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A theorem in Banach algebras and its applications

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If A is a complex Banach algebra which is also a Bezout domain, it is shown that for any prime p and a non-negative integer n, p^n is not a topological divisor of zero. Using the above result it is shown that a complex Banach algebra which is a principal ideal domain is isomorphic to the complex field.

1. Introduction

The object of this paper is to show that if A is a Banach algebra which is also a Bezout domain, then for any prime p, p^n is not a topological divisor of zero. As a corollary of the above theorem, it is shown that a complex Banach algebra which is a principal ideal domain is isomorphic to the field $\mathbb{C}$ of complex numbers. A few applications are given involving some concrete algebras.

Section 2 introduces a few definitions and theorems which are used in Section 3. The main results are proved in Section 3.

2.

An integral domain A is a commutative ring with an identity $1 \neq 0$, which has no divisors of zero. A Bezout domain A is an integral domain in which for every two elements a and b the greatest common divisor d exists with

(2.1) $d = ar + bs$ for some r and s in A.

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Every principal ideal domain is an example of a Bezout domain. There are examples of Bezout domains which are not principal ideal domains.

For the main definitions and results in Banach algebras we refer to Larsen [4], Naĭmark [5], and Rickart [6].

DEFINITION 2.1. Let A be a complex Banach algebra with an identity denoted by 1, where A is not necessarily commutative. An element $a \in A$ is called a left topological divisor of zero, if there exists a sequence $\{z_n\}$, $z_n \in A$, such that $\|z_n\| = 1$ for all n with $\|az_n\| \to 0$ as $n \to \infty$. A right topological divisor of zero is similarly defined. A topological divisor of zero is defined to be either a right or a left topological divisor of zero.

The next theorem stated in the context of integral domains is a modification of Theorem 1.6.2 given in Larsen [4] (p. 46).

THEOREM 2.1. *Let A be a complex Banach algebra with an identity 1. If A is an integral domain, the following conditions are equivalent:*

(i) $x \in A$ is not a topological divisor of zero;

(ii) the principal ideal (x) is closed in A;

(iii) there exists a constant $K > 0$, such that for all $y \in A$,

$$\|xy\| \geq K\|y\| . \tag{2.2}$$

THEOREM 2.2 (Rickart [6]). *Let A be a complex Banach algebra with an identity 1. If A has no topological divisors of zero other than the zero element, then A is isomorphic to the field $\mathbb{C}$ of complex numbers.*

The proof of the next theorem is obtained by using Theorem 2.1 and Theorem 2.2.

THEOREM 2.3. *Let A be a complex Banach algebra which is also an integral domain. If for each $a \in A$, $a \neq 0$, the ideal (a) is closed, then A is isomorphic to the complex field $\mathbb{C}$.*

3.

The results stated in the introduction are now proved in this section.

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THEOREM 3.1 (Fundamental Theorem). *Let A be a complex Banach algebra which is also a Bezout domain. Let p be a prime element of A. For any non-negative integer n, p^n is not a topological divisor of zero or, equivalently, the principal ideal (p^n) is closed.*

Proof. We consider three cases.

Case 1. For $n = 0$, the result is trivial since $p^0 = 1$.

Case 2. For $n = 1$, we consider the principal ideal (p). It is easily seen using the Bezout property, that in a Bezout domain the ideal (p) is maximal since p is a prime. Since a maximal ideal in a Banach algebra is closed it follows that (p) is closed and hence, equivalently, p is not a topological divisor of zero.

Case 3. For any positive integer n, consider p^n. As p is not a topological divisor of zero by Case 2, it follows from Theorem 2.1 that there exists a positive constant K (see (2.2)) such that

$$\|py\| \geq K\|y\| \text{ , for all } y \in A \text{ .}$$

Now for any arbitrary $y \in A$,

$$\|p^n y\| = \|p(p^{n-1}y)\| \geq K\|p^{n-1}y\| \geq K^2\|p^{n-2}y\| \geq \ldots \geq K^n\|y\| \text{ .} \tag{3.1}$$

Since $K^n > 0$, by Theorem 2.1 *(iii)* it follows that p^n is not a topological divisor of zero. The proof is now complete.

Let A be a unique factorization domain. For any non-zero non-unit element $a \in A$, let the prime factorization of a be

$$a = u.p_1^{n_1} \ldots p_m^{n_m} \text{ ,} \tag{3.2}$$

where u is a unit, $p_1, \ldots, p_m$ are distinct primes, and $n_1, \ldots, n_m$ are positive integers. The relation (3.2) is equivalent to the following:

$$(a) = \bigcap_{j=1}^{m} \left(p_j^{n_j}\right) \text{ .} \tag{3.3}$$

THEOREM 3.2 (A Gelfand-Mazur like theorem). *Let A be a complex Banach algebra, which is also a principal ideal domain. Then A is isomorphic to the field $\mathbb{C}$ of complex numbers.*

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Proof. In view of Theorem 2.3, it suffices to show that for any $a \in A$, $a \neq 0$, the ideal (a) is closed. If a is a unit, this is trivial. For any non-zero non-unit element a with factorization (3.2) we have

$$(a) = \bigcap_{j=1}^{m} \left(p_j^{n_j}\right) .$$

Since each p_j is a prime, by Theorem 3.1, $\left(p_j^{n_j}\right)$ is closed and consequently (a) is closed. This completes the proof of the theorem.

Not every unique factorization domain is a Bezout domain. In fact it can be easily shown that a unique factorization domain with the Bezout property is a principal ideal domain. The following conjecture still remains open.

CONJECTURE. A complex Banach algebra which is also a unique factorization domain is isomorphic to $\mathbb{C}$.

The following partial answer to the above conjecture is noteworthy.

THEOREM 3.3. *Let A be a complex Banach algebra which is also a unique factorization domain. If for every prime element $p \in A$, the ideal (p) is closed, then A is isomorphic to $\mathbb{C}$.*

Proof. By hypothesis, for each prime element $p \in A$, the ideal (p) is closed. By Theorem 2.1 it follows that p is not a topological divisor of zero. Using the Case 3 argument of Theorem 3.1, it follows that p^n is not a topological divisor of zero for any non-negative integer n or equivalently that $\left(p^n\right)$ is closed. Using the same kind of argument as in Theorem 3.2, it follows that for any non-zero non-unit $a \in A$, the ideal (a) is closed, the result being trivial in the case of a unit. The result now follows from Theorem 2.3.

APPLICATIONS. (1) Let $P(x)$ be the algebra of all complex polynomials in one variable. Since it is a principal ideal domain under the usual algebraic operations it follows that $P(x)$ can not be made into a Banach algebra under any norm. However a stronger result can be deduced concerning $P(x)$. In any Banach space the spectrum of a bounded linear operator mapping the space to itself is a compact subset of $\mathbb{C}$ and hence bounded. Each element $f \in P(x)$ can be regarded as a linear operator of $P(x)$ to itself by the following mapping:

$$T_f(g) = fg \text{ , for any arbitrary } g \in P(x) \text{ .}$$

If $P(x)$ were to be a Banach space, it can not have elements with unbounded spectrum. But $P(x)$ has many elements with unbounded spectrum. Hence $P(x)$ can not even be made into a Banach space.

(2) Let $\mathbb{C}(x)$ be the algebra of all formal power series over the field $\mathbb{C}$, taken under the usual algebraic operations. It is shown in Hungerford [2] that $\mathbb{C}(x)$ is a principal ideal domain which is also a local domain, in the sense that it has a unique maximal ideal. In view of Theorem 3.2 it follows that $\mathbb{C}(x)$ can not be made into a Banach algebra under any norm. It is also remarkable that every element of $\mathbb{C}(x)$ has a bounded spectrum, which can be easily established. This example shows that Theorem 3.2 can not be deduced from the spectral theorem, which says that the spectrum of any element in a Banach algebra is a compact subset of $\mathbb{C}$.

(3) Let Ω be the collection of all complex sequences of the form $a = \{a_n\}_{n=0}^{\infty} = (a_0, a_1, \ldots, a_n, \ldots)$. We define a multiplication on Ω by the rule:

for $a = \{a_n\}_{n=0}^{\infty}$ and $b = \{b_n\}_{n=0}^{\infty}$ the nth term $(a.b)_n$ of the product $a.b$ is defined by

$$(a.b)_n = \sum_{t=0}^{n-1} a_t(b_{n-t}-b_{n-t-1}) + a_n b_0 \quad \text{if } n \geq 1 \text{ ,}$$

and

$$(a.b)_0 = a_0 b_0 \quad \text{for } n = 0 \text{ .}$$

Ω taken under the above multiplication and usual addition and scalar multiplication is an algebra, which is an integral domain (see [1]). Using essentially the same kind of technique as in $\mathbb{C}(x)$ it can be shown that Ω is a principal ideal domain. It follows from Theorem 3.2 that Ω can not be made into a Banach algebra under any norm.

REMARK 3.1. Srinivasan [7] has given a different proof of Theorem 3.2.

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