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LETTERS TO THE EDITOR

A VARIANT OF THE EHRENFEST MODEL

K. W. H. VAN BEEK* AND
A. J. STAM,* *Rijksuniversiteit Groningen*

Abstract

The Ehrenfest model is modified by drawing r balls at a time. The stationary distribution is the same as for $r = 1$.

STATIONARY DISTRIBUTION; MARKOV CHAIN

A set of N balls, labeled $1, \cdots, N$, is divided over two urns, I and II. A random sample of size $r < N$ is drawn without replacement from $\{1, \cdots, N\}$ and after that every ball whose label is in the sample is removed from its urn and put into the other urn. Then this procedure is repeated. If $r = 1$ we have the classical Ehrenfest model for diffusion of a gas between two connected containers, see [1]–[5]. As state space we may take the set $E = \{0, 1\}^N$ of all sequences $x = (x_1, \cdots, x_N)$ where $x_i = 1$ means that ball i is in urn I and $x_i = 0$ means that it is in II. The process then is Markovian. We write $p(x, y)$, $x \in E$, $y \in E$, for its transition probabilities. The lumped process whose state is the number of balls in I has state space $F = \{0, \cdots, N\}$ and is also Markovian.

If $r = 1$ the stationary distribution is uniform on E and for the lumped process it is the binomial $(N, \frac{1}{2})$ distribution on F that corresponds to the uniform distribution on E. We shall prove that similar results hold for $r > 1$.

A one-step transition $x \to y$ may be described as follows. A random sample A of size r is taken from $\{1, \cdots, N\}$ and every x_i with $i \in A$ is changed, or replaced by $y_i = x_i + 1$ [mod 2]. So

$$(1) \qquad \sum_{i=1}^{N} y_i = \sum_{i=1}^{N} x_i + r, \quad [\text{mod } 2].$$

Let r be odd. Then by (1) every state in E has even period. Since drawing the same sample twice in succession is possible, every state has period 2. Let $B = \{i_1, \cdots, i_{r+1}\}$. It is possible to draw the samples $B - \{i_k\}$, $k = 2, \cdots, r + 1$, in succession. Then x_{i_1} is changed r times and x_{i_h}, $h \geqq 2$, is changed $r - 1$ times, which results in changing x_{i_1} only. This shows that all states communicate. Let E_0 and E_1 be the subsets of E with $\sum x_i$ even and $\sum x_i$ odd, respectively. By (1) every one-step transition is either from E_0 to E_1 or from E_1 to E_0. So E_0 and E_1 are the cyclically moving classes.

Now let r be even. Then (1) implies that E_0 and E_1 are closed. When $x \in E_0$, $y \in E_0$ or $x \in E_1$, $y \in E_1$, the number of i with $x_i \neq y_i$ is even. So if $r = 2$ a suitable sequence of samples leads from x to y. But if $r > 2$ two successive samples of the form $H \cup \{i\}$ and $H \cup \{j\}$, $i \neq j$, $i \notin H$, $j \notin H$, have the same effect as the single sample (i, j) when $r = 2$.

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Postal address: Mathematisch Instituut, Rijksuniversiteit, Postbus 800, 9700 AV Groningen, The Netherlands.

So x and y communicate, i.e. E_0 and E_1 are irreducible. As for odd r we have $p^{(2)}(x, x) > 0$. Let B be as above. Since it is possible to draw the $r+1$ samples $B - \{i_k\}$, $k = 1, \cdots, r+1$, in succession, $p^{(r+1)}(x, x) > 0$. So the states are aperiodic.

For any $y \in E$ the set $E(y)$ of x with $p(x, y) > 0$ contains exactly $\binom{N}{r}$ elements, viz. those x with $x_i \neq y_i$ for r values of i, and $p(x, y) = \binom{N}{r}^{-1}$, $x \in E(y)$.

Let r be even. Then $E(y) \subset E_i$ if $y \in E_i$, $i = 1, 0$, and the stationary equations

$$\pi(y) = \sum_{x \in E(y)} \pi(x) p(x, y), \tag{2}$$

admit two distinct solutions: the uniform distributions $\pi(y) = 2^{1-N}$ on E_0 and on E_1.

Let r be odd. Then $E(y) \subset E_1$ if $y \in E_0$ and $E(y) \subset E_0$ if $y \in E_1$. So the right-hand side of (2) transforms the uniform distribution on E_0 into the uniform distribution on E_1 and vice versa. The (only) stationary distribution of the process is uniform on E.

For the lumped process the above results lead to the following conclusions. If r is even, $F_0 = \{k \in F : k \text{ even}\}$ and $F_1 = \{k \in F : k \text{ odd}\}$ are irreducible closed sets of aperiodic states and the stationary distributions on F_0 and F_1 have probability $\binom{N}{k} 2^{1-N}$ at k. If r is odd all states communicate and have period 2. The cyclically moving classes are F_0 and F_1. The stationary distribution is binomial $(N, \frac{1}{2})$.

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