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Dynamical Analysis of a Stage-Structured Model for Lyme Disease with Two Delays

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Abstract. In this paper, a nonlinear stage-structured model for Lyme disease is considered. The model is a system of differential equations with two time delays. We derive the basic reproductive rate $R_0(\tau_1, \tau_2)$. If $R_0(\tau_1, \tau_2) < 1$, then the boundary equilibrium is globally asymptotically stable. If $R_0(\tau_1, \tau_2) > 1$, then there exists a unique positive equilibrium whose local asymptotic stability and the existence of Hopf bifurcations are established by analyzing the distribution of the characteristic values. An explicit algorithm for determining the direction of Hopf bifurcations and the stability of the bifurcating periodic solutions is derived by using the normal form and the center manifold theory. Some numerical simulations are performed to confirm the correctness of theoretical analysis. Finally, some conclusions are given.

1 Introduction

In the natural world, many species have a life history that takes them through two stages: the juvenile stage and the adult stage. Individuals in each stage are identical in biological characteristics and some vital rates (rates of survival, development, and reproduction) of individuals in a population almost always depend on stage structure. Aiello and Freedman [1] suggested and analyzed a stage structure model with constant maturation time delay for single species, see [1,2,12,15]. The authors in [17] considered a model with maturation delay and completely studied the stability properties and bifurcation analysis.

Lyme disease is the world's most common bacterial tick-borne infection [11]. Lyme disease is caused by a spirochete (*Borrelia burgdorferi*) which is most commonly present in ticks [3]. Analyzing the spread of vector-borne disease can be relatively complex when the vector's acquisition of a pathogen and subsequent transmission to a host occur in different life stages. A contemporary example is Lyme disease. The processes underlying the spread of Lyme disease involve a spirochete, a tick (with larval, nymph and adult stages), and two (or more) vertebrate hosts. For more ecological background, see [3, 5, 6, 16].

The life cycle of ticks encompasses several stages over a period of two years. The eggs hatch in the summer and become larvae. If a larva is successful in feeding on a host, it molts into a nymph. The nymphs that survive the winter and that are able to find a blood meal during the spring, molt into adults. Adult female ticks deposit eggs.

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Ticks are assumed to hatch uninfected. The ticks in their larval and nymphal stages prefer to feed on mice whereas the adults prefer deer. Ticks that are unable to find a host die off [5, 6]. A long-lived tick vector acquires infection during the larval blood meal and transmits it as a nymph.

Most models for vector-borne diseases ignore vector population dynamics. Caraco et al. [5] model a vector-borne disease where the vector's stage structure effects the transmission dynamics. Zhao [19] studied the global dynamics of this spatial model for Lyme disease. Zhang and Zhao [18] proposed a reaction-diffusion model to study transmission dynamics of Lyme disease while taking into account seasonality. Next, we introduce in detail the model proposed by Caraco et al. [5] Let $M(t)$ and $m(t)$ be densities of susceptible and pathogen-infected mice, $V(t)$ be densities of larvae infesting susceptible mice, $N(t)$ and $n(t)$ be the densities of susceptible and infectious questing nymphs, $A(t)$ and $a(t)$ be the densities of uninfected and pathogen-infected adult ticks, respectively. Then the model proposed by Caraco et al. for Lyme disease is governed by the following system:

$$
(1.1)\qquad \begin{cases}
\dot{M}(t) = r_M(M(t)+m(t))\Big(1-\dfrac{(M(t)+m(t))}{K_M}\Big) - \mu_M M(t) \\
\qquad\qquad - \alpha_M\beta_M M(t)n(t), \\
\dot{m}(t) = \alpha_M\beta_M M(t)n(t) - \mu_M m(t), \\
\dot{V}(t) = (A(t)+a(t))(\alpha_H f H - c(A(t)+a(t))) - V(t)(\alpha_M(M(t) \\
\qquad\qquad + m(t)) + \mu_V + D), \\
\dot{N}(t) = V(t)(\alpha_M M(t) + D) + (1-\beta)\alpha_M m(t)V(t) - N(t)(\mu_N \\
\qquad\qquad + \alpha_M(M(t)+n(t)) + \gamma P), \\
\dot{n}(t) = \beta\alpha_M m(t)V(t) - n(t)(\mu_N + \alpha_M(M(t)+m(t)) + \gamma P), \\
\dot{A}(t) = \alpha_M(M(t) + (1-\beta)m(t))N(t) - \mu_A A(t), \\
\dot{a}(t) = \alpha_M((M(t)+m(t))n(t) + \beta N(t)m(t)) - \mu_A a(t).
\end{cases}
$$

All parameters are positive constants and have the following biological interpretations [5]: r_M is the intrinsic rate of mice, K_M is carrying capacity, μ_M is the mortality rate among mice, α_M is the rate at which juvenile ticks attack mice, β_M $(0<\beta_M<1)$ is a mouse's susceptibility to pathogen infection when bitten by an infectious nymph, α_H is the rate at which adult ticks attack deer, the total population sizes of deer is a constant denoted by H, c denotes the scales self-regulation in tick reproduction, f denotes the larvae hatching per adult tick-deer interaction in absence of tick self-regulation, D is the rate at which larval ticks attack hosts other than mice, μ_V is the mortality rate per larva, β $(0<\beta<1)$ is a tick's susceptibility to infection when feeding on an infected mouse, γP is the rate at which nymphs bite humans, μ_A is the mortality rate per adult tick.

As mentioned in [19], let $\bar{M} = M + m$, $\bar{V} = V$, $\bar{N} = N + n$, $\bar{A} = A + a$. In view of system (1.1), we obtain the following system:

$$\begin{cases} \dot{\bar{M}}(t) = r_M \bar{M}(t)\Big(1 - \dfrac{\bar{M}(t)}{K_M}\Big) - \mu_M \bar{M}(t), \\ \dot{\bar{V}}(t) = \bar{A}(t)(\alpha_H f H - c\bar{A}(t)) - \bar{V}(t)(\alpha_M \bar{M}(t) + \mu_V + D), \\ \dot{\bar{N}}(t) = \bar{V}(t)(\alpha_M \bar{M}(t) + D) - \bar{N}(t)(\mu_N + \alpha_M \bar{M}(t) + \gamma P), \\ \dot{\bar{A}}(t) = \alpha_M \bar{M}(t)\bar{N}(t) - \mu_A \bar{A}(t). \end{cases}$$

Sometimes mathematical models in epidemiology can be formulated as systems of autonomous differential equations that can be rewritten as smaller asymptotically autonomous systems with a limit system that is considerably easier to handle than the original one. By a standard result on logistic type equation, it is easily to show that the mice population are asymptotically constant Q, *i.e.*,

$$\lim_{t\to+\infty} \bar{M}(t) = K_M\Big(1 - \frac{\mu_M}{r_M}\Big) \triangleq Q.$$

This gives rise to the following limiting system [5], the population dynamics of a single species with three stages is modeled by the system

$$\begin{cases} \dot{\bar{V}}(t) = \bar{A}(t)(\alpha_H f H - c\bar{A}(t)) - \bar{V}(t)(\alpha_M Q + \mu_V + D), \\ \dot{\bar{N}}(t) = \bar{V}(t)(\alpha_M Q + D) - \bar{N}(t)(\mu_N + \alpha_M Q + \gamma P), \\ \dot{\bar{A}}(t) = \alpha_M Q \bar{N}(t) - \mu_A \bar{A}(t). \end{cases}$$

In population dynamics, the need to incorporate a time delay is often the result of the existence of some stage-structure. Thus, we propose the following model.

(1.2)
$$\begin{cases} \dot{\bar{V}}(t) = \bar{A}(t)(\alpha_H f H - c\bar{A}(t)) - e^{-\mu_V \tau_1}(\alpha_M Q + D)\bar{V}(t-\tau_1) - \mu_V \bar{V}(t), \\ \dot{\bar{N}}(t) = e^{-\mu_V \tau_1}(\alpha_M Q + D)\bar{V}(t-\tau_1) - e^{-\mu_N \tau_2}(\alpha_M Q + \gamma P)\bar{N}(t-\tau_2) - \mu_N \bar{N}(t), \\ \dot{\bar{A}}(t) = e^{-\mu_N \tau_2}\alpha_M Q \bar{N}(t-\tau_2) - \mu_A \bar{A}(t). \end{cases}$$

Here the population is divided into three stage structure, *i.e.*, larval, nymph and adult. The larva that was born at $t - \tau_1$ and still survives at time t, transforming from larva to nymph, is given by the term $e^{-\mu_V \tau_1}(\alpha_M Q + D)\bar{V}(t-\tau_1)$. The nymph which was born at $t - \tau_2$ and still survives at time t, transforming from nymph to adult is given by the term $e^{-\mu_N \tau_2}\alpha_M Q \bar{N}(t-\tau_2)$.

To reduce the number of system parameters, we non-dimensionalize system (1.2) with the following scaling:

$$x(t) = \frac{\bar{V}(t)}{c}, \quad y(t) = \frac{\bar{N}(t)}{c(\alpha_M Q + D)}, \quad z(t) = \bar{A}(t),$$

$$\bar{a} = \frac{\alpha_H f H}{c}, \quad \bar{D} = \alpha_M Q + D,$$

$$\bar{d} = \alpha_M Q + \gamma P, \quad \bar{b} = \alpha_M Q c(\alpha_M Q + D).$$

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Using the above variables and dropping bars from the resulting equation, we obtain

$$\begin{cases} \dot{x}(t) = (a - z(t))z(t) - e^{-\mu_V \tau_1} D x(t - \tau_1) - \mu_V x(t), \\ \dot{y}(t) = e^{-\mu_V \tau_1} x(t - \tau_1) - e^{-\mu_N \tau_2} d y(t - \tau_2) - \mu_N y(t), \\ \dot{z}(t) = e^{-\mu_N \tau_2} b y(t - \tau_2) - \mu_A z(t). \end{cases} \tag{1.3}$$

The dynamics of system (1.3) will be investigated in a suitable phase space and a bounded feasible region. We denote by C the Banach space of continuous functions $\phi: [-\sigma, 0] \to R^3$ with the sup-norm

$$\|\phi\| = \max\{ \sup|\phi_1(\theta)|, \sup|\phi_2(\theta)|, \sup|\phi_3(\theta)| \},$$

where $-\sigma \le \theta \le 0$, $\phi = (\phi_1, \phi_2, \phi_3)$, and $\sigma = \max\{\tau_1, \tau_2\}$. Further, let

$$C_+ = \{ \phi = (\phi_1, \phi_2, \phi_3) \in C, \phi_i \ge 0, \text{ for all } \theta \in [-\sigma, 0], i = 1, 2, 3 \}.$$

The initial conditions of system (1.3) are given as

$$x(\theta) = \phi_1(\theta), y(\theta) = \phi_2(\theta), z(\theta) = \phi_3(\theta), \theta \in [-\sigma, 0], \tag{1.4}$$

where $\phi = (\phi_1, \phi_2, \phi_3) \in C_+$.

By using the basic theory of delay differential equations (see, for example, [8]), it is not difficult to show that for the initial conditions given above, the solution

$$(x(t), y(t), z(t))$$

of the system (1.3) corresponding to initial conditions (1.4) exists, is unique, nonnegative, and bounded on $[0, +\infty)$.

The organization of this paper is organized as follows. In Section 2, sufficient conditions for the global stability of the trivial equilibrium are established. By analyzing the corresponding characteristic equations, we investigate the stability of the positive equilibrium and existence of Hopf bifurcation. In Section 3, the direction and stability of periodic solutions bifurcating from the Hopf bifurcation of (1.3) are determined by using the normal form theory and center manifold argument presented in Hassard et al. [9]. Some numerical simulations are carried out to illustrate the analysis results in Section 4. Some conclusions are given in Section 5.

2 Stability Analysis and Hopf Bifurcation

The dynamics of the ticks are described by the quantity

$$R_0(\tau_1, \tau_2) = \frac{ab}{\mu_A (D + \mu_V e^{\mu_V \tau_1})(d + \mu_N e^{\mu_N \tau_2})}.$$

The quantity $R_0(\tau_1, \tau_2)$ is called the basic reproduction number. If we initially consider one larva, then the basic reproductive number of the ticks is the average number of the next generation larvae that this produces. This has the rather obvious biological interpretation that if the product of the losses from each tick stage is greater than the product of the gains to each tick stage, then the ticks will die out; if not, they will persist.

There are two possible equilibria in this model. There is the boundary equilibrium $E_0 = (0,0,0)$ where no ticks are sustained, and if $R_0(\tau_1,\tau_2) > 1$, system (1.3) has a positive equilibrium $E^* = (x^*, y^*, z^*)$ where

$$x^* = \frac{\mu_A(d+\mu_N e^{\mu_N\tau_2})e^{\mu_V\tau_1}}{b}z^*,$$
$$y^* = \frac{\mu_A e^{\mu_N\tau_2}}{b}z^*,$$
$$z^* = a - \frac{\mu_A(D+\mu_V e^{\mu_V\tau_1})(d+\mu_N e^{\mu_N\tau_2})}{b}.$$

First, we discuss the stability of $E_0 = (0,0,0)$.

Theorem 2.1 *For any $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that $R_0(\tau_1,\tau_2) \leq 1$, E_0 is globally asymptotically stable.*

Proof We assume that $R_0(\tau_1,\tau_2) \leq 1$ and prove that E_0 is globally asymptotically stable. Using the following Lyapunov function, which has been widely used in the mathematical biology literature (see [7,10,13] and references therein),

$$W(x,y,z) = W_1(t) + W_2(t) + W_3(t)$$

with

$$W_1(t) = x(t) + (D+\mu_V e^{\mu_V\tau_1})y(t) + \frac{(D+\mu_V e^{\mu_V\tau_1})(d+\mu_N e^{\mu_N\tau_2})}{b}z(t),$$
$$W_2(t) = \mu_V\int_{t-\tau_1}^{t} x(s)\,ds,$$
$$W_3(t) = (D+\mu_V e^{\mu_V\tau_1})\mu_N\int_{t-\tau_2}^{t} y(s)\,ds.$$

Then $W(t) \geq 0$, $W(t) = 0$ if and only if $x(t) = 0, y(t) = 0, z(t) = 0$, and W is a positive definite infinity functional. The derivative of W along solutions of system (1.3) is

$$W'|_{(1.3)} = \Big(\frac{\mu_A(D+\mu_V e^{\mu_V\tau_1})(d+\mu_N e^{\mu_N\tau_2})}{b}(R_0(\tau_1,\tau_2)-1) - z(t)\Big)z(t) \leq -z(t)^2.$$

It can be seen that if $R_0(\tau_1,\tau_2) \leq 1$, then $\frac{dW(t)}{dt} \leq 0$ and $\frac{dW(t)}{dt} = 0$ if and only if $z(t) = 0$. It is easy to show that E_0 is the largest invariant set in $\big\{(x,y,z) \big| \frac{dW(t)}{dt} = 0\big\}$. Therefore, when $R_0(\tau_1,\tau_2) \leq 1$ the boundary equilibrium E_0 is globally asymptotically stable from the Lyapunov–Lasalle invariance principle. ■

Theorem 2.2 *For any $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that $R_0(\tau_1,\tau_2) > 1$, E_0 is unstable.*

Proof The characteristic equation of system (1.3) at E_0 is

$$\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + (q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau_1} + (p_2\lambda^2 + p_1\lambda + p_0)e^{-\lambda\tau_2} + (r_1\lambda + r_0)e^{-\lambda(\tau_1+\tau_2)} = 0. \tag{2.1}$$

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where

$$\begin{aligned}
&b_2 = \mu_V + \mu_N + \mu_A, \quad b_1 = \mu_A\mu_V + \mu_A\mu_N + \mu_V\mu_N, \quad b_0 = \mu_V\mu_N\mu_A,\\
&q_2 = De^{-\mu_V\tau_1}, \quad q_1 = (\mu_A + \mu_N)De^{-\mu_V\tau_1}, \quad q_0 = \mu_N\mu_A De^{-\mu_V\tau_1},\\
&p_2 = de^{-\mu_N\tau_2}, \quad p_1 = (\mu_A + \mu_V)de^{-\mu_N\tau_2}, \quad p_0 = \mu_V\mu_A de^{-\mu_N\tau_2},\\
&r_1 = Dde^{-(\mu_V\tau_1+\mu_N\tau_2)}, \quad r_0 = (Dd\mu_A - ab)e^{-(\mu_V\tau_1+\mu_N\tau_2)}.
\end{aligned}$$

For $\tau_1 = \tau_2 = 0$, we have that

$$b_2 + q_2 + p_2 > 0, \qquad b_0 + q_0 + p_0 + r_0 > 0,$$

and

$$(b_2 + q_2 + p_2)(b_1 + q_1 + p_1 + r_1) - (b_0 + q_0 + p_0 + r_0) > 0,$$

since $R_0(0,0) > 1$. This shows that the roots of (2.1) have negative real parts for $\tau_1 = \tau_2 = 0$. If $R_0(\tau_1,\tau_2) > 1$, let

$$g(\lambda) = \lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + (q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau_1} + (p_2\lambda^2 + p_1\lambda + p_0)e^{-\lambda\tau_2} + (r_1\lambda + r_0)e^{-\lambda(\tau_1+\tau_2)}.$$

Note that

$$g(0) = \mu_A(D + \mu_V e^{\mu_V\tau_1})(d + \mu_N e^{\mu_N\tau_2})\big(1 - R_0(\tau_1,\tau_2)\big)\, e^{-(\mu_V\tau_1+\mu_N\tau_2)} < 0$$

by $R_0(\tau_1,\tau_2) > 1$ and $\lim_{\lambda\to+\infty} g(\lambda) = +\infty$. It follows from the continuity of the function $g(\lambda)$ on $(-\infty,+\infty)$ that the equation $g(\lambda) = 0$ has at least one positive root. Hence, the characteristic equation (2.1) has at least one positive real root. Hence, E_0 is unstable if $R_0(\tau_1,\tau_2) > 1$. ∎

Next we discuss the stability of E^* and the existence of the local Hopf bifurcation.

We shall regard τ_1 and τ_2 as parameters to study the stability switches of the positive equilibrium E^* when $R_0(\tau_1,\tau_2) > 1$. The characteristic equation of system (1.3) at E^* is

$$\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + (q_2\lambda^2 + q_1\lambda + q_0)e^{-\lambda\tau_1} + (p_2\lambda^2 + p_1\lambda + p_0)e^{-\lambda\tau_2} + (r_1\lambda + r^*)e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{2.2}$$

where $r^* = \big(Dd\mu_A - b(a-2z^*)\big)\, e^{-(\mu_V\tau_1+\mu_N\tau_2)}$. Obviously, system (1.3) has two delays, τ_1 and τ_2 in coefficients. Then the analysis is very complicated if we choose the two delays as parameters. Hence, in this section we discuss that $\tau_1 = 0$ and $\tau_2 \ge 0$. Next, we discuss the case of $\tau_1 \ge 0$ and $\tau_2 = 0$.

For $\tau_1 = \tau_2 = 0$, the characteristic equation (2.2) can be rewritten as follows

$$\lambda^3 + (b_2 + q_2 + p_2)\lambda^2 + (b_1 + q_1 + p_1 + r_1)\lambda + b_0 + q_0 + p_0 + r^* = 0. \tag{2.3}$$

By the Routh-Hurwitz criterion we know that if

$$b_0 + q_0 + p_0 + r^* > 0$$

and

$$(b_2 + q_2 + p_2)(b_1 + q_1 + p_1 + r_1) - (b_0 + q_0 + p_0 + r^*) > 0$$

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hold, then all roots of (2.3) have negative real parts.

If $R_0(0,0) > 1$ holds, then $b_0 + q_0 + p_0 + r^* > 0$. If

$$R_0(0,0) < \overline{R}_0(0,0) = 2 + \frac{(b_2 + q_2 + p_2)(b_1 + q_1 + p_1 + r_1) - (b_0 + q_0 + p_0 + Dd\mu_A)}{\mu_A(D+\mu_V)(d+\mu_N)}$$

holds, then $(b_2+q_2+p_2)(b_1+q_1+p_1+r_1)-(b_0+q_0+p_0+r^*) > 0$. Consequently, when $1 < R_0(0,0) < \overline{R}_0(0,0)$, E^* is locally asymptotically stable in the case of $\tau_1 = \tau_2 = 0$.

Returning to (2.2), when $R_0(0,\tau_2) > 1$, there exists E^*, *i.e.*, when $\tau_2 \in [0, \bar{\tau}_2)$, there exists the positive equilibrium E^* where

$$\bar{\tau}_2 = \frac{1}{\mu_N} \ln\Big(\frac{ab}{\mu_N \mu_A (D+\mu_V)} - \frac{d}{\mu_N} \Big).$$

Equation (2.2) takes the general form

$$P(\lambda,\tau_2) + Q(\lambda,\tau_2)e^{-\lambda\tau_2} = 0 \tag{2.4}$$

with

$$\begin{aligned} P(\lambda,\tau_2) &= \lambda^3 + (b_2+q_2)\lambda^2 + (b_1+q_1)\lambda + b_0 + q_0, \\ Q(\lambda,\tau_2) &= p_2\lambda^2 + (p_1+r_1)\lambda + p_0 + r^*. \end{aligned} \tag{2.5}$$

In the following, we investigate the existence of purely imaginary roots $\lambda = i\omega(\omega > 0)$ to equation (2.4). Equation (2.4) takes the form of a third-degree exponential polynomial in λ with all the coefficients of P and Q depending on τ_2. Beretta and Kuang [4] established a geometrical criterion which gives the existence of purely imaginary roots of a characteristic equation with delay-dependant coefficients.

In order to apply the geometrical criterion due to Beretta and Kuang, we can easily verify the following conclusions for all $\tau_2 \in [0, \bar{\tau}_2)$,

(i) $P(0,\tau_2) + Q(0,\tau_2) \neq 0$,
(ii) $P(i\omega,\tau_2) + Q(i\omega,\tau_2) \neq 0$,
(iii) $\limsup\Big\{ \Big| \frac{Q(\lambda,\tau_2)}{P(\lambda,\tau_2)} \Big|; |\lambda| \to \infty, \mathrm{Re}\lambda \geq 0 \Big\} < 1$,
(iv) $F(\omega,\tau_2) := |P(i\omega,\tau_2)|^2 - \big|Q(i\omega,\tau_2)\big|^2$ has a finite number of zeros,
(v) Each positive root $\omega(\tau_2)$ of $F(\omega,\tau_2) = 0$ is continuous and differentiable in τ_2 whenever it exists.

Now let $\lambda = i\omega(\omega > 0)$ be a root of (2.4). Substituting it into (2.4) and separating the real and imaginary parts yields

$$\begin{aligned} \omega^3 - (b_1+q_1)\omega &= (p_1+r_1)\omega\cos\omega\tau_2 - (-p_2\omega^2 + p_0 + r^*)\sin\omega\tau_2, \\ (b_2+q_2)\omega^2 - (b_0+q_0) &= (-p_2\omega^2 + p_0 + r^*)\cos\omega\tau_2 + (p_1+r_1)\omega\sin\omega\tau_2. \end{aligned} \tag{2.6}$$

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From (2.6) it follows that

(2.7)

$$\sin\omega\tau_2 = \frac{\begin{array}{l}(p_1+r_1)((b_2+q_2)\omega^2-(b_0+q_0))\omega \\ \qquad -(-p_2\omega^2+p_0+r^*)(\omega^2-(b_1+q_1))\omega\end{array}}{(p_1+r_1)^2\omega^2+(-p_2\omega^2+p_0+r^*)^2},$$

$$\cos\omega\tau_2 = \frac{\begin{array}{l}((b_2+q_2)\omega^2-(b_0+q_0))(-p_2\omega^2+p_0+r^*) \\ \quad +(p_1+r_1)(\omega^2-(b_1+q_1))\omega^2-(-p_2\omega^2+p_0+r^*)(\omega^2-(b_1+q_1))\omega\end{array}}{(p_1+r_1)^2\omega^2+(-p_2\omega^2+p_0+r^*)^2},$$

where we deliberately omit the dependence of the parameter on τ_2.

By the definitions of $P(\lambda,\tau_2)$ and $Q(\lambda,\tau_2)$ as in (2.5), and applying property (i), (2.7) can be written as

$$\sin\omega\tau_2 = \mathrm{Im}\Big(\frac{P(i\omega,\tau_2)}{Q(i\omega,\tau_2)}\Big),\quad \cos\omega\tau_2 = -\mathrm{Re}\Big(\frac{P(i\omega,\tau_2)}{Q(i\omega,\tau_2)}\Big),$$

which yields $|P(i\omega,\tau_2)|^2 = |Q(i\omega,\tau_2)|^2$. That is,

$$F(\omega,\tau_2) = \omega^6 + a_2(\tau_2)\omega^4 + a_1(\tau_2)\omega^2 + a_0(\tau_2) = 0, \tag{2.8}$$

where

$$\begin{aligned}
a_2(\tau_2) &= (b_2+q_2)^2 - 2(b_1+q_1) - p_2^2,\\
a_1(\tau_2) &= -2(b_2+q_2)(b_0+q_0) + (b_1+q_1)^2 + 2p_2(p_0+r^*) - (p_1+r_1)^2,\\
a_0(\tau_2) &= (b_0+q_0)^2 - (p_0+r^*)^2.
\end{aligned}$$

Let $z=\omega^2$. Then

$$h(z) = z^3 + a_2(\tau_2)z^2 + a_1(\tau_2)z + a_0(\tau_2) = 0. \tag{2.9}$$

After some simplification, we have

$$\begin{aligned}
a_2(\tau_2) &= \mu_N^2+\mu_V^2+\mu_A^2+D^2+2\mu_V D-(de^{-\mu_N\tau_2})^2,\\
a_1(\tau_2) &= (\mu_N\mu_A)^2+(\mu_V\mu_A)^2+(\mu_N\mu_V)^2+(D\mu_A)^2+(D\mu_N)^2+2D\mu_V(\mu_A^2+\mu_N^2)\\
&\quad +2abde^{-2\mu_N\tau_2}-4\mu_A de^{-2\mu_N\tau_2}(D+\mu_V)(d+\mu_N e^{\mu_N\tau_2})\\
&\quad -d^2e^{-2\mu_N\tau_2}(\mu_V^2+\mu_A^2+D^2+2\mu_V D),\\
a_0(\tau_2) &= \big(abe^{-\mu_N\tau_2}-\mu_A(D+\mu_V)(de^{-\mu_N\tau_2}+\mu_N)\big)\\
&\quad \times\big(-abe^{-\mu_N\tau_2}+\mu_A(D+\mu_V)(de^{-\mu_N\tau_2}+3\mu_N)\big).
\end{aligned}$$

From [14], we have the following results.

(i) When $a_0(\tau_2)<0$, (2.9) has at least one positive root. From the expression of $a_0(\tau_2)$, it is easy to obtain that if $R_0(0,0) > R_0^* := 1+2\frac{\mu_N}{d+\mu_N}$, then there exists $\tau_\omega\in[0,\overline{\tau}_2)$ such that $a_0(\tau_2)<0$, $\tau_2\in[0,\tau_\omega)$, where $\tau_\omega = \bar{\tau}_2 - \frac{\ln 3}{\mu_N}$.

(ii) When $a_0(\tau_2)\ge 0$ and $\triangle = a_2^2(\tau_2)-3a_1(\tau_2)<0$, (2.9) has no positive real root. Therefore (2.4) has no purely imaginary root.

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(iii) When $a_0(\tau_2) \geq 0$, (2.9) has positive roots if and only if $z = (-a_2(\tau_2) + \sqrt{\Delta})/3 > 0$ and $h(z) \leqslant 0$. By the analysis of condition (i), we know that when $R_0(0,0) \leq R_0^*$, then $a_0(\tau_2) \geq 0$.

For case (i) and case (iii), for simplicity, we assume that (2.9) has a unique positive root denoted by X and (2.8) has a positive root given by $\omega = \sqrt{X}$.

Define

$$I = \Big\{ \tau_2 \in [0, \bar{\tau_2}) : R_0(0,0) > R_0^* \text{ or } 1 < R_0(0,0) \leq R_0^*, \\ z = (-a_2(\tau_2) + \sqrt{\Delta})/3 > 0 \text{ and } h(z) \leqslant 0 \Big\}.$$

For $\tau_2 \in I$, there exists an $\omega = \omega(\tau_2) > 0$ such that $F(\omega(\tau_2), \tau_2) = 0$. Then let $\theta(\tau_2) \in [0, 2\pi)$ be defined for $\tau_2 \in I$ by

$$\sin\theta(\tau_2) = \frac{(p_1 + r_1)\big((b_2 + q_2)\omega^2 - (b_0 + q_0)\big)\omega - (-p_2\omega^2 + p_0 + r_0)\big(\omega^2 - (b_1 + q_1)\big)\omega}{(p_1 + r_1)^2\omega^2 + (-p_2\omega^2 + p_0 + r_0)^2},$$

$$\cos\theta(\tau_2) = \frac{\big((b_2 + q_2)\omega^2 - (b_0 + q_0)\big)(-p_2\omega^2 + p_0 + r_0) + (p_1 + r_1)\big(\omega^2 - (b_1 + q_1)\big)\omega^2}{(p_1 + r_1)^2\omega^2 + (-p_2\omega^2 + p_0 + r_0)^2}.$$

And the relation between the argument θ and $\omega\tau_2$ for $\tau_2 > 0$ must be

$$\omega\tau_2 = \theta(\tau_2) + 2n\pi, \quad n = 0, 1, 2, \ldots.$$

Hence we can define the maps: $\tau_n : I \to R_{+0}$ given by

$$\tau_n(\tau_2) = \tau_2 - \frac{\theta(\tau_2) + 2n\pi}{\omega(\tau_2)}, \quad \tau_n > 0, \quad n = 0, 1, 2, \ldots,$$

where the positive root ω of (2.9) exists in I.

Let us introduce the functions

$$S_n(\tau_2) = \tau_2 - \frac{\theta(\tau_2) + 2n\pi}{\omega(\tau_2)}, \quad n = 0, 1, 2, \ldots,$$

which are continuous and differentiable in τ_2.

The following theorem is from [4].

Theorem 2.3 *Assume that the function $S_n(\tau_2)$ has a positive root $\tau_2^0 \in I$ for some $n \in N_0$, then a pair of simple purely imaginary roots $\pm i\omega^*$ of equation (2.4) exists at $\tau = \tau_2^0$, and*

$$\delta(\tau_2^0) := \text{Sign}\Big\{ \frac{d\,\text{Re}(\lambda)}{d\tau_2}\Big|_{\lambda = i\omega^*} \Big\} = \text{Sign}\Big\{ \frac{dS_n(\tau_2)}{d\tau_2}\Big|_{\tau_2 = \tau_2^0} \Big\}.$$

Therefore, this pair of simple conjugate purely imaginary roots crosses the imaginary axis from left to right if $\delta(\tau_2^0) > 0$, and crosses the imaginary axis from right to left if $\delta(\tau_2^0) < 0$.

We can easily observe that $S_0(\tau_2) < 0$. Moreover, for all $\tau_2 \in I$, $S_0(\tau_2) > S_n(\tau_2)$ with $n \in N$. Therefore, if S_0 has no zero in I, then the function S_n has no zero in I, and if the function $S_n(\tau_2)$ has positive zeros $\tau_2 \in I$ for some $n \in N$, there exists at least one zero satisfying $\frac{dS_n(\tau_2)}{d\tau_2} > 0$.

Applying Theorem 2.3 and the Hopf bifurcation theorem for functional differential equations [8], we can conclude the existence of a Hopf bifurcation as stated in the following theorem.

Theorem 2.4 *Assume $R_0(0,\tau_2) > 1$; for system (1.3), the following conclusions hold.*

(i) *If the function $S_0(\tau_2)$ has no positive zero in I, then the positive equilibrium E^* is locally asymptotically stable for all $\tau_2 \geq 0$.*

(ii) *If the function $S_n(\tau_2)$ has positive zeros in I, there exists τ_2^0, for some $n \in N$, such that the positive equilibrium E^* is asymptotically stable for $\tau_2 \in [0, \tau_2^0)$ and becomes unstable for τ_2 staying in some right neighborhood of τ_2^0, with a Hopf bifurcation occurring when $\tau_2 = \tau_2^0$.*

In addition, in the case of $\tau_1 \geq 0$ and $\tau_2 = 0$, we have the following results. When $\tau_1 \in [0, \bar{\tau}_1)$, there exists E^* where

$$\bar{\tau}_1 = \frac{1}{\mu_V} \ln\Big(\frac{ab}{\mu_V \mu_A (d + \mu_N)} - \frac{D}{\mu_V} \Big).$$

Equation (2.2) takes the general form

$$P(\lambda, \tau_1) + Q(\lambda, \tau_1) e^{-\lambda \tau_1} = 0. \tag{2.10}$$

Using the same methods as previously, we obtain

$$F(\omega, \tau_1) = \omega^6 + c_2(\tau_1)\omega^4 + c_1(\tau_1)\omega^2 + c_0(\tau_1) = 0, \tag{2.11}$$

where

$$c_0(\tau_1) = \big(abe^{-\mu_V \tau_1} - \mu_A (De^{-\mu_V \tau_1} + \mu_V)(d + \mu_N) \big) \times \big(-abe^{-\mu_V \tau_1} + \mu_A (De^{-\mu_V \tau_1} + 3\mu_V)(d + \mu_N) \big).$$

Let $z = \omega^2$. Then

$$g(z) = z^3 + c_2(\tau_1) z^2 + c_1(\tau_1) z + c_0(\tau_1) = 0. \tag{2.12}$$

From [14], we have the following results.

(i) When $c_0(\tau_1) < 0$, (2.12) has at least one positive root. From the expression of $c_0(\tau_1)$, it is easy to obtain that if $R_0(0,0) > R_0^{**} := 1 + 2\frac{\mu_V}{D + \mu_V}$, then there exists $\tau_\omega^* \in [0, \bar{\tau}_1)$ such that $c_0(\tau_1) < 0$, $\tau_1 \in [0, \tau_\omega^*)$ where $\tau_\omega^* = \bar{\tau}_1 - \frac{\ln 3}{\mu_V}$.

(ii) When $c_0(\tau_1) \geq 0$ and $\triangle^* = c_2^2(\tau_1) - 3c_1(\tau_1) < 0$, (2.12) has no positive real root. Therefore (2.10) has no purely imaginary root.

(iii) When $c_0(\tau_1) \geq 0$, (2.12) has positive roots if and only if

$$z^* = (-c_2(\tau_1) + \sqrt{\Delta^*})/3 > 0$$

and $g(z^*) \leqslant 0$. By the analysis of condition (i), we know that if $R_0(0,0) \leq R_0^{**}$, then $c_0(\tau_1) \geq 0$.

For case (i) and case (iii), for simplicity, we assume that (2.12) has a unique positive root denoted by X^* and (2.11) has a positive root given by $\omega = \sqrt{X^*}$.

Define

$$I^* = \Big\{ \tau_1 \in [0, \bar{\tau}_1) : R_0(0,0) > R_0^{**}; \text{or } 1 < R_0(0,0) \le R_0^{**}, \\ z^* = (-c_2(\tau_1) + \sqrt{\Delta^*})/3 > 0, \text{ and } g(z^*) \leqslant 0 \Big\}.$$

For $\tau_1 \in I^*$, there exists an $\omega = \omega(\tau_1) > 0$ such that $F(\omega(\tau_1), \tau_1) = 0$. Then let $\theta(\tau_1) \in [0, 2\pi)$ be defined for $\tau_1 \in I^*$ by

$$\sin\theta(\tau_1) = \frac{\begin{array}{r}(q_1 + r_1)\big((b_2 + p_2)\omega^2 - (b_0 + p_0)\big)\omega \\ - (-q_2\omega^2 + q_0 + r_0)\big(\omega^2 - (b_1 + p_1)\big)\omega\end{array}}{(q_1 + r_1)^2\omega^2 + (-q_2\omega^2 + q_0 + r_0)^2},$$

$$\cos\theta(\tau_1) = \frac{\begin{array}{r}\big((b_2 + q_2)\omega^2 - (b_0 + p_0)\big)(-q_2\omega^2 + q_0 + r_0) \\ + (q_1 + r_1)\big(\omega^2 - (b_1 + p_1)\big)\omega^2\end{array}}{(q_1 + r_1)^2\omega^2 + (-q_2\omega^2 + q_0 + r_0)^2}.$$

Using the same methods as [4], let us introduce the functions $S_n(\tau_1) = \tau_1 - \frac{\theta(\tau_1) + 2n\pi}{\omega(\tau_1)}$, $n = 0, 1, 2, \ldots$, that are continuous and differentiable in τ_1.

Theorem 2.5 *Assume that the function $S_n(\tau_1)$ has a positive root $\tau_1^0 \in I^*$ for some $n \in N_0$. Then a pair of simple purely imaginary roots $\pm i\omega^{**}$ of equation (2.10) exists at $\tau = \tau_1^0$ and*

$$\delta(\tau_1^0) := \operatorname{Sign}\Big\{ \frac{d\operatorname{Re}(\lambda)}{d\tau_1}\Big|_{\lambda = i\omega^{**}} \Big\} = \operatorname{Sign}\Big\{ \frac{dS_n(\tau_1)}{d\tau_1}\Big|_{\tau_1 = \tau_1^0} \Big\}.$$

Therefore, this pair of simple conjugate purely imaginary roots crosses the imaginary axis from left to right if $\delta(\tau_1^0) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau_1^0) < 0$.

We can easily observe that $S_0(\tau_1) < 0$. Moreover, for all $\tau_1 \in I^*$, $S_0(\tau_1) > S_n(\tau_1)$ with $n \in N$. Therefore, if S_0 has no zero in I^*, then the function S_n has no zero in I^* and if the function $S_n(\tau_1)$ has positive zeros $\tau_1 \in I^*$ for some $n \in N$, there exists at least one zero satisfying $\frac{dS_n(\tau_1)}{d\tau_1} > 0$.

Theorem 2.6 *Assume $R_0(\tau_1, 0) > 1$, for system (1.3), the following conclusions hold:*

(i) *If the function $S_0(\tau_1)$ has no positive zero in I^*, then the positive equilibrium E^* is locally asymptotically stable for all $\tau_1 \ge 0$.*
(ii) *If for some $n \in N$ the function $S_n(\tau_1)$ has positive zeros in I^*, then there exists τ_1^0, such that the positive equilibrium E^* is asymptotically stable for $\tau_1 \in [0, \tau_1^0)$ and becomes unstable for τ_1 staying in some right neighborhood of τ_1^0 with a Hopf bifurcation occurring when $\tau_1 = \tau_1^0$.*

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3 Direction and Stability of the Hopf Bifurcation

In the previous section, we obtain the conditions which guarantee that system (1.3) undergoes the Hopf bifurcation at the positive equilibrium E^* under the conditions of Theorem 2.4 (ii). In this section, using the normal form theory and the center manifold argument presented by [9] we can establish an explicit formula for determining the direction and stability of periodic solutions bifurcating from the positive equilibrium E^* at a Hopf bifurcation value. We obtain the conditions which guarantee that system (1.3) undergoes the Hopf bifurcation at the positive equilibrium E^* when $\tau_2 = \tau_2^0$. In this section, using the normal form theory and the center manifold argument presented by Hassard et al. [9], we can establish an explicit formula to determine the direction and stability of periodic solutions bifurcating from the positive equilibrium E^* at a Hopf bifurcation value, say $\tau_2 = \tau_2^0$. Furthermore, let $\bar{\tau}_2 = \tau_2^0 + \mu, \mu \in R$. Then $\mu = 0$ is the Hopf bifurcation value for (1.3). We choose the phase space as $C = C([-\tau_2^0, 0], R^3)$; system (1.3) is transformed into the following functional differential equation in C.

$$\dot{u}_t = L_\mu(u_t) + f(\mu, u_t), \tag{3.1}$$

where $u_t(\theta) = u(t+\theta) \in C$, and $L_\mu: C \to R^3$, $F: R \times C \to R^3$ are given, respectively, by $L_\mu \phi = A\phi(0) + B\phi(-\tau_2^0)$, where

$$A = \begin{pmatrix} -(D+\mu_V) & 0 & a-2z^* \\ 1 & -\mu_N & 0 \\ 0 & 0 & -\mu_A \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -de^{-\mu_N \tau_2^0} & 0 \\ 0 & be^{-\mu_N \tau_2^0} & 0 \end{pmatrix},$$

$$f(\mu, \phi) = \begin{pmatrix} -\phi_3^2(0) \\ 0 \\ 0 \end{pmatrix},$$

where $\phi = (\phi_1, \phi_2, \phi_3)^T$.

By the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-\tau_2^0, 0]$, such that

$$L_\mu \phi = \int_{-\tau_2^0}^{0} d\eta(\theta, \mu)\phi(\theta)\,, \big(\phi \in C([-\tau_2^0, 0], R^3)\big).$$

In fact, we can choose $\eta(\theta, \mu) = A\delta(\theta) - B\delta(\theta + \tau_2^0)$, where

$$\delta(\theta) = \begin{cases} \tau_2^0, & \theta = 0, \\ 0, & \theta \neq 0. \end{cases}$$

For $\phi \in C^1([-\tau_2^0, 0], R^3)$ define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta) = \begin{cases} \dot{\phi}(\theta), & \theta \in [-\tau_2^0, 0), \\ \int_{-\tau_2^*}^{0} d\eta(\xi, \mu)\phi(\xi), & \theta = 0. \end{cases}$$

For $\phi \in C^1([-\tau_2^0, 0], R^3)$, let

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-\tau_2^0, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Then system (3.1) is equivalent to the following operator equation

$$\dot{u_t} = A(\mu)u_t + R(\mu)u_t,$$

where $u_t = (x(t+\theta), y(t+\theta), z(t+\theta))^T(-\tau_2^0 \le \theta \le 0)$.

For $\psi \in C^1([0,\tau_2^0],(R^{3*})$ and $\phi \in C([-\tau_2^0,0],R^3)$ define

$$A^*\psi(s) = \begin{cases} -\dot{\psi}(s), & s \in (0,\tau_2^0], \\ \int_{-\tau_2^0}^0 d\eta^T(t,0)\psi(-t), & s = 0, \end{cases}$$

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{-\tau_2^0}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi-\theta)\, d\eta(\theta)\phi(\xi)\, d\xi,$$

where $\eta(\theta) = \eta(\theta,0)$. We know that A^* and $A = A(0)$ are adjoint operators.

From the discussion in Section 2, we know that $\pm i\omega^*$ are eigenvalues of $A(0)$ and therefore they are also eigenvalues of A^*.

It is not difficult to verify that the vectors

$$q(\theta) = (q_1(0), q_2(0), 1)^T e^{i\omega^*\theta} \quad (\theta \in [-\tau_2^0, 0])$$
$$q^*(s) = \bar{D}\big(1, q_2^*(0), q_3^*(0)\big)\, e^{i\omega^* s} \quad (s \in [0,\tau_2^0])$$

are eigenvalues of $A(0)$ and A^* corresponding to the eigenvalue $i\omega^*$ and $-i\omega^*$, respectively. Furthermore, let $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\theta)\rangle = 0$, where

$$q_1(0) = \frac{a - 2z^*}{i\omega^* + \mu_V + D}, \qquad q_2(0) = b(i\omega^* + \mu_A)e^{(i\omega^*+\mu_N)\tau_2^0},$$

$$q_2^*(0) = -i\omega^* + \mu_V + D, \quad q_3^*(0) = \frac{\bar{a} - 2z^*}{-i\omega^* + \mu_A},$$

$$D = \big\{ q_1(0) + \bar{q}_2^*(0)q_2(0) + \bar{q}_3^*(0) - \tau_2^0 q_2(0)(d\bar{q}_2^*(0) - b\bar{q}_3^*(0))e^{-(i\omega^*+\mu_N)\tau_2^0}\big\}^{-1}.$$

Now using the same notations as in Hassard et al. [9], we can obtain the coefficients that will be used to determine the important quantities:

$$g_{20} = -2\big\{ q_1(0) + \bar{q}_2^*(0)q_2(0) + \bar{q}_3^*(0) - \tau_2^0 q_2(0)(d\bar{q}_2^*(0) - b\bar{q}_3^*(0))e^{-(i\omega^*+\mu_N)\tau_2^0}\big\}^{-1},$$
$$g_{11} = -2\big\{ q_1(0) + \bar{q}_2^*(0)q_2(0) + \bar{q}_3^*(0) - \tau_2^0 q_2(0)(d\bar{q}_2^*(0) - b\bar{q}_3^*(0))e^{-(i\omega^*+\mu_N)\tau_2^0}\big\}^{-1},$$
$$g_{02} = -2\big\{ q_1(0) + \bar{q}_2^*(0)q_2(0) + \bar{q}_3^*(0) - \tau_2^0 q_2(0)(d\bar{q}_2^*(0) - b\bar{q}_3^*(0))e^{-(i\omega^*+\mu_N)\tau_2^0}\big\}^{-1},$$
$$g_{21} = -2\big\{ q_1(0) + \bar{q}_2^*(0)q_2(0) + \bar{q}_3^*(0) - \tau_2^0 q_2(0)(d\bar{q}_2^*(0) - b\bar{q}_3^*(0))e^{-(i\omega^*+\mu_N)\tau_2^0}\big\}^{-1}, \\ \times\big(W_{20}^{(3)}(0) + 2W_{11}^{(3)}(0)\big),$$

where

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*}q(0)e^{i\omega^*\theta} + \frac{i\bar{g}_{02}}{3\omega^*}\bar{q}(0)e^{-i\omega^*\theta} + E_1 e^{2i\omega^*\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^*}q(0)e^{i\omega^*\theta} + \frac{i\bar{g}_{11}}{\omega^*}\bar{q}(0)e^{-i\omega^*\theta} + E_2,$$

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and

$$E_1 = \begin{pmatrix} 2i\omega^* + \mu_V + D & 0 & -a + 2z^* \\ -1 & 2i\omega^* + \mu_N + de^{-(2i\omega^*+\mu_N)\tau_2^0} & 0 \\ 0 & -be^{-(2i\omega^*+\mu_N)\tau_2^0} & 2i\omega^* + \mu_A \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} \mu_V + D & 0 & -a + 2z^* \\ -1 & \mu_N + de^{-\mu_N\tau_2^0} & 0 \\ 0 & -be^{-\mu_N\tau_2^0} & \mu_A \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}.$$

Furthermore, substituting E_1 and E_2 into $W_{20}(\theta)$ and $W_{11}(\theta)$, respectively, g_{21} can be expressed by the parameters. Based on the above analysis, we can see that each g_{ij} can be determined by the parameters. Thus we can compute the following quantities:

$$C_1(0) = \frac{i}{2\omega^*}\Big(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\Big) + \frac{g_{21}}{2}, \quad \mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\lambda'(\tau_2^0)},$$

$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 Im\lambda'(\tau_2^0)}{\omega^*}, \quad \beta_2 = 2\operatorname{Re}\{C_1(0)\}.$$

Hence, we have the following result.

Theorem 3.1 (i) *μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0$ (resp., < 0), the Hopf bifurcation is supercritical (resp., subcritical).*

(ii) *β_2 determines the stability of the bifurcation periodic solutions. The bifurcation periodic solutions are orbitally stable (resp., unstable) if $\beta_2 < 0$ resp.,(> 0).*

(iii) *T_2 determines the period of the bifurcating periodic solutions. The period increases (resp., decreases) if $T_2 > 0$ (resp., < 0).*

4 Numerical Simulations

In this section, we carry out some simulations of system (1.3) to illustrate the theoretical results obtained in Section 2.

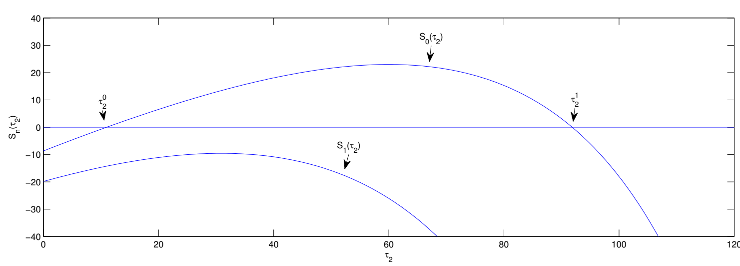

Figure 1: The plot of $S_n(\tau_2)$, $n = 0, 1$.

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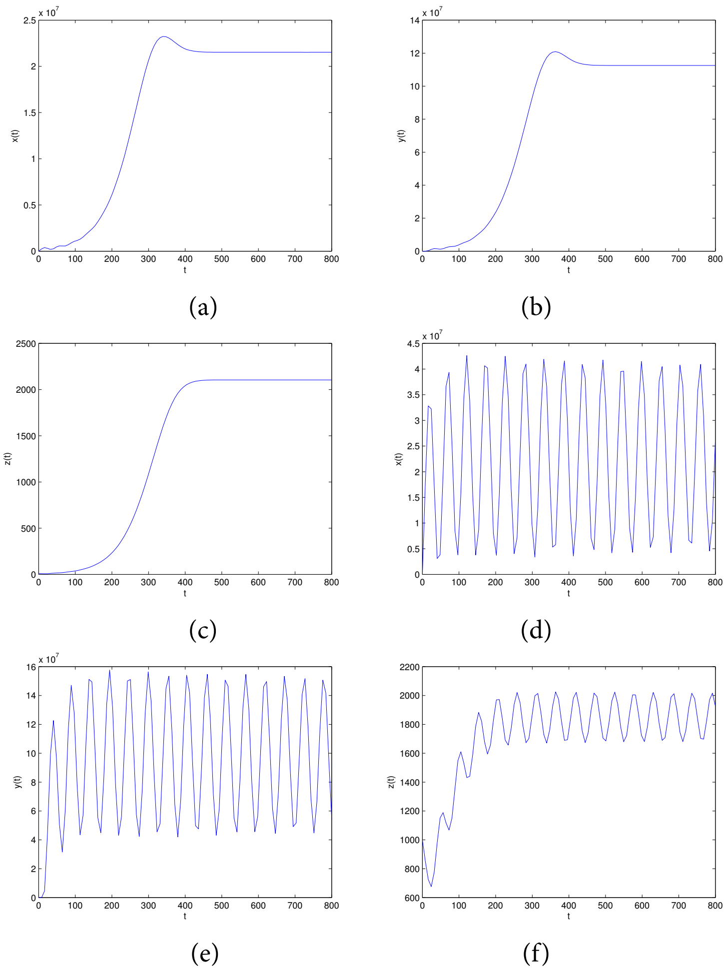

Figure 2: The positive equilibrium E^* of the system (11) is locally asymptotically stable for $\tau_2 = 5 < \tau_2^0$ (see (a)-(c)), unstable and there exists a stable periodic solution for $\tau_2 = 14 \in (\tau_2^0, \tau_2^1)$ (see (d)-(f)). The initial conditions are $\phi_1 \equiv 1000$, $\phi_2 \equiv 1000$ and $\phi_3 \equiv 1000$.

Choosing parameters as $\alpha_H = 0.0055$, $f = 0.01$, $H = 600$, $c = 0.000105$, $\alpha_M = 0.02$, $Q = 3$, $\mu_V = 0.08$, $D = 0.0001$, $\mu_N = 0.06$, $\gamma P = 0.005$, $\mu_A = 0.02$. For $\tau_1 = 0$ and $\tau_2 \geq 0$, the functions $S_n(\tau_2)$ are plotted in Figure 1. In Figure 1 , there are two Hopf bifurcation values for τ_2, say $\tau_2^0 < \tau_2^1$. The first occurs when $S_0(\tau_2)$ crosses 0 at $\tau_2 = \tau_2^0 = 11.0345$ and the second occurs when $S_0(\tau_2)$ crosses 0 at $\tau_2 = \tau_2^1 = 93.3577$. The stability of the endemic equilibrium E^* switches at τ_2^0 and τ_2^1. In Figures 2 (a)–(f) we show the trajectory plots of the $(x(t), y(t), z(t))$ with the initial functions for two values of $\tau_2 = 5(< \tau_2^0)$ and $\tau_2 = 14(\in (\tau_2^0, \tau_2^1))$. We show that the positive equilibrium

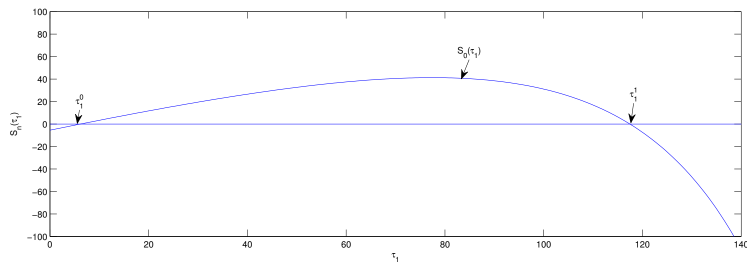

Figure 3: The plot of $S_n(\tau_1)$, $n = 0$.

E^* is asymptotically stable for both $\tau_2 = 5$ and a stable periodic solution appears for $\tau_2 = 14$.

For $\tau_1 \geq 0$ and $\tau_2 = 0$, the functions $S_n(\tau_1)$ are plotted in Figure 3 where there are two Hopf bifurcation values for τ_1, say $\tau_1^0 < \tau_1^1$. The first occurs when $S_0(\tau_1)$ crosses 0 at $\tau_1 = \tau_1^0 = 8.2317$ and the second occurs when $S_0(\tau_1)$ crosses 0 at $\tau_1 = \tau_1^1 = 118.37418$. The stability of the endemic equilibrium E^* switches at τ_1^0 and τ_2^1. In Figures 4(g)–(l) we show the trajectory plots of the $(x(t), y(t), z(t))$ with the initial functions for two values of $\tau_1 = 2(< \tau_1^0)$ and $\tau_1 = 10(\in (\tau_1^0, \tau_1^1))$. We show that the positive equilibrium E^* is asymptotically stable for both $\tau_1 = 2$ and a stable periodic solution appears for $\tau_1 = 10$.

5 Conclusion

In this paper, we study a nonlinear three stage-structured model for Lyme disease with two delays. We show that for any $\tau_1 \geq 0$ and $\tau_2 \geq 0$ such that $R_0(\tau_1, \tau_2) \leq 1$, the boundary equilibrium E_0 of system (1.3) is globally asymptotically stable. Next we study the effect of delay on the stability of the positive equilibrium. To analyze the characteristic equation with two delays, we first focus on the case when one of the delay τ_1 equals to zero and obtain a critical value for the delay τ_2. When $\tau_2 < \tau_2^0$, all roots of the characteristic equation have negative parts and when $\tau_2 = \tau_2^0$, purely imaginary roots appear. Secondly, we focus on the case when one of the delay τ_2 equals to zero and obtain a critical value for the delay τ_1. When $\tau_1 < \tau_1^0$, all roots of the characteristic equation have negative parts and when $\tau_1 = \tau_1^0$, purely imaginary roots appear. Next, by using the center manifold and normal forms theory, regarding τ_2 as a parameter, we investigate the direction and stability of the Hopf bifurcation. Explicit algorithms for determining the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions are derived. Finally, we carry out some numerical simulations to support the analysis results.

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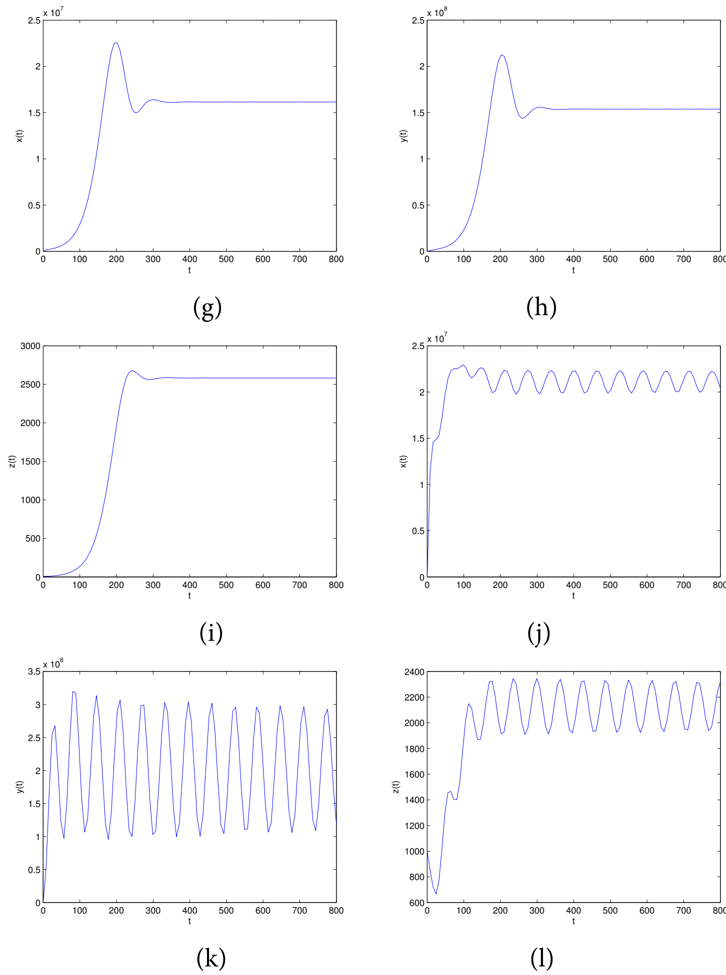

Figure 4: The positive equilibrium E^* of the system (11) is locally asymptotically stable for $\tau_1 = 2 < \tau_1^0$ (see (g)-(i)), unstable and there exists a stable periodic solution for $\tau_1 = 10 \in (\tau_1^0, \tau_1^1)$ (see (j)-(l)). The initial conditions are $\phi_1 \equiv 1000$, $\phi_2 \equiv 1000$ and $\phi_3 \equiv 1000$.

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