of the Félix-Halperin characterization of* cat*.* Let Coalg_r be the category of differential cocommutative r-reduced coalgebras which are free as R-modules. Let $\overline{\mathrm{Lie}}_s$ be the full subcategory of Lie_s of Lie algebras which are free as R-modules. Then, for a mild system R_*, the full subcategory Ho-$\overline{\mathrm{Lie}}_s$ of $\overline{\mathrm{Lie}}_s$ in Ho-R_*-Lie_s is equivalent to Ho-R_*-Lie_s.

We have adjoint functors

$$\mathcal{L} \colon \mathrm{Coalg}_r \leftrightarrows \overline{\mathrm{Lie}}_s : \mathcal{C}.$$

which—after tensorizing with $\mathbb{Z}[1/2]$—become the classical functors. If R_* is a mild system, the category Coalg_r can be endowed with the structure of a cofibration category such that the above adjoint functors induce equivalences

$$\text{Ho-}\mathrm{Coalg}_r \leftrightarrows \text{Ho-}\overline{\mathrm{Lie}}_s.$$

In particular $\mathcal{L}\mathcal{C}(L, \partial) \longrightarrow (L, \partial)$ is a weak equivalence. (Comp. [20]).

Given $D \in \mathrm{Coalg}_r$, let $P_n D$ be the n-th term in the primitive filtration of D, *i.e.*, if $\bar{\Delta}$ denotes the reduced diagonal, then $P_n D := \operatorname{kernel}\big((\mathrm{id} \otimes \bar{\Delta} \otimes \cdots \otimes \bar{\Delta}) \circ \cdots \circ (\mathrm{id} \otimes \bar{\Delta}) \circ \bar{\Delta}\big)$ where the composition consists of n factors.

THEOREM 4. *Let R_* be tame. Then $P_n(D) \longrightarrow D$ represents an n-*LS*-map.*

PROOF. It suffices to assume $D = \mathcal{C}(L, \partial)$, $L \in \overline{\mathrm{Lie}}_s$, L cofibrant, and to show that $\mathcal{L}(P_n D) \longrightarrow \mathcal{L}(D)$ is an n-LS-map, respectively the composition $\mathcal{L}(P_n D) \longrightarrow \mathcal{L}(D) \xrightarrow{\sim} L$. This is the content of the next result.

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PROPOSITION 3.1. *Let $(L,\partial)\in\overline{\mathrm{Lie}}_s$ be cofibrant. Then $L\Big(P_n\big(C(L,\partial)\big)\Big)\longrightarrow(L,\partial)$ is an n-LS-map in R_*-Lie$_s$, R_* tame.*

The proof is based on the following fact.

LEMMA 3.2. *For mild R_* there exists a homotopy in R_*-Ch$_s$ between the injection $L\Big(P_n\big(C(L,\partial)\big)\Big)\longrightarrow L\Big(P_{n+1}\big(C(L,\partial)\big)\Big)$ and the composition ψ of the restriction of $LC(L,\partial)\longrightarrow(L,\partial)$ with the injection $(L,\partial)\longrightarrow L\Big(P_{n+1}\big(C(L,\partial)\big)\Big)$.*

PROOF OF PROPOSITION 3.1. First we observe that $L\big(P_nC(L,\partial)\big)\longrightarrow(L,\partial)$ is surjective and split surjective in homology. Denote by K_n its kernel and construct the following diagram

$$\begin{array}{ccc} K_n & \longrightarrow & F_n \\ \big\downarrow & & \big\downarrow \\ L\big(P_nC(L,\partial)\big) & \xrightarrow{\sim} & E_n \\ \big\downarrow & & \big\downarrow\!\!\!\downarrow \\ (L,\partial) & \xrightarrow{\sim} & (L,\partial)_f \end{array}$$

by factorizing $L\big(P_nC(L,\partial)\big)\longrightarrow(L,\partial)_f$ appropriately, F_n being the fibre of $E_n\twoheadrightarrow(L,\partial)_f$. It follows that $K_n\longrightarrow F_n$ is a weak equivalence.

For $n=1$ we have $L\big(P_1C(L,\partial)\big)\cong\mathbb{L}(L,\partial)\longrightarrow L$ and it follows from Proposition 2.5 that $E_1\longrightarrow L_f$ is an 1-LS-fibration with R_*-cat$(F_1)\leq 1$.

Suppose inductively that $L\big(P_nC(L,\partial)\big)\longrightarrow(L,\partial)$ is an n-LS-map such that its kernel K_n has R_*-cat$(K_n)\leq 1$. From the lemma we deduce that $K_n\longrightarrow E_{n+1}$ is homotopically trivial in R_*-Ch$_s$. There is (W,d) such that K_n is homotopically equivalent to $\mathbb{L}(W,d)$; the corresponding class $\mathbb{L}(W,d)\longrightarrow E_{n+1}$ in Ho-R_*-Ch$_s$ is trivial.

Recall [19] that if $\overline{\mathrm{Ch}}_s$ denotes the full subcategory of abelian Lie algebras in Lie$_s$ then $\mathbb{L}(\_)$ and the forgetful functor F

$$\mathbb{L}\colon\overline{\mathrm{Ch}}_s\leftrightarrows\overline{\mathrm{Lie}}_s\colon F$$

are adjoint and induce adjoint functors on the homotopy categories. Thus $(W,d)\longrightarrow E_{n+1}$ is trivial in Ho-R_*-Ch$_s$; we deduce that $\mathbb{L}(W,d)\longrightarrow E_{n+1}$ is trivial in Ho-R_*-Lie$_s$. By Theorem 3 we conclude that $E_{n+1}\longrightarrow L_f$ is an $(n+1)$-LS-fibration. Note that the theorem applies, because $L\big(P_{n+1}C(L,\partial)\big)$ is the cofibre of the appropriate morphism $\mathbb{L}\Big(s^{-1}\big(P_{n+1}C(L,\partial)/P_nC(L,\partial)\big)\Big)\longrightarrow K_n$. It remains to show that R_*-cat$(K_{n+1})=R_*$-cat$(F_{n+1})\leq 1$.

The exact homotopy sequence in Ho-R_*-Lie$_s$

$$\longrightarrow[\Sigma F_n,E_{n+1}]\longrightarrow[\Sigma F_n,L_f]\longrightarrow[F_n,F_{n+1}]\longrightarrow[F_n,E_{n+1}]\longrightarrow[F_n,L_f]$$

decomposes into short sequences, because $\bar{\Omega}Y\longrightarrow\bar{\Omega}X$ admits a section (if Y,X fibrant in R_*-$\mathcal{S}_r$ represent E_{n+1},L_f resp.). Thus $F_n\longrightarrow F_{n+1}$ is homotopically trivial, hence F_{n+1} being a cofibre of a map into F_n we have R_*-cat$(F_{n+1})\leq 1$.

PROOF OF LEMMA 3.2. We shall deduce the result from Proposition 3.3 below about the Bar construction B and the Cobar construction Ω.

With the arguments for Lemmas 2.6, 2.7 of [20] one can show that

(i) $g_n\colon P_n\big(C(L,\partial)\big) \longrightarrow P_n\Big(B\big(U(L,\partial)\big)\Big)$ is a mild quasi-equivalence (*i.e.* $H_{r+i}(g_n;R_i)$ is an isomorphism and $H_{r+i+1}(g_n;R_i)$ an epimorphism for all $i \geq 0$; and

(ii) $\Omega(g_n)$ is a mild quasi-equivalence.

By Proposition 3.3 there is a chain homotopy between $L\big(P_nC(L,\partial)\big) \longrightarrow L\big(P_{n+1}C(L,\partial)\big)$ and ψ considered as chain maps into $\Omega\Big(P_{n+1}\big(\mathrm{BU}(L,\partial)\big)\Big)$. Since $\Omega(g_{n+1})$ is a weak equivalence in R_*-Ch_s, the maps are homotopic in R_*-Ch_s as maps into $\Omega\big(P_{n+1}C(L,\partial)\big)$. Recall (see below) that as algebra $\Omega\big(P_{n+1}C(L,\partial)\big)$ is isomorphic to the tensor algebra $T\big(s^{-1}\overline{P}_{n+1}C(L,\partial)\big)$. Let $T^k\big(s^{-1}\overline{P}_{n+1}C(L,\partial)\big)$ denote the subspace generated by the tensors of length k and define $\Omega'\big(P_{n+1}C(L,\partial)\big) = \oplus_{k\geq 0} T^k\big(s^{-1}\overline{P}_{n+1}C(L,\partial)\big)\otimes R_k$, similarly define $L'\big(P_{n+1}C(L,\partial)\big) \subset \Omega'\big(P_{n+1}C(L,\partial)\big)$. There is a retraction of chain complexes $\Omega'\big(P_{n+1}C(L,\partial)\big) \longrightarrow L'\big(P_{n+1}C(L,\partial)\big)$. Moreover, the inclusions $L\big(P_{n+1}C(L,\partial)\big) \longrightarrow L'\big(P_{n+1}C(L,\partial)\big)$ and $\Omega\big(P_{n+1}C(L,\partial)\big) \longrightarrow \Omega'\big(P_{n+1}C(L,\partial)\big)$ are weak equivalences in R_*-Ch_s. Thus the assertion follows.

3.2. *The functors Bar and Cobar.* Let (A,d) be an augmented graded differential associative algebra over a ring R such that A is free as R-module. Denote by $\rho_A\colon \Omega\big(B(A)\big) \longrightarrow A$ the counit of the adjunction given by the bar construction B and cobar construction Ω [12]. The essential definitions concerning B and Ω will be recalled in the course of the proof below.

PROPOSITION 3.3. *There exists a chain homotopy between the canonical injection $\Omega\big(P_i(BA)\big) \longrightarrow \Omega\big(P_{i+1}(BA)\big)$ and the composition ρ of the restriction of ρ_A with the canonical injection $A \longrightarrow \Omega\big(P_{i+1}(BA)\big)$.*

PROOF. Let $\overline{A}$ be the augmentation ideal of A. The underlying algebra of $\Omega(BA)$ is $T\Big(s^{-1}\big(\overline{T}(s\overline{A})\big)\Big)$ (where $T(\ )$ denotes the tensor algebra and s the suspension of chain complexes). We will use different symbols for the two tensor products involved. Thus a homogeneous element $w \in T\Big(s^{-1}\big(\overline{T}(s\overline{A})\big)\Big)$ will be written as

$$\begin{aligned} w &= w^1\tilde{\otimes}w^2\tilde{\otimes}\cdots\tilde{\otimes}w^p \\ &= s^{-1}(sa_1^1\otimes\cdots\otimes sa_{n_1}^1)\tilde{\otimes}s^{-1}(sa_1^2\otimes\cdots\otimes sa_{n_2}^2)\tilde{\otimes}\cdots\tilde{\otimes}s^{-1}(sa_1^p\otimes\cdots\otimes sa_{n_p}^p), \end{aligned}$$

where $w^i \in s^{-1}\big(\overline{T}(s\overline{A})\big)$, $a_j^i \in \overline{A}$. The differential D on $\Omega(BA)$ is of the form $D(w) = D_1(w) + D_2(w) + D_3(w)$ where

$$\begin{aligned} D_1(w) &= \Sigma_{i,j}(-1)^{\varepsilon(i,j)}\cdots\tilde{\otimes}s^{-1}(sa_1^i\otimes\cdots\otimes sa_j^i)\tilde{\otimes}s^{-1}(sa_{j+1}^i\otimes\cdots\otimes sa_{n_i}^i)\tilde{\otimes}\cdots \\ D_2(w) &= \Sigma_{i,j}(-1)^{\varepsilon(i,j)}\cdots\tilde{\otimes}s^{-1}(sa_1^i\otimes\cdots\otimes sa_j^i\otimes sda_{j+1}^i\otimes\cdots\otimes sa_{n_i}^i)\tilde{\otimes}\cdots \\ D_3(w) &= \Sigma_{i,j}(-1)^{\varepsilon(i,j)}\cdots\tilde{\otimes}s^{-1}\big(sa_1^i\otimes\cdots\otimes s(a_j^i.a_{j+1}^i)\otimes\cdots\otimes sa_{n_i}^i\big)\tilde{\otimes}\cdots \end{aligned}$$

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with

$$\varepsilon(i,j) = \sum_{l=1}^{i-1}(|sa_1^l \otimes \cdots \otimes sa_{n_l}^l| + 1) + \sum_{l=1}^{j} |sa_1^i \otimes \cdots \otimes sa_l^i|.$$

The following linear map $h\colon \Omega(BA) \longrightarrow \Omega(BA)$ will allow to construct the homotopy we are looking for:

$$h(w) = \begin{cases} 0 & \text{if } n_1 > 1 \text{ or} \\ & \text{if } n_1 = 1 \text{ and } p = 1 \\ (-1)^{|a_1^1|+1}\Big(s^{-1}(sa_1^1 \otimes sa_1^2 \otimes \cdots \otimes sa_{n_2}^2)\tilde{\otimes} w^3 \tilde{\otimes} \cdots \tilde{\otimes} w^p\Big) & \text{if } n_1 = 1 \text{ and } p > 1. \end{cases}$$

Easy calculations establish the following properties of h:

(i) if $n_1 \geq 3$, then $(hD + Dh)(w) = hDw = w$;

(ii) if $n_1 = 2$ and $p > 1$, then

$$(hD+Dh)(w) = w+(-1)^{|a_1^1|(1+|a_2^1|)}\Big\{s^{-1}\Big(s(a_1^1\cdot a_2^1)\otimes sa_1^2\otimes\cdots\otimes sa_{n_2}^2\Big)\tilde{\otimes} w^3\tilde{\otimes}\cdots\tilde{\otimes} w^p\Big\};$$

(iii) if $n_1 = 2$ and $p = 1$, then $(hD + Dh)(w) = w$;

(iv) if $n_1 = 1$ and $p = 1$, then $(hD + Dh)(w) = 0$; if $n_1 = 1$ and $p > 1$, then

$$(hD + Dh)(w) = w - \Big\{s^{-1}\Big(s(a_1^1 \cdot a_1^2) \otimes sa_2^2 \otimes \cdots \otimes sa_{n_2}^2\Big)\tilde{\otimes} w^3 \tilde{\otimes} \cdots \tilde{\otimes} w^p\Big\};$$

By induction over the tensor length in $\tilde{\otimes}$ we conclude from these formulas that for each w there exists a strictly positive integer $j(w)$ such that

$$(hD + Dh - \mathrm{id})^{j(w)} = (-1)^{j(w)}\rho(w)$$

In particular, if $n_1 = n_2 = \cdots = n_p = 1$ and $p > 1$ we have

$$(hD + Dh - \mathrm{id})^{p-1} = (-1)^{p-1}s^{-1}\Big(s(a_1^1.a_1^2 \cdots a_1^p)\Big).$$

Using $h \circ h = D \circ D = 0$ we get the general formula

$$\begin{aligned} \rho(w) - w &= \sum_{\mu=1}^{j(w)}(-1)^{\mu}\{(hD)^{\mu} + (Dh)^{\mu}\} \\ &= \sum_{\mu=1}^{j(w)}(-1)^{\mu}\{[(hD)^{\mu-1}h]D + D[(hD)^{\mu-1}h]\} \end{aligned}$$

Hence, the homotopy we are looking for can be defined as a sum of terms $(hD)^{\alpha}h$, $\alpha \geq 0$, provided we can show that

$$(hD)^{\alpha}h\Big(\Omega\big(P_i(BA)\big)\Big) \subset \Omega\big(P_{i+1}(BA)\big).$$

Denote by $\Omega\big(P_{i+1}(BA)\big)_{(1)}$ the submodule of $\Omega\big(P_{i+1}(BA)\big)$ generated by the homogeneous elements w such that $n_j \leq i$ for $j \geq 2$ and $n_1 > 1$. We will in fact prove by induction that

$$(hD)^{\alpha}h\Big(\Omega\big(P_i(BA)\big)\Big) \subset \Omega\big(P_{i+1}(BA)\big)_{(1)}.$$

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The formula is true for $\alpha = 0$. To perform the induction it suffices therefore to establish that

$$(hD)\Big(\Omega\big(P_{i+1}(BA)\big)_{(1)}\Big) \subset \Omega\big(P_{i+1}(BA)\big)_{(1)}.$$

This follows immediately from formulas (i), (ii), (iii) above.

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