ON THE RANGE OF FINITE EMBEDDINGS OF A FINITE AMALGAM

by ABDUL MAJEED

(Received 12 October, 1970)

Certain questions about the range of finite embeddings of a finite amalgam were discussed in [3]. Another pertinent question is the following.

If a finite reduced amalgam has both an infinite and a finite embedding, does it have a maximal finite embedding such that all other finite embeddings are its homomorphic images?

We give a counter example to answer this question in the negative. The finite amalgam considered will involve a group from the family of groups of the type $(l, m; n, k)$ discussed by Coxeter [2] having the following presentation:

$$(l, m; n, k) = \text{gp}\,\{g, h; g^l = h^m = (gh)^n = (g^{-1}h)^k = 1\}. \tag{1}$$

These groups may be regarded as factor groups of

$$\text{gp}\,\{g, h; g^l = h^m = (gh)^n = 1\}, \tag{2}$$

which is known to be finite if

$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n} > 1$$

and infinite otherwise.

When $l = m = n = 3$, then (2) becomes

$$G_3 = \text{gp}\,\{g, h; g^3 = h^3 = (gh)^3 = 1\} \tag{3}$$

and is therefore infinite. It is shown in [1] that every element of G_3 is expressible in the form

$$h^r(gh^{-1})^p(g^{-1}h)^q \qquad (p, q, r \text{ integers}).$$

Therefore the most general factor group of G_3 is given by

$$(gh^{-1})^p(g^{-1}h)^q = 1, \tag{4}$$

which takes the form $(gh^{-1}g^{-1}h)^q = 1$ when $p = q$. The relation $(gh^{-1}g^{-1}h)^q = 1$ implies $(g^{-1}h)^{3q} = 1$. The group

$$(3, 3; 3, k) = \text{gp}\,\{g, h; g^3 = h^3 = (gh)^3 = (g^{-1}h)^k = 1\} \tag{5}$$

is finite of order $3k^2$ for every $k \geqq 2$ and is a central quotient group of the group H_3 given by

$$H_3 = \text{gp}\,\{g, h; g^3 = h^3 = (gh)^3 = (gh^{-1}g^{-1}h)^k = 1\}.$$

The relation (4) shows that there is no minimal normal subgroup of G_3.

https://doi.org/10.1017/S0017089500001555 Published online by Cambridge University Press

Consider now the finite amalgam A formed by the groups

$$A = \mathrm{gp}\,\{a, b; a^2 = b^2 = (ab)^3 = 1\},$$
$$B = \mathrm{gp}\,\{b, c; b^2 = c^2 = (bc)^3 = 1\},$$
$$C = \mathrm{gp}\,\{c, a; c^2 = a^2 = (ca)^3 = 1\}.$$

The free embedding of A is

$$F = \mathrm{gp}\,\{a, b, c; a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = 1\}.$$

Take $bc = g$, $ca = h$; then $ba = gh$. Then F has an alternative presentation:

$$F = \mathrm{gp}\,\{g, h, c; g^3 = h^3 = (gh)^3 = (gc)^2 = (ch)^2 = c^2 = 1\}.$$

Since

$$c^{-1}gc = c\,.\,bc\,.\,c = cb = (bc)^{-1} = g^{-1}$$

and

$$c^{-1}hc = c\,.\,ca\,.\,c = ac(ca)^{-1} = h^{-1},$$

F is a split extension of G_3 by a cycle of order 2. Therefore F is an infinite embedding of A.

Also, for each integer k, the normal closure N_k of $(g^{-1}h)^k$ in G_3 is normal in F. The relation $(g^{-1}h)^k = 1$ implies $(cbca)^k = 1$. For $k > 2$, N_k is tidy with respect to A, B, C. F/N_k, therefore, embeds the amalgam A. But

$$F/N_k = \mathrm{gp}\,\{g, h, c; g^3 = h^3 = (gh)^3 = (g^{-1}h)^k = (gc)^2 = (ch)^2 = c^2 = 1\},$$

being an extension of $(3, 3; 3, k)$ of order $3k^2$ by a cycle of order 2, is finite. Thus A has a finite embedding, namely F/N_k. Since the element c normalises G_3, all the finite embeddings of A are determined by the normal subgroups of G_3. As shown above, since G_3 has no minimal normal subgroup, A has no maximal finite embedding having each of the finite embeddings F/N_k as its homomorphic images.

The author is grateful to the referee of [3] for suggesting this problem.

REFERENCES

1. W. Burnside, *Theory of groups of finite order* (Cambridge, 1911).
2. H. S. M. Coxeter, The abstract groups $G^{m,n,p}$, *Trans Amer. Math. Soc.* **45** (1939), 73–150.
3. A. Majeed, On embeddable finite amalgams of groups, *Glasgow Math. J.* **13** (1972), 135–141.

UNIVERSITY OF THE PUNJAB
LAHORE, PAKISTAN