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How do glaciers respond to climate? Perspectives from the simplest models

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ABSTRACT. We study the approximations underlying the macroscopic theory of glacier response to climate, and illustrate some basic properties of glacier response using two simple examples, a block on an inclined plane and a block with a more realistic terminal region. The properties include nonlinearity and the usefulness of linear approximations, sensitivity to bed slope, timescale, stability, characteristic elevations of the equilibrium line at which the properties of the response change, the limit of fast response in a changing climate, the minimum sustainable size, increased sensitivity to climate change as a glacier retreats, and finally the similarity in the responses of simple glaciers with the same product of length and square of bed slope.

1. INTRODUCTION

A simplified approach to glacier response to climate is often useful. One possibility is to use scaling methods to extrapolate from measured to many unmeasured glaciers (e.g. Radić and Hock, 2011). One can also model the response of individual glaciers (e.g. Oerlemans, 2005; Lüthi and others, 2010). Another possibility is to use a simplified approach to shed light on some of the basic physical processes which influence the response, processes which may be less obvious in more accurate but more complicated numerical models, yet when understood may suggest which generalizations and extrapolations are justified. A related possibility is to use a simplified approach, together with the simplest models, to study and to illustrate semi-quantitatively some of the most general properties of the response of valley glaciers. Such is the focus of this paper. We regard it as a useful step in the improvement of methods for extrapolation from measured to unmeasured glaciers, our longer-term objective. We begin by examining the approximations which underlie many of the simplified approaches, which we call 'macroscopic'.

2. MACROSCOPIC APPROXIMATIONS

2.1. Restricted and unrestricted macroscopic approximations

If there is any single property common to macroscopic approximations for predicting response, it is the imposition of constraints on the form of the ice thickness distribution, which would be calculated as part of a more complete approach. We call this the 'restricted' macroscopic approximation when the thickness distribution is specified either directly or via a functional relationship between map area A and ice volume V. We call the latter an 'A–V functional relationship'. It turns out that these are equivalent, which is most easily seen if the glacier has a vertical headwall and constant width W. Since $V = W\int_o^L h(X)\,\mathrm{d}X$, where L is the length, h is the ice thickness and X is measured from the terminus, it follows that $\mathrm{d}V/\mathrm{d}L = Wh(L)$ or $\mathrm{d}V/\mathrm{d}A = h(L)$, the ice thickness at the headwall. However, this holds for any L, so it is permissible to write $Wh(X) = [\mathrm{d}V/\mathrm{d}L]_{L=X}$. Retreat, for example, would be equivalent to removing the upper part of a glacier, and moving the remainder back until it touches the headwall (Fig. 1). Thus an A–V relationship specifies the thickness distribution along the glacier, and therefore the shape. This can be generalized to non-uniform width if the thickness $h(X)$ is replaced by the thickness averaged across the glacier at a given X. The basic idea is that an A–V relationship determines the ice thickness distribution or puts strong constraints upon it.

The best-known example is

$$V = cA^{\mu} \tag{1}$$

where μ is a constant. This is usually taken to represent an ensemble of glaciers, although here we focus on a single glacier (Chen and Ohmura, 1990; Bahr and others, 1997; Lüthi, 2009), in which case the dependence of c upon slope and other factors is accounted for. Differentiation gives

$$h(X) = \mathrm{d}V/\mathrm{d}A = \mu c\{A(X)\}^{\mu-1} \tag{2}$$

where $h(X)$ is now the thickness averaged across the glacier at a given X, and $A(X)$ is the map area below X. The implied shape of this glacier when it has different sizes but constant width is illustrated in Figure 1.

The functional relationship in the restricted approximation suggests that area responds instantly to volume, at least in the simplest case of the block model discussed below. This is reminiscent of perfect plasticity. One result is that in this model the surface cannot respond correctly on short timescales, such as those associated with the propagation of kinematic waves, perhaps on the order of a decade for a small valley glacier (e.g. Harrison and others, 2003). In what we call the 'unrestricted' macroscopic approximation one requires not a functional relationship between A and V but a differential equation which includes time derivatives of A (Harrison and others, 2003; Lüthi, 2009) and possibly of V. The resulting flexibility in the surface configuration permits the description of changes on relatively short timescales, although at the expense of more parameters and a higher-order governing differential equation. In the macroscopic approximations the A–V relationship is usually taken to be independent of climate, which ultimately implies that the shape is taken to be the same in advance and retreat. This is a potentially serious limitation, inherent in all restricted approximations including the often-used Eqn (1).

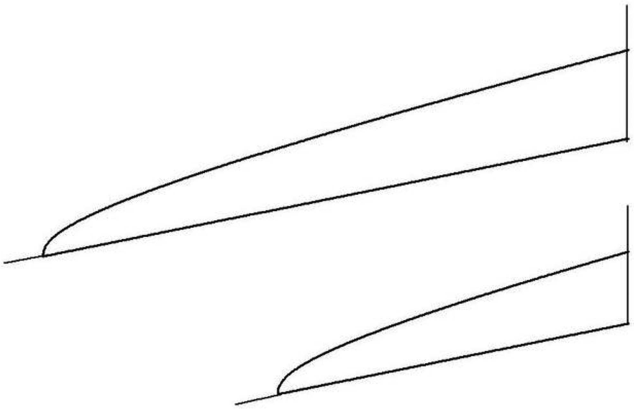

Fig. 1. Longitudinal profiles showing preservation of shape during advance or retreat. The thickness distribution is that implied by the scaling relation, Eqn (1).

In the restricted approximation, the ice thickness distribution specified by the A–V relationship, together with the bed topography, determine the surface configuration of the glacier. Once the mass-balance rate distribution $\dot{b}$ is specified, it can be integrated over this surface (which changes with time) to give the glacier-wide balance rate $\dot{B}$. Finally, to conserve mass this is set equal to the rate of change of volume (using ice volume units for balance rate), giving a simple first-order differential equation for the response in the restricted approximation:

$$\dot{B} = \frac{\mathrm{d}V}{\mathrm{d}t} = \int \dot{b}\,\mathrm{d}A \tag{3}$$

where the integral is taken over the surface of the glacier. Strictly speaking, the integral should be taken down to the elevation of the equilibrium line if it lies below the terminus. Otherwise glaciers would never nucleate from zero size because there would be no glacier surface over which to integrate. Notice that it is the integral of $\dot{b}(A)$ which governs $\mathrm{d}V/\mathrm{d}t$ in Eqn (3). This is a result of having specified an A–V relationship, and as consequence the approximation tends to be insensitive to the spatial pattern of $\dot{b}$. The same cannot be said for the unrestricted approximation, because it does not force the same A–V constraint.

In what follows we focus on the restricted approximation, preferring to sacrifice the detail of the fast component of response for the resulting simplicity and the perspective which it offers on the nature of glacier response.

2.2. Separation of climatic and geometric effects on balance rate

A fundamental property of the response is that the glacier-wide mass-balance rate depends not only upon climate, but upon the instantaneous geometry of a glacier as well. It is instructive to consider this issue from the point of view of the restricted macroscopic approximation. In the spirit of that approximation we can write the glacier-wide balance rate $\dot{B}$ as a function of some climate parameter (or parameters) C, together with A and V:

$$\dot{B} = \dot{B}(C, A, V) \tag{4}$$

which implies that

$$\mathrm{d}\dot{B} = \frac{\partial \dot{B}}{\partial C}\mathrm{d}C + \frac{\partial \dot{B}}{\partial A}\mathrm{d}A + \frac{\partial \dot{B}}{\partial V}\mathrm{d}V \tag{5}$$

Following Lüthi (2009) $\partial\dot{B}/\partial A$ can be identified with the balance rate at the terminus $\dot{b}_t$, and $\partial\dot{B}/\partial V$ with the balance rate elevation gradient, which we take to be constant (Appendix A). Also, A and V are not independent but are related by a functional relationship in our restricted macroscopic approximation. Therefore it is unambiguous to define a 'thickness parameter' H as

$$\frac{\mathrm{d}A}{\mathrm{d}V} \equiv \frac{1}{H} \tag{6}$$

which may vary as the glacier evolves. This relationship would be more complicated in the unrestricted approximation. At any rate, Eqn (5) in the restricted approximation becomes

$$\mathrm{d}\dot{B} = \frac{\partial \dot{B}}{\partial C}\mathrm{d}C + \left(\frac{\dot{b}_t}{H} + \dot{g}\right)\mathrm{d}V \tag{7}$$

Equation (7) gives a convenient separation of the effects on balance rate of climate (first term on the right-hand side) and of surface geometry (second term), although the separation is not really complete because of the occurrence of $\dot{b}_t$ and $\dot{g}$ in the latter. The effect can be further decomposed into the effects of the terminus position (the $\dot{b}_t/H$ term) and that of surface elevation (the $\dot{g}$ term). Since $\dot{b}_t$ is usually negative, at least as long as the equilibrium line is above the terminus, and the $\dot{g}$ term is usually positive, their sum can have either sign depending on which term dominates. In the block model considered below, the separation of terminus position and surface elevation effects is less obvious because it is concentrated at the terminus, but the same formalism still holds if thickness change is defined as volume change divided by area.

2.3. Timescale and stability

Following Harrison and others (2001) and Lüthi (2009), it is useful to define a volume timescale τ_V by

$$\tau_V \equiv \frac{1}{\frac{-\dot{b}_t}{H} - \dot{g}} \tag{8}$$

$\dot{b}_t$ is usually not constant but changes with climate and the elevation of the terminus. This timescale is an extension of that of Jóhannesson and others (1989), useful when the balance rate can be expressed as a function of elevation. Timescales have also received a great deal of attention from Nye (1960), Weertman (1964), Bahr and others (1998), Pfeffer and others (1998) and Oerlemans (2001), for example. Essentially the same timescale occurs in the unrestricted macroscopic approximation, and is implicit in full numerical theories.

If the terminus position term $-\dot{b}_t/H$ dominates in Eqns (7) and (8), τ_V will be positive; if the surface elevation term $\dot{g}$ dominates, τ_V will be negative; if they are equal it will be infinite. It is easiest to see the net effect of these terms when complications due to climate change are negligible, in other words, in the constant-climate scenario. Then $\mathrm{d}C = 0$ in Eqn (7) and

$$\mathrm{d}\dot{B} \equiv d\left(\frac{\mathrm{d}V}{\mathrm{d}t}\right) = -\frac{1}{\tau_V}\mathrm{d}V \qquad \text{(constant climate)} \tag{9}$$

in which Eqn (8) has been used. If at some instant $\tau_V > 0$ and a glacier is advancing so that $\mathrm{d}V > 0$, $\mathrm{d}\dot{B}$ will be negative. This means that the balance rate $\dot{B}$ is decreasing so the

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advance occurs at a decreasing rate. Similarly, if $\tau_V > 0$ but the glacier is retreating, it is doing so at a decreasing rate. Thus for positive τ_V we call the instantaneous response 'stable', and for negative τ_V, 'unstable'. Stability could also be discussed in terms of the sign of the second derivative of volume, which is negative for stability (Eqn (9)). The terminology applies only at a particular instant in time, and could be more complicated in the face of strong rates of climate change.

The timescale τ_V has another and simpler application when the changes of interest are so small that τ_V is approximately constant. Then, expanding $\dot{B}$ in powers of ΔV one gets

$$\dot{B} \approx \dot{B}' - \frac{1}{\tau_V}\Delta V \qquad \text{(small changes)} \qquad (10)$$

where $\dot{B}'$ is the 'reference surface' balance rate, the balance rate as it would be if the geometry of the surface did not change from its original value (Elsberg and others, 2001). Equation (10) can also be obtained more formally starting from Eqn (7). Using the fact that $\mathrm{d}V/\mathrm{d}t = \mathrm{d}\Delta V/\mathrm{d}t = \dot{B}$, one gets

$$\frac{\mathrm{d}\Delta V}{\mathrm{d}t} + \frac{\Delta V}{\tau_V} \approx \dot{B}' \qquad \text{(small changes)} \qquad (11)$$

In this situation τ_V has the conventional meaning of a time constant. If in addition the climate and therefore $\dot{B}'$ are constant, and initially the glacier is close to steady state with climate, the solution is

$$\Delta V \approx \tau_V \dot{B}'\left(1 - \mathrm{e}^{-\frac{t}{\tau_V}}\right) \qquad \text{(small changes near steady state and climate constant)} \qquad (12)$$

Harrison and others (2001) noted that τ_V occurs not only as a conventional time constant but also as part of the amplitude of the response.

2.4. Characteristic elevations of the equilibrium line

An example of negative τ_V is of special interest. Consider a special elevation of an equilibrium line low on a glacier such that $\tau_V = \infty$. This is possible because the magnitude of $-\dot{b}_t$ in Eqn (8) would be relatively small because the equilibrium line would be relatively close to the terminus. Let us call this the 'transition' elevation of the equilibrium line. Now imagine that the equilibrium line at some time is actually lower, and that its elevation does not change with time; τ_V would be negative and the response unstable by our definition. As time goes on, the glacier would advance, a larger ablation area would develop, $-\dot{b}_t$ would become larger as the terminus becomes lower, and finally τ_V would become positive, so the response stabilizes.

At the other extreme, there is a 'critical' equilibrium-line elevation, this one high on the glacier, above which the glacier would ultimately vanish (Lüthi, 2009). We call the transition and critical elevations the 'characteristic' elevations of the equilibrium line. Their values depend on the A–V relationship (the 'model'), and are of most practical interest for glaciers with small slope, as illustrated below.

2.5. Nonlinearity and other general properties of the response

Another property of the response for large changes in volume is its nonlinearity. It is worth noting that, with today's rapidly changing glaciers, a linearized response theory valid for small changes may have limited usefulness. The nonlinearity considered in simple models occurs because the balance rate, even if it varies only linearly with elevation, is a linear or more complicated function of horizontal position once it is evaluated on the sloping surface of the glacier. Thus its integral over the surface to give the glacier-wide value $\dot{B}$ produces a higher-order result (Oerlemans, 2003). Of course there are other sources of nonlinearity, most obviously those associated with the basal boundary condition (e.g. Jouvet and others, 2011) and the nonlinear rheology. In the latter case, the approximation of near-plasticity can be taken as motivation for the constant thickness in the block model discussed below.

Despite its limitations there are some interesting implications from this, the simplest type of nonlinearity. For example, one can see in Eqn (7) that nonlinearity occurs if $-\dot{b}_t$ depends upon V. In fact it does even if the climate is constant, because the glacier terminus changes elevation. Neglecting this effect (by holding $-\dot{b}_t$ constant in a process of linearization) results in an overestimate of the amount of advance and an underestimate of the amount of retreat, as illustrated below.

It is interesting that the nonlinearity and other general properties of response discussed above are independent of the form of the A–V relationship, or 'model', and at least in part, of the macroscopic approach itself. However, it has been assumed implicitly in the above discussion that the balance rate gradient $\dot{g}$ is constant; some implications are given in Appendix A. There are other general properties of the response, such as slope dependence and sensitivity to climate change, but they are discussed below in the context of examples of the A–V models.

3. THE BLOCK ON A UNIFORM SLOPE WITH CONSTANT CLIMATE

So far the approach has been general for any restricted macroscopic system. We now consider a particular model, a block on a uniform slope. It is the simplest model which can be used to illustrate the most basic properties of glacier response, and is therefore a good starting point despite limitations due to its unrealistically steep face. These will be considered further below.

3.1. Governing differential equation for the block model

The bed topography is a uniformly inclined plane with small slope β terminated by a vertical headwall. The width W is constant. The A–V relationship is a functional one, so we are assuming that the restricted macroscopic approximation is valid. The relationship is

$$V = HA \qquad (13)$$

where H is a constant. Since according to the above discussion the thickness is $\mathrm{d}V/\mathrm{d}A$, we see that this is a block of constant thickness H, as shown in Figure 2. It would be the special case of $\mu = 1$ in Eqn (1).

The balance rate $\dot{b}$ is assumed to be zero on the vertical frontal cliff but is otherwise a linear function of elevation z:

$$\dot{b} = \dot{g}(z - z_{\mathrm{ela}}) \qquad (14)$$

where z_{ela} is the elevation of the equilibrium line above point O in Figure 2 and $\dot{g}$ is a constant balance rate gradient (Appendix A).

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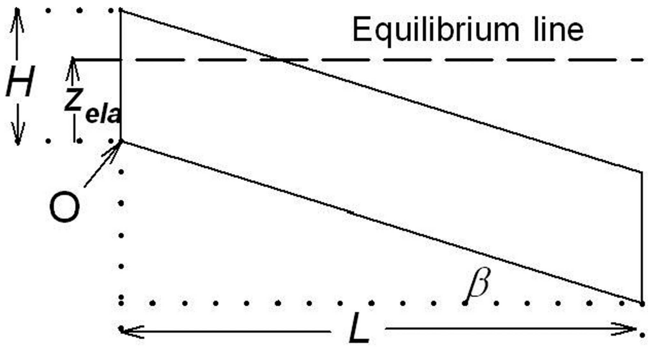

Fig. 2. Longitudinal section of block with length L and thickness H showing the equilibrium-line elevation z_{ela} with respect to the highest bed point O. β is the slope of the bed.

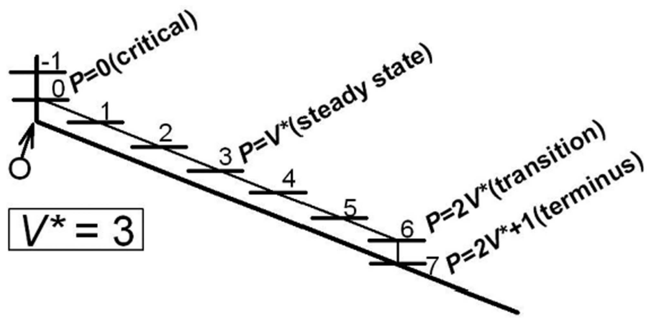

Fig. 3. Short lines representing intersection of the equilibrium line with the block surface, labeled by the corresponding value of the equilibrium-line parameter P (Eqn (19)).

On the surface of a block with low slope

$$z = H - \beta x \qquad \text{(on surface)} \tag{15}$$

where x is the horizontal coordinate with zero at point O in Figure 2. Substitution into Eqn (14) and integration with respect to x (Eqn (3)) give the glacier-wide balance rate $\dot{B}$ (the terms have been reordered):

$$\dot{B} = \frac{dV}{dt} = \dot{g}\left((H - z_{ela})WL - \beta W \frac{L^2}{2}\right) \tag{16}$$

where L is the length of the glacier. It is simplest to eliminate L, using $V = HA = HLW$, the A–V relationship for the block. The result is

$$\frac{dV}{dt} = \dot{g}\left((H - z_{ela})\frac{V}{H} - \frac{\beta}{2WH^2}V^2\right) \tag{17}$$

which is the governing differential equation describing the block response in terms of V. As anticipated above, it is nonlinear.

3.2. Non-dimensional forms for the block model

The differential equation can be put in the simplified non-dimensional form

$$\frac{dV^*}{dt^*} = V^*(P - V^*) \tag{18}$$

by the following definitions:

$$P \equiv 1 - \frac{z_{ela}}{H} \tag{19}$$

and the units

$$t_b \equiv \frac{1}{\dot{g}} \tag{20}$$

$$V_b \equiv \frac{2WH^2}{\beta} \tag{21}$$

where the subscript b stands for block. The length unit is

$$L_b \equiv \frac{2H}{\beta} \tag{22}$$

which follows from the V_b definition and $V = HWL$ for the block. A useful result is easily confirmed:

$$L^* = V^* \tag{23}$$

which means that V^* could be replaced by L^* in Eqn (18); this is because the block is so simple. Dimensional and non-dimensional units will both be used in what follows. Equation (18) is the logistic equation, which describes supply-limited growth.

It is straightforward to show that V_b is the volume and L_b the length which are ultimately reached when the equilibrium line is fixed and passes through point O in Figures 2 and 3, in other words, when $z_{ela} = 0$ or equivalently, $P = 1$. Then the equilibrium line bisects the upper surface of the block; this is because $\dot{g}$ is constant. A block with this size and slope could be considered 'short' in the sense that the cliff face is still a prominent aspect of its morphology. In contrast, if $V \gg V_b$ (or $L \gg L_b$) the cliff face will not be so prominent and we may hope that the block and models that produce a more realistic terminus region will give roughly the same predictions. This conjecture is supported by a more realistic model described below. In summary, cases for which

$$V^*(\text{or } L^*) \gg 1 \tag{24}$$

('long' glaciers) are likely the most reliable when making semi-quantitative inferences about real glaciers using the block model.

We see from the simplification resulting in Eqn (18) that it is often more convenient to specify the elevation of the equilibrium line not by z_{ela} but by the 'equilibrium-line parameter' P defined by Eqn (19). P is the elevation of the equilibrium line below the highest point of the block, measured in ice thickness units H. $P = 0$ when the equilibrium line intersects the highest point of the block, and unlike z_{ela} it increases when the equilibrium line is lowered. For example, in a climate favorable to the glacier, the equilibrium line would be low and P would be a relatively large positive number. P is negative when the equilibrium line intersects the headwall above the highest ice. It is straightforward to work out values of P for characteristic elevations of the equilibrium line, as shown for the example $L^* = V^* = 3$ in Figure 3.

Finally, several timescales will be defined in what follows, and like t_b they are proportional to the factor $1/\dot{g}$.

3.3 Solution and properties of the block model for constant climate

3.3.1. Solution

We have defined the block model and now need to specify the time dependence of the climate in terms of the elevation of the equilibrium line z_{ela} or equivalently, the equilibrium-line parameter P. Our first and simplest scenario is that of constant climate, so z_{ela} (and P) are constant after time $t = 0$. The initial volume V_0 is not necessarily in 'steady state' at $t = 0$, by which we mean 'in adjustment with climate' then.

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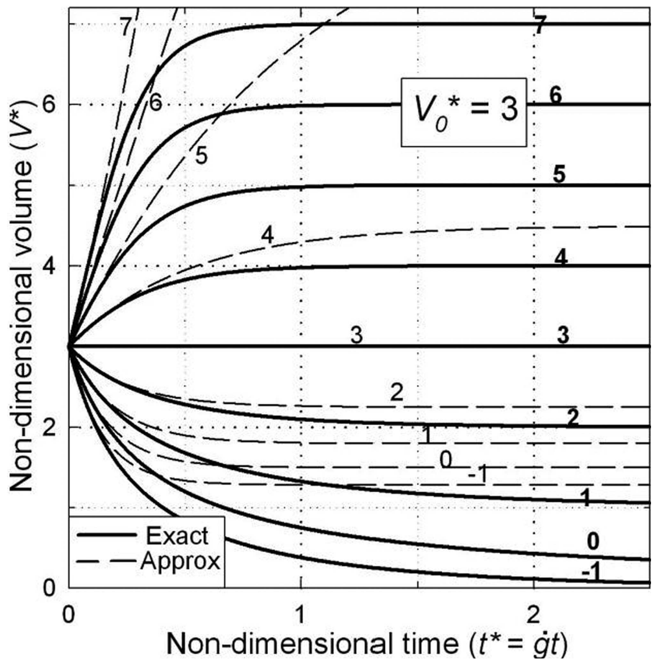

Fig. 4. Dependence of non-dimensional volume of the block upon non-dimensional time $\dot{g}t$ for constant equilibrium-line elevations z_{ela} and therefore constant equilibrium-line parameter P. Each curve is labeled by its value of P. The initial non-dimensional volume V_0^* is 3. The solid curves are from the exact theory; the broken curves from a version of the theory linearized about the initial volume.

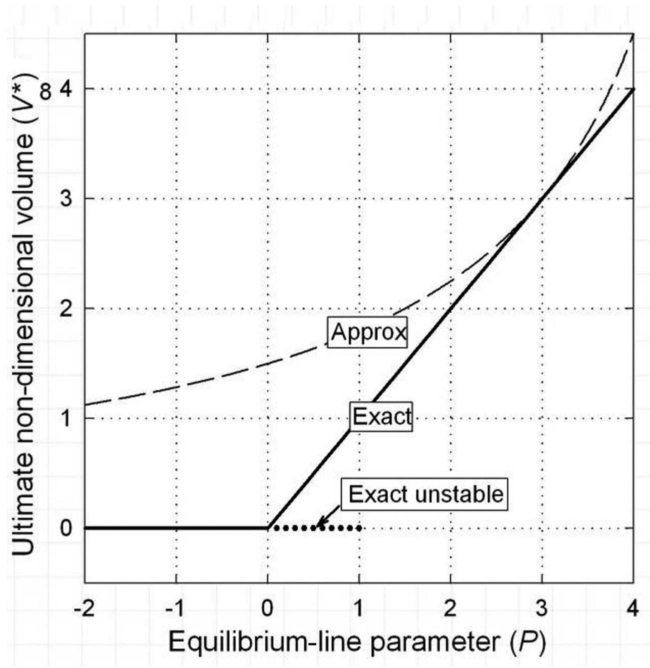

Fig. 5. Ultimate non-dimensional volume of the block V_∞^* resulting when the equilibrium-line elevation z_{ela}, and therefore P, are constant. The solid and dotted lines are from the exact theory; there are two possible values in the range $0 \le P \le 1$, depending upon whether or not the original volume is zero, but the dotted one is unstable. The broken curve is from a version of Eqn (18) linearized about $V_0^* = 3$.

It is worth noting that this does not necessarily imply a step change in climate at $t = 0$; in fact the climate could be the same before and after $t = 0$, but V_0, a relic of climate conditions prior to $t = 0$, would not necessarily be in adjustment with this climate. In either case the resulting response is given by Eqn (18) with P constant. It has an exact solution:

$$V^* = \frac{P}{1 + \left(\frac{P}{V_0^*} - 1\right) e^{-Pt^*}} \tag{25}$$

The existence of an exact solution simplifies our quest to illustrate basic properties of response. An immediate example is the behavior of V^* implied by Eqn (25) for large time. One can show from it (or directly from Eqn (18)) that

$$V^* - V_0^* \approx 1 - \frac{P}{V_0^*} e^{-Pt^*} \qquad (\text{large } Pt^*) \tag{26}$$

which never approaches the common form $1 - e^{-Pt^*}$ unless $P \approx V_0^*$, the special case in which the volume is always near steady state (Eqn (12)).

Note that the response depends on P (constant in this climate scenario) and of course the initial condition V_0. We recall that there is another climate parameter, the balance rate gradient $\dot{g}$, but we take it to be constant in space (Appendix A) and also in time, so we can absorb it into the definition of the non-dimensional time t^* via Eqn (20). The solid curves in Figure 4 show the response predicted for an initial non-dimensional volume $V_0^* = 3$ as a function of $t^* \equiv gt$. This value is representative of several glaciers with which we are familiar. The individual curves are labeled by the equilibrium-line parameter P. The maximum P shown is $2V^* + 1$ which ensures that the equilibrium line lies on or above the block (Fig. 3). One sees that the block ultimately vanishes for $P \le 0$, which is when the equilibrium line lies at or above the highest point of the block. The critical elevation for the block is thus at $P = 0$. Finally, we do not expect numerically realistic results for small V^* because of Eqn (24).

The broken curves in Figure 4 demonstrate the role of nonlinear effects in the response. They result from a version of Eqn (18) that is linearized about the initial volume V_0^* (Appendix B), a choice made to approximate the response when $V^* \approx V_0^*$. One sees, as anticipated above, that it overestimates the amount of advance and underestimates the amount of retreat.

3.3.2. Steady state

In a constant-climate scenario, steady-state volume and length are ultimately reached after a sufficiently long time. We call these $V_\infty^* = L_\infty^*$. They are found by setting the derivative equal to zero in Eqn (18):

$$V_\infty^* = P \qquad P \ge 0 \tag{27a}$$

$$V_\infty^* = 0 \qquad P \le 1 \tag{27b}$$

Figure 5 shows these two values as a function of P. There are two because of the nonlinearity of the theory, but both occur only if $0 \le P \le 1$ (Fig. 2). The lower or $V_0^* = 0$ branch of the V_∞^* solution requires that the initial volume V_0^* be exactly zero. It terminates at $P = 1$, because for $P > 1$ (the equilibrium line below the base of the headwall) a glacier would nucleate. The discussion of the lower branch is rather academic, because it is unstable. Any small deviation from it, even a single snowflake in the strict block model, will cause growth to the upper branch. Formally this is because

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$dV/dt > 0$ between the two branches. The broken curve in Figure 5 shows the prediction of a version of the theory linearized about the value $P = 3$. It would be accurate only near steady state, in which $P \approx V_0{}^* = 3$. We see again that it overestimates the amount of advance and underestimates the amount of retreat. It is also seen from the figure that the critical elevation, the lowest equilibrium line (or maximum P) at which the block glacier ultimately vanishes, is $P = 0$.

Although the time-dependent theory is nonlinear, Eqn (27a) is linear. An interesting consequence is that the magnitude of the ultimate response is symmetric in advance and retreat. This means that if the block is initially in steady state and experiences a climate change δP, the amount of advance will be the same as the amount of retreat for $-\delta P$. However, using the block amounts to using a restricted approximation as defined above, in which the thickness H would be the same in advance and retreat. The limitations of the block or of any other restricted approximation, such as Eqn (1), should be kept in mind as noted above.

3.3.3. Timescale and implications

The timescale τ_V for the block response can be found from its definition in Eqn (8); one must first work out the elevation of the terminus as a function of V in order to calculate $\dot{b}_t$. The result is

$$\dot{g}\tau_V = \frac{1}{2V^* - P} \tag{28}$$

We discussed above how in general the characteristics of the response are determined by the sign of τ_V, which is easily illustrated with this example, taking the climate to be constant. For $P > 2V^*$ (low equilibrium line), $\tau_V < 0$, so the response is unstable until V grows to reverse this inequality (at $V^* = P/2$) and stabilize the response. At this point the equilibrium line would be at what we called the transition elevation, which is at the top of the cliff face of the block (Fig. 3). For all greater elevations of the equilibrium line $\tau_V > 0$ so the response would be stable. The critical elevation, above which the glacier eventually vanishes, corresponds to the minimum P at which $V_\infty^* = 0$. This is at $P = 0$ (Eqn (27b)), which is when the equilibrium line is at the highest point of the ice. However, this is a property of the block and not a general conclusion. Note that the transition elevation depends on V^*, and therefore on the instantaneous size of the glacier, but the critical elevation does not. As noted above, stability could also be discussed in terms of the sign of the second derivative of volume. For the block $d^2V^*/dt^{*2} = -1/\dot{g}\tau_V$ by Eqns (28) and (18).

Because of its dependence on V^*, τ_V is usually time-dependent, even when climate is constant. Near $V^* = 0$ Eqn (28) gives

$$\dot{g}\tau_V \approx -\frac{1}{P} \qquad (\text{near } V^* = 0) \tag{29}$$

Near steady state, for which $P \approx V_0{}^* \approx V^*$, we have

$$\dot{g}\tau_V \approx \frac{1}{V_0{}^*} \approx \frac{1}{P} \qquad (\text{near steady state}) \tag{30}$$

3.3.4. Effective timescale in the constant-climate scenario

From Eqn (26) it follows that τ_V is a timescale characterizing the entire response by a simple exponential behavior $1 - e^{-Pt^*}$ only when changes are small. It is instructive to define a version of the timescale which is positive and constant for any magnitude of change in the steady-state climate scenario, using the block model. Our purpose is to define an effective value which can be used to investigate the response speed as a function of P and $V_0{}^*$ in the constant-climate scenario.

To do so we introduce the volume change ΔV, the difference between the volume at time t and its initial value:

$$\Delta V^* = V^* - V_0{}^* \tag{31}$$

An effective timescale τ_E can be defined as the time at which ΔV^* reaches a value that can be set by the choice of a parameter λ:

$$\Delta V^* = \lambda \Delta V_\infty{}^* \tag{32}$$

where ΔV_∞^* is the ultimate change in volume, $V_\infty{}^* - V_0{}^*$. For example, if $\lambda = 1/2$, the timescale would be defined as that required for the volume change ΔV^* to reach half of its ultimate value. The choice of λ is rather arbitrary, but to make the new timescale reduce to τ_V when the glacier is close to steady state, we choose $\lambda = 1 - e^{-1} = 0.632$. This is the e-folding time of Leysinger Vieli and Gudmundsson (2004), except that we define it to be the same in advance and retreat, and even when the glacier ultimately vanishes.

We first focus on the situation $P \geq 0$. Then $V_\infty{}^* = P$ (Eqn (27a)) and

$$\Delta V_\infty{}^* = P - V_0{}^*, \qquad P \geq 0 \tag{33}$$

Combining Eqns (31) and (32) gives

$$V^* - V_0{}^* = \lambda(P - V_0{}^*), \qquad P \geq 0 \tag{34}$$

We can substitute for V^* from Eqn (25) and solve for t^*, which we define to be the non-dimensional effective timescale $\dot{g}\tau_E$ for this case, finally getting

$$\dot{g}\tau_E = \frac{1}{P} \ln\left\{1 + \frac{\lambda}{1-\lambda}\left(\frac{P}{V_0{}^*}\right)\right\}, \qquad P \geq 0 \tag{35}$$

Notice that this applies to either advance or retreat as long as the ultimate volume is non-negative, which is why it requires $P \geq 0$. If the glacier is always near steady state, $P/V_0{}^* \approx 1$, $\tau_E \approx \tau_V$ and Eqn (35) reduces to Eqn (30), as it should. The expression corresponding to Eqn (35) for the situation $P \leq 0$, in which $V_\infty^* = 0$, is found to be

$$\dot{g}\tau_E = \frac{1}{P} \ln\left\{1 - \frac{\lambda \frac{P}{V_0{}^*}}{\frac{P}{V_0{}^*} - (1-\lambda)}\right\}, \qquad P \leq 0 \tag{36}$$

The non-dimensional effective timescale $\dot{g}\tau_E$ is shown as a function of P and $V_0{}^*$ in Figure 6. One sees that, for a given P, τ_E decreases with increasing $V_0{}^*$ so $\partial\tau_E/\partial V_0{}^* < 0$. This means that for a given P the response is faster for larger initial non-dimensional volume $V_0{}^*$ (e.g. Bahr and others, 1998; Lüthi, 2009). One also sees that τ_E decreases with increasing $|P|$ (lower equilibrium line if $P > 0$; higher for $P < 0$). In other words $\partial\tau_E/\partial|P| < 0$ so the response is faster for larger $|P|$, given an initial volume $V_0{}^*$. The results depend on $V_0{}^*$ and P, both of which contain the slope, as we shall see below. Thus comparisons apply to a block on a given slope or blocks on equal slopes. Finally, the rates of advance and retreat starting from a given P could be compared, but like the amplitude of the ultimate response discussed above (after Fig. 5) the result is sensitive to the use of the block model and to other factors. A more rigorous model solved numerically indicates that, for large enough glaciers, retreat is faster than advance (Leysinger Vieli and Gudmundsson, 2004).

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3.3.5. Dependence of block response upon slope

The use of non-dimensional variables greatly simplifies our analysis, but it hides the dependence of the response upon bed slope β. In addition, H itself must be a function of slope, which is determined once the rheology of ice is specified. We take advantage of the fact that

$$\sigma \approx \rho g_{\text{grav}} H \beta \tag{37}$$

where σ is the basal shear stress, roughly 1 bar ($\approx 10^5$ Pa) for many mountain glaciers, and ρg_{grav} is the weight density of ice. Then the 'rheological parameter' $\widetilde{H}$ defined by

$$\widetilde{H} \equiv \frac{\sigma}{\rho g_{\text{grav}}} \tag{38}$$

is a constant, ~10 m. The slope dependence of H becomes

$$H \equiv \frac{\widetilde{H}}{\beta} \approx \frac{10}{\beta} \text{ (in meters)} \tag{39}$$

so the volume and length scales V_b and L_b (Eqns (21) and (22)), together with the equilibrium-line parameter P (Eqn (19)), can be rewritten as

$$V_b = \frac{2W\widetilde{H}^2}{\beta^3} \tag{40}$$

$$L_b = \frac{2\widetilde{H}}{\beta^2} \tag{41}$$

$$P = 1 - \beta\frac{z_{\text{ela}}}{\widetilde{H}} \tag{42}$$

These relations indicate that the response is extremely sensitive to the slope, especially when it is small. We recall from Eqn (24) that the most realistic performance of the block will be when $L \gg L_b$. From Eqn (41) $L_b = 65$ km for $\beta = 1°$ and 2.6 km for 5°.

When $P > 0$, the volume and length of the block eventually approach

$$V_\infty = \frac{2W\widetilde{H}^2}{\beta^3}\left(1 - \frac{\beta z_{\text{ela}}}{\widetilde{H}}\right) \tag{43}$$

$$L_\infty = \frac{2\widetilde{H}}{\beta^2}\left(1 - \frac{\beta z_{\text{ela}}}{\widetilde{H}}\right) \tag{44}$$

as calculated from Eqn (27a) with the help of the scales expressed in terms of $\widetilde{H}$. It is useful to define climate sensitivities S_V and S_L as the negative derivatives of these two quantities with respect to z_{ela}. A more physical meaning of these sensitivities is given below but for the moment they say how the ultimate size of a block glacier would vary if one constant-climate regime (defined by a constant equilibrium-line elevation z_{ela}) were replaced by another:

$$S_V = \frac{2W\widetilde{H}}{\beta^2} \tag{45}$$

$$S_L = \frac{2}{\beta} \tag{46}$$

It follows that these two relations hold for finite changes because of the linear dependence of Eqns (43) and (44) on z_{ela}, which is in turn due to the linear dependence of $V_\infty{}^* = L_\infty{}^*$ on P discussed above. In retrospect, the high sensitivity for a low-slope glacier implied by these relations makes sense because it is obvious that a small rise in equilibrium line might completely remove the accumulation area. $S_{\bar{h}}$, a normalized version of S_V defined by $S_{\bar{h}} \equiv S_V/A$, has the advantage of non-dimensionality and is interpreted as the average thickness change per change in elevation of the equilibrium line. It is

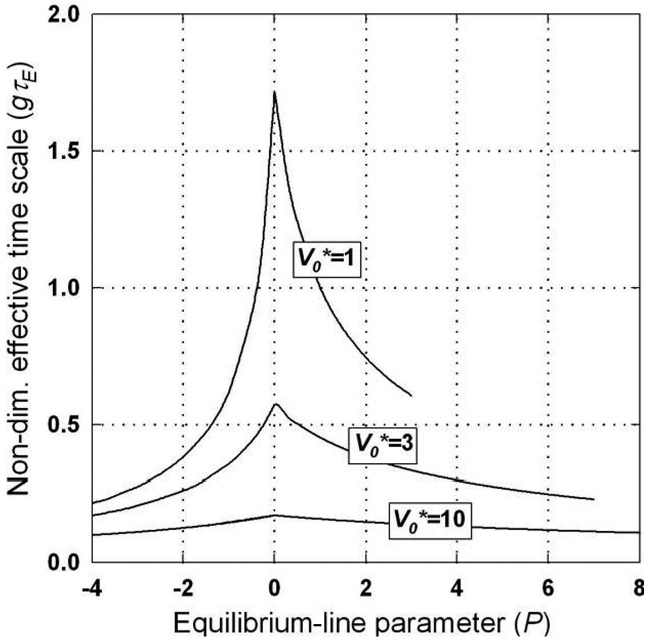

Fig. 6. Dependence of the non-dimensional effective timescale $\dot{g}\tau_E$ for the block upon constant equilibrium-line parameter P and initial non-dimensional volume $V_0{}^*$.

$$S_{\bar{h}} = \frac{2\widetilde{H}}{\beta^2 L} \tag{47}$$

When $S_{\bar{h}} = 1$, a 1 m rise in equilibrium line would ultimately result in the same average thickness decrease. Temperature (T) sensitivities can be obtained by multiplication with $\mathrm{d}z_{\text{ela}}/\mathrm{d}T$, which can be expressed in terms of a characteristic lapse rate.

Although we have considered only the case of constant slope β, nevertheless we can see that because of the inverse dependence upon β^2 and β, these equations imply a huge change in sensitivity when a glacier advances down a steep mountainside onto a relatively flat plane or valley. A mountain glacier may achieve steady state while just reaching the valley bottom, but a small additional lowering of the equilibrium line will lead to an increase in H and major growth, to the extent that the original mountain glacier becomes relatively unimportant except for its role in triggering the formation of a larger ice mass. This is an old idea (e.g. Oerlemans 2002), which suggests a useful generalization: when the slope varies, its value in the lower glacier is the most important.

Figure 6 showed the dependence of the effective timescale τ_E for the block upon P and $V_0{}^*$. We still need to take explicit account of the implied slope β and size dependence in these quantities, and it would be most informative to have a simple, even if approximate, expression. To achieve simplicity we consider the situation in which the block is near steady state. Then $\dot{g}\tau_V \approx 1/V_0{}^*$ (Eqn (30)). Use of Eqn (21) and $V = LWH$ then gives

$$\dot{g}\tau_E \approx \dot{g}\tau_V \approx \frac{2\widetilde{H}}{\beta^2 L} \quad \text{(near steady state)} \tag{48}$$

The quantity $\beta^2 L$ determines the response time, at least

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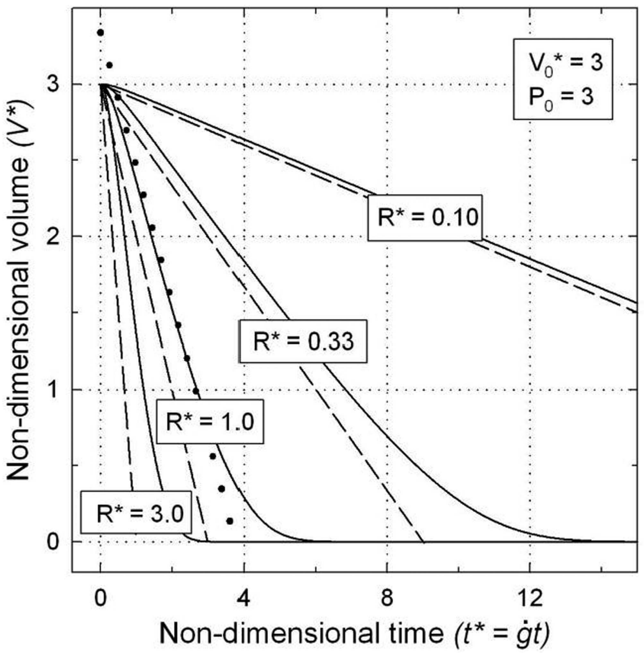

Fig. 7. Evolution of non-dimensional volume V^* of the block as a function of non-dimensional time $\dot{g}t$ when the elevation of the equilibrium line varies linearly with time; the initial volume and equilibrium-line parameter (V_0, P_0) are the same (3,3), which means that the glacier is initially in adjustment with climate. The solid curves show the exact response for four different non-dimensional rates R^* of climate change. The broken lines show the limit of 'fast' response. The dotted line is an analytical approximation to the response for $R^* = 1$.

approximately. Thus although the larger of two glaciers on equal slopes will have the smaller response time, it does not follow in general that large glaciers respond faster than small ones, because of the sensitivity to slope. Because large glaciers tend to have small slopes, they tend to respond relatively slowly (Leysinger Vieli and Gudmundsson, 2004; Lüthi, 2009).

The horizontal bed limit $\beta = 0$ is instructive. Then the nonlinear term in Eqn (18) vanishes, as it must, because we saw that slope was the source of the nonlinearity. The solution is

$$V = V_0 e^{Pt^*} \qquad (\beta = 0) \tag{49}$$

If $P > 0$ the block grows indefinitely; if $P < 0$ it ultimately vanishes. If $P = 0$ it does not change from its initial value (which from Eqn (19) requires $z_{\text{ela}} = H$), but this is unstable against small changes in P. In other words, for $\beta = 0$ the block is unstable against small changes in the equilibrium-line elevation. This is the simplest example of the 'instability' of an ice sheet on a horizontal bed (Böðvarsson, 1955; Weertman, 1961), although strictly speaking a block can never represent a real ice mass on a horizontal bed, even of ice-sheet scale, because Eqn (24) cannot be satisfied for finite volume.

Finally, it is worth pointing out that glaciers are characterized not only by slope but by elevation; z_{ela} and P as used in our simple models refer to the elevation of the equilibrium line above point O in Figure 3. To compare the behavior of glaciers when z_{ela} is referred to some constant datum such as sea level, one must take into account the elevations of point O, which usually differ from glacier to glacier.

4. THE BLOCK ON A UNIFORM SLOPE WITH VARYING CLIMATE

4.1. Fast response

So far we have used the block model for the constant-climate scenario. Next, we keep the block model but consider a scenario in which the climate, as represented by the parameter P, changes linearly with time. The motivation is to consider a potentially useful approximation in the case of slow climate change, in which the glacier might be treated as if it were always in steady state, that is, in perfect adjustment with climate. Then its response would be 'fast'. The block model can give an idea of the resulting error. The block approximation for fast response would be

$$V^* \approx P(t) \tag{50}$$

according to Eqn (18) with $dV^*/dt^* = 0$. We compare this limit of fast response with the exact response as calculated numerically from Eqn (18) when P has the simple time dependence

$$P = P_0 - R^* t^* \tag{51}$$

where P_0 is the initial value of P and R^* is a constant non-dimensional rate of change defined by $R^* \equiv dP/dt^*$. The equilibrium line would be rising for positive R^*. The fast response for this case is then given by

$$V^*\,(\text{fast}) = P_0 - R^* t^* \tag{52}$$

This is shown in Figure 7 by the broken lines. The solid curves are the exact responses calculated numerically from Eqn (18) with P given by Eqn (51). The results are shown for several values of R^* and for an initial $V_0{}^* = P_0 = 3$, which means that the block is in steady state initially. Because of Eqn (24), we do not expect the curves to be realistic for small V^*.

An interesting feature of fast response is its connection to the three sensitivity factors S_V, S_L and $S_{\bar{h}}$ defined above. These factors gave the ultimate difference of volume, length or thickness in terms of the difference in the elevation of the equilibrium line between two constant-climate regimes. Now imagine one of these regimes changing into another so slowly that the block remains in steady state with the climate. This would require fast response. Then the volume and equilibrium-line elevation in Eqn (43), for example, would become well-defined functions of time. Differentiation with respect to time yields the same sensitivity factor as before, but now it relates the rate of change of volume to the rate of change of the equilibrium-line elevation. Although it is exact only in the limit of fast response, it has a clearer physical meaning than our original definition.

4.2. The approximation of fast response

How good is the approximation of fast response? We saw this in Figure 7 for some non-dimensional examples, but it would be useful to have a criterion, even if approximate, to guide the general case. To this end Eqns (18) and (52) can be used to formulate a differential equation for the difference

$$v^* = V^* - V^*(\text{fast}) \tag{53}$$

from which one can show that

$$v^* \approx \frac{R^*}{P} = \frac{R^*}{P_0 - R^* t^*}, \quad \frac{R^*}{P^2} \ll 1 \tag{54}$$

but only if any initial transient, such as that shown in Figure 7, has decayed. In that example $P_0 = L_0{}^* = V_0{}^* = 3$, which means that initially the block was in steady state

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(Eqn (27a)). Even so, there is a small transient associated with switching on the climate change at $t^* = 0$; one can show that it decays on the timescale $1/P_0$. The transient can be bypassed by setting the initial condition $v^* = R^*/P_0$. Then for small time,

$$v^* \approx \frac{R^*}{P_0}\left(1 + \frac{R^*}{P_0}t^*\right), \quad \frac{R}{P^2} \ll 1 \tag{55}$$

The V^* resulting from Eqn (55) predicts a linear response, but with a different intercept from the fast response case, and a different slope. This is shown by the dotted line in Figure 7 for the example $(P_0, R^*) = (3.0, 1.0)$, which gives an adequate fit. The deviations at small time are due to the transient, and at large time are not significant because the block model fails (Eqn (24)).

It is advantageous to express the intercept and slope in terms of their ratios to the corresponding fast response quantities. The result is simple:

$$\frac{V_0{}^*}{V_0{}^*(\text{fast})} \approx 1 + \frac{R^*}{P_0{}^2} \tag{56}$$

$$\frac{\mathrm{d}V_0{}^*/\mathrm{d}t^*}{\mathrm{d}V_0{}^*(\text{fast})/\mathrm{d}t^*} \approx 1 - \frac{R^*}{P_0{}^2} \tag{57}$$

In other words, the approximation of fast response introduces two errors: an offset in the intercept (the initial volume), and an error in the rate of change (after the transient). These ratios are the same with the dimensions inserted on the left-hand sides, but we need to express R^* in terms of something more familiar, the rate of change of the height of the equilibrium line $r \equiv \mathrm{d}z_{\text{ela}}/\mathrm{d}t$. From the definitions of P and R^* (Eqns (42) and (51)) this gives

$$\frac{R^*}{P_0{}^2} = \frac{r\beta}{P_0{}^2\widetilde{H}\dot{g}} \tag{58}$$

This single quantity describes both aspects of the error in the fast approximation. Although this form is convenient for our calculations in which P_0 is specified, to see the full and complex dependence upon slope β, one must substitute for $P_0(\beta)$ using Eqn (42). The result is

$$\frac{R^*}{P_0{}^2} = \frac{r\beta}{\left(1 - \frac{z_{\text{ela}}(0)}{\widetilde{H}}\beta\right)^2 \widetilde{H}\dot{g}} \tag{59}$$

To put numbers into Eqn (58), assume a global warming rate of 0.01 K a^{-1} and a lapse rate of 0.065 K m^{-1}; this implies $r \approx 1.5$ m. Take $\dot{g} = 0.01$ a^{-1}, $\widetilde{H} = 10$ m and $\beta = 4^\circ = 0.07$ rad. Then $R^* \approx 1.0$, which is why this value was singled out for attention in Figure 7. For our $P_0 = 3$ example, $R^*/P_0{}^2 \approx 0.12$. By Eqns (56) and (57) this means that in this example the fast approximation gives an effective initial volume (the intercept) too small by 12%, and an adjustment rate $\mathrm{d}V^*/\mathrm{d}t$ too large by the same amount. This agrees with the exact calculation in Figure 3, which justifies our identifying $r\beta/P_0{}^2\widetilde{H}\dot{g}$ as the single quantity that characterizes, at least approximately, the wide range of behavior seen in Figure 7. However, this quantity depends upon many parameters. Also, one sees in the figure that, as time goes on, the difference between the exact and fast responses may become large. Thus we feel that the usefulness of the fast approximation depends upon the application and should be decided on a case-by-case basis.

5. THE PARABOLIC GLACIER ON A UNIFORM SLOPE

5.1. The model

A model with a curved surface avoids many of the difficulties associated with the cliff face of the block. One possibility would be to use the model described by Eqn (1). However, because it requires the indefinite increase in thickness with length, it is perhaps more realistic to require a constant basal shear stress approximated by

$$\sigma \approx \rho g_{\text{grav}} H\alpha \tag{60}$$

This is like Eqn (37), except that the ice surface slope α is no longer equal to the bed slope β (which we keep constant), and the ice thickness H is no longer constant. For small slopes α can be related to β and H

$$\sigma \approx \rho g_{\text{grav}} H\left(\beta + \frac{\mathrm{d}H}{\mathrm{d}x}\right) \tag{61}$$

For a given σ, Eqn (61) defines the configuration of the surface, which we call 'parabolic', although it is truly parabolic only in the limit of vanishing β. When Eqn (24) is satisfied, the predictions of block and parabola coincide. The A–V relationship is a functional one as in the block model, and only the constant-climate scenario has been considered. Even then the details are tedious. Here we simply describe what we judge to be one of the most interesting differences from the block model, a non-zero minimum sustainable size (Lüthi, 2009), which is determined by the critical elevation of the equilibrium line.

5.2. Critical elevation and length

Not surprisingly, the behavior of V_∞ is more complicated than shown in Figure 5 for the block. In the parabolic model there may not be any non-zero steady state V_∞ for constant climate when the equilibrium line is high on the glacier. In other words, the critical elevation is lower than the highest ice, unlike the situation with the block. Recalling that z_{ela} is measured up from point O in Figure 2, one finds that for the block glacier the critical elevation is at the highest point of the ice $z_{\text{ela}}^{(\text{crit})} = H$ or

$$z_{\text{ela}}^{(\text{crit})} = \frac{\widetilde{H}}{\beta} \qquad (\text{block}) \tag{62}$$

by Eqn (39), while for the parabolic glacier it is found to be

$$z_{\text{ela}}^{(\text{crit})} = 0.273\frac{\widetilde{H}}{\beta} \qquad (\text{parabola}) \tag{63}$$

This is about 60 m for a slope of 5°, and 310 m for 1°, about a quarter of that for the block on the same slope. In other words, the critical elevation for the parabolic model is considerably lower than for the block.

To the critical elevation of the equilibrium line in Eqn (63) corresponds the steady-state length of the parabolic glacier $L_\infty^{(\text{crit})}$:

$$L_\infty^{(\text{crit})} = 0.346\frac{\widetilde{H}}{\beta^2} \qquad (\text{parabola}) \tag{64}$$

It is about 350 m for a bed slope of 5°, and 11 km for a bed slope of 1°, compared with zero for the block. This defines a highly slope-dependent minimum sustainable length under constant-climate conditions. The corresponding accumulation–area ratio at steady state is roughly 0.55 when the balance gradient is constant. Equation (64) gives the non-zero size which a parabolic glacier will reach when the equilibrium line stabilizes just below the critical elevation. If

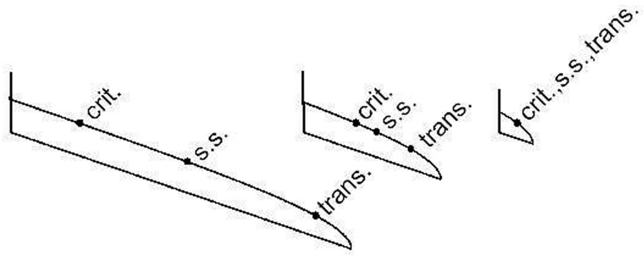

Fig. 8. Profiles of a glacier with constant basal stress at three different stages, together with the intersections of its surface with the two characteristic elevations of the equilibrium line (critical (crit.) and transition (trans.)), and of the equilibrium line at steady state (s.s.). On the right the glacier has come to a steady state with the equilibrium line just below the critical elevation.

the elevation is slightly higher, the parabolic glacier may still have an accumulation area for a time, and perhaps even a nearly healthy accumulation–area ratio, but its extinction is assured even if the climate does not change. In common with the block (Fig. 6), the parabolic glacier responds slowly when the equilibrium line is near the critical elevation.

The parabolic glacier and the characteristic elevations of its equilibrium line are shown in Figure 8. At left and center one sees that the transition elevation, as well as the steady-state elevation of the equilibrium line, depend on the size of the glacier, while the critical elevation does not. At the right the glacier has come to a steady state with the equilibrium line fixed just below the critical elevation. It has the minimum sustainable size. The transition elevation coincides with these two.

5.3. Sensitivity factors

This behavior implies a highly variable and therefore nonlinear behavior of the sensitivity factors in the vicinity of the critical elevation, which are defined (as they were for the block) as the negative derivatives of V_∞ and L_∞ with respect to z_{ela} when the climate is constant or changes very slowly. Figure 9 compares length sensitivities of block and parabola. They coincide for glaciers that are long in the sense that Eqn (24) is satisfied. This is about 2.6 km for a bed slope of 5° and 65 km for a bed slope of 1°. Perhaps there are real glaciers with respectable accumulation–area ratios whose equilibrium lines are above the critical level, but these may not include the few whose mass balances are monitored, because these tend to be relatively steep. The prediction is sensitive to bed slope and to the model used. At any rate, as a glacier retreats, its sensitivity to climate increases, at least if there is no significant increase in bed slope or the relation between balance rate and elevation. A more typical version of the balance rate gradient than our constant $\dot{g}$ version lowers the critical elevation (Appendix A).

6. SUMMARY

The non-dimensional quantity $2\widetilde{H}/\beta^2 L$, or its reciprocal, occurs so often in the block model that it is worth assigning a symbol for it:

$$K \equiv \frac{\beta^2 L}{2\widetilde{H}} \tag{65}$$

K has already occurred under a different name: $K = L^* = V^*$ by Eqns (41) and (23). There are several other

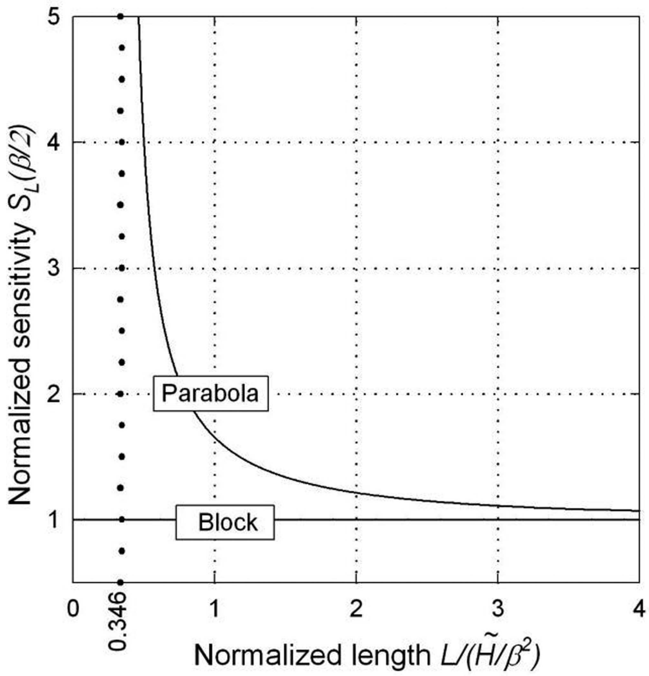

Fig. 9. Length sensitivity S_L (Eqn (46)) versus length L, normalized as labeled, for both the block and the parabolic model glaciers. β is the bed slope and $\widetilde{H}$ is ~10 m.

examples. One is the non-dimensional timescale near steady state, $\dot{g}\tau_E \approx \dot{g}\tau_V = 1/K$ (Eqn (48)); this connection is reminiscent of the observation that τ_V tends to determine the amplitude as well as the timescale of the response (Harrison and others, 2001). Another example is the thickness sensitivity $S_{\overline{h}} = 1/K$ (Eqn (47)). K also enters the definition of a 'long' glacier, for which Eqn (24) can be rewritten as $K \gg 1$. K is somewhat model-dependent, but its behavior can be worked out for the more general parabolic model. It is also best defined near steady state, which some of the connections require. Nevertheless, its common occurrence suggests that it has potential as a scaling parameter. Its use would be to relate the responses of different glaciers with different size and slope. Recalling that $\widetilde{H}$ is a rheological parameter (≈10 m by Eqn (39)), $2\widetilde{H}$ in Eqn (65) is approximately constant, so the geometric dependence of K is determined by $\beta^2 L$ alone in the block model. Thus this simple quantity can be taken as the scaling candidate. In the block model the bed slope β is the same as the surface slope α. This not true in general, but for scaling purposes one can try to approximate β by the more easily observable α.

We have now completed the program outlined in our abstract, which was to provide insight into characteristic aspects of glacier response. Most of the results are simple although not immediately obvious. We first considered the macroscopic approximation, and the general conclusions which can be drawn from it without specifying a particular model. Next we illustrated and extended these ideas with the help of the simplest model, a block on a uniformly inclined plane. We tried to keep its limitations in mind, considering for example a more general model with a more realistic terminus. It is our impression that, of all the response properties described, the sensitivity to bed slope (or more rigorously to $1/\beta^2 L$) is the most important, especially since it does not appear, at least explicitly, in some treatments of

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response. The simplicity of our approach is advantageous, but the question is whether it affects what we have said about the nature of response. The answer is necessarily somewhat subjective, but we believe that the conclusions are substantially correct. They are in essential agreement with those of Lüthi (2009), although his approach was more quantitative and his emphasis and coverage were somewhat different. We are less confident about the usefulness of $\alpha^2 L$ (or Lüthi's version of it) as a scaling parameter. The most obvious reason is that we have considered only simple glaciers. Finally, it should be remembered that the price for simplicity, which resulted from our use of the restricted macroscopic approximation, is failure to predict the response accurately on short timescales.

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APPENDIX A. VARIABILITY OF THE BALANCE RATE GRADIENT

The simplicity of assuming a constant balance rate gradient $\dot{g}$ is evident, but at least in the block model it is straightforward to allow for different $\dot{g}$'s in ablation and accumulation areas ($\dot{g}_{\text{abl}}$ and $\dot{g}_{\text{acc}}$). It can be shown that to the right-hand side of Eqn (18) one adds the term $\frac{P^2}{4}\left(\frac{\dot{g}_{\text{acc}}}{\dot{g}_{\text{abl}}} - 1\right)$ and sets $t^* \equiv \dot{g}_{\text{abl}} t$. This new governing equation, like the old, can be solved in convenient closed form, and all the studies for the constant $\dot{g}$ case repeated. An interesting point is that the new term does not contribute any new nonlinearity to the governing equation. We summarize some of the results for the special case $\dot{g}_{\text{abl}} = 2\dot{g}_{\text{acc}}$ in which the above quantity to be added for the block becomes $-P^2/8$. The patterns in Figures 4–6 are shifted, although there are few fundamental changes. First, in the constant-climate scenario one finds that, instead of Eqn (27a), $V_\infty{}^* = 0.854P$. But this is still linear, which means that the symmetry in the ultimate amplitude in advance and retreat, as discussed above with certain reservations, is preserved, at least for the block. Second, the asymptotic approach to steady state seen in Figure 4 is lost, although the significance of this behavior as predicted by the block model is doubtful in any case because of Eqn (24). Third, the critical elevation of the equilibrium line is lowered (as it is also in the case of the parabolic model); it becomes lower than the highest point of the ice. Fourth, the structure of the curves of Figure 6 is unchanged.

Finally, it is worth pointing out that in large ice masses $\dot{g}$ is a function not only of elevation but also of horizontal position.

APPENDIX B. LINEAR APPROXIMATIONS

The role of the nonlinearity in the response to climate can be judged by comparison with a linearized version of the theory. This is obtained by expressing the volume as $V^* = V_r{}^* + \Delta V_r{}^*$ where $V_r{}^*$ is some constant reference volume, near which the approximation should be valid.

There are an infinite number of linear approximations because the choice of V_r^* is arbitrary. Substitution into Eqn (18) for the block gives

$$\frac{d\Delta V_r^*}{dt^*} + (2V_r^* - P)\Delta V_r^* + \Delta V_r^{*2} = PV_r^* - V_r^{*2} \qquad \text{(B1)}$$

which is still exact. Linearity is obtained by neglecting the ΔV_r^{*2} term.

The effect of neglecting the ΔV_r^{*2} term was illustrated in Figure 4, where $V_r^* = V_0^*$. It can be seen from that figure, and also from Eqn (B1), that the solution with ΔV_r^{*2} neglected resembles a growing exponential for $P > 2V_0^*$, and thus the linear approximation fails completely. $P = 2V_0^*$ happens to be the value of P corresponding to the transition elevation of the equilibrium line at the initial volume. Rather similar conditions apply for $V_r^* = P$, a choice one would make to optimize the approximation near (Eqn (27a)). In this case the approximation grows exponentially for $P < 0$, which is when the equilibrium line is above the critical elevation.

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