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The Gelfand–Graev representation of classical groups in terms of Hecke algebras

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Abstract. Let G be a p-adic classical group. The representations in a given Bernstein component can be viewed as modules for the corresponding Hecke algebra—the endomorphism algebra of a progenerator of the given component. Using Heiermann's construction of these algebras, we describe the Bernstein components of the Gelfand–Graev representation for $G = \mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$, and $\mathrm{O}(2n)$.

1 Introduction

Let F be a non-Archimedean local field of residue characteristic q. Let G be the group of F-points of a connected, split reductive algebraic group defined over F; in particular, the group G contains a Borel subgroup. Let U be the unipotent radical of the Borel subgroup, and fix a nondegenerate (Whittaker) character $\psi : U \to \mathbb{C}^\times$. The Gelfand–Graev representation of G is $\mathrm{c}-\mathrm{ind}_U^G(\psi)$, where $\mathrm{c}-\mathrm{ind}$ stands for induction with compact support. The goal of this paper is to give an explicit description of the Bernstein components of the Gelfand–Graev representation.

Let us briefly describe what is known. Let K be a special maximal compact subgroup of G, and let I be an Iwahori subgroup contained in K. Let $\mathcal{H}$ be the Iwahori–Hecke algebra of I-biinvariant functions on G, and let $\mathcal{H}_K$ be the subalgebra consisting of functions supported on K. Then $\mathcal{H}_K$ is isomorphic to the group algebra of the Weyl group W of G, and thus it has a one-dimensional representation ε (the sign character). As an $\mathcal{H}$-module, $(\mathrm{c}-\mathrm{ind}_U^G\psi)^I$ is isomorphic to the projective $\mathcal{H}$-module [10]

$$\mathcal{H} \otimes_{\mathcal{H}_K} \varepsilon.$$

If $G = \mathrm{GL}_n$, then a similar statement holds for all Bernstein components with appropriate Hecke algebras arising from Bushnell–Kutzko types [11]. We build on methods of that paper. We finish this paragraph by mentioning a recent article of Mishra and Pattanayak [20] that considers Bernstein components of $\mathrm{c}-\mathrm{ind}_U^G(\psi)$ corresponding to representations induced from the Borel subgroup. Their result is formulated in terms of Hecke algebras arising from types constructed by Roche.

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For a general G, one does not have a complete theory of types and corresponding Hecke algebras, but there is a replacement: endomorphism algebras of pro-generators of Bernstein components.

It turns out that these algebras are more suited for the problem at hand. In more detail, let $P = MN$ be a parabolic subgroup of G, and let σ be an irreducible cuspidal representation of M. Let M° be the subgroup of M consisting of all $m \in M$ such that $|\chi(m)| = 1$ for all smooth characters $\chi : M \to \mathbb{C}^\times$. Let σ_0 be an irreducible summand of σ restricted to M°. Then $i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0))$ is a projective G-module generating a single Bernstein component. Here, i_P^G denotes normalized parabolic induction. Let

$$\mathcal{H} = \mathrm{End}_G(i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0))).$$

Observe that we have a natural inclusion

$$\mathcal{A} = \mathrm{End}_M(\mathrm{c}-\mathrm{ind}_{M_0}^M(\sigma_0)) \subseteq \mathcal{H}.$$

For every G-module π,

$$\mathfrak{F}(\pi) = \mathrm{Hom}_G(i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0)), \pi)$$

is naturally a right $\mathcal{H}$-module. The functor $\mathfrak{F}$ is an equivalence between the Bernstein component generated by $i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0))$ and the category of right $\mathcal{H}$-modules.

Now, assume that σ is ψ-generic. Let

$$\Pi = \mathfrak{F}(\mathrm{c}-\mathrm{ind}_U^G(\psi)).$$

It is not difficult to see, using Bernstein's second adjointness, that $\Pi \cong \mathcal{A}$, as $\mathcal{A}$-modules. Thus, understanding Π reduces to understanding $\mathcal{H}$-modules isomorphic to $\mathcal{A}$. This was done for GL_n in [11]. We extend this computation to $\mathcal{H}$ for $G = \mathrm{SO}(2n+1, F)$, $\mathrm{Sp}(2n, F)$, and $\mathrm{O}(2n, F)$. For classical groups, the algebra $\mathcal{H}$ has been computed by Heiermann [17]; more recently, Solleveld [25] has studied the same algebra in a more general setting. If $G = \mathrm{SO}(2n+1, F)$, $\mathrm{Sp}(2n, F)$, or $\mathrm{O}(2n, F)$, it turns out that the algebra $\mathcal{H}$ is a tensor product of affine Hecke algebras, each of which is isomorphic to the Iwahori–Hecke algebra of GL_k or to an algebra of type $\tilde{C}_k$ with semisimple rank $k \le n$, with unequal parameters. (Note that we work with the full orthogonal group $\mathrm{O}(2n)$ instead of $\mathrm{SO}(2n)$; this is because the case of $\mathrm{SO}(2n)$ is significantly more involved due to the complicated structure of the R-group [14].) Assume that $\mathcal{H}$ corresponds to the affine type $\tilde{C}_n$. The diagram has two special vertices, denoted by 0 and n. Corresponding to them, we have two finite subalgebras $\mathcal{H}_0$ and $\mathcal{H}_n$ of $\mathcal{H}$. We prove that any $\mathcal{H}$-module isomorphic to $\mathcal{A}$ is necessarily

$$\mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_0 \text{ or } \mathcal{H} \otimes_{\mathcal{H}_n} \varepsilon_n$$

for a one-dimensional representation ε_0 or ε_n. Here, we moved to more familiar language of left $\mathcal{H}$-modules. This is harmless indeed, since $\mathcal{H}$ is isomorphic to its opposite algebra; this follows from the Iwahori–Matsumoto relations. Finally, we determine precisely the isomorphism class of Π.

Next, we apply the classification of $\mathcal{H}$-modules isomorphic to $\mathcal{A}$ to study the Gan–Gross–Prasad restriction problem from $\mathrm{O}(m+1, F)$ to $\mathrm{O}(m, F)$. Fix an irreducible supercuspidal representation of $\mathrm{O}(m+1, F)$. Then, for every maximal ideal $\mathcal{J}$ in the

Bernstein center of $\mathrm{O}(m, F)$, there exists at most one irreducible quotient annihilated by $\mathcal{J}$. This is a generalization of a similar result for general linear groups where only Whittaker generic representations of $\mathrm{GL}(n)$ appear as quotients of supercuspidal representations of $\mathrm{GL}(n+1)$.

We finish this paper with an appendix where we show that $\mathcal{H}$ is isomorphic with the Hecke algebra arising from the type constructed by Stevens.

2 Preliminaries

2.1 Notation

Throughout the paper, F will denote a non-Archimedean local field of residue characteristic q and uniformizer ϖ, equipped with the absolute value $|\cdot|$ normalized in the usual way.

We let G denote the special odd orthogonal group, the symplectic group, or the (full) even orthogonal group. If we want to emphasize the rank, we use G_n to denote $\mathrm{SO}(2n+1, F)$, $\mathrm{Sp}(2n, F)$, or $\mathrm{O}(2n, F)$. By $\mathrm{Rep}(G)$, we denote the category of smooth complex representations of G.

For an arbitrary group H, we let $X(H)$ denote the group of complex characters of H.

2.2 Parabolic subgroups

If G is the disconnected group $\mathrm{O}(2n, F)$, then, following [15], we consider only parabolic subgroups $P = MN$ such that M has supercuspidal (modulo center) representations. Explicitly, this means that

$$M = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_k}(F) \times \mathrm{O}(2n_0, F);$$

however, we do not allow $n_0 = 1$ if $\mathrm{O}(2n, F)$ is split.

2.3 Unramified characters

If M is a Levi subgroup of G, we let $M^\circ = \bigcap_\chi \ker|\chi|$, the intersection taken over the set of all rational characters $\chi : M \to F^\times$. We say that a (complex) character χ of M is unramified if it is trivial on M°; we let $X^{\mathrm{nr}}(M)$ denote the group of all unramified characters on M. Then M/M° is a free $\mathbb{Z}$-module of finite rank, and the group $X^{\mathrm{nr}}(M) = X(M/M^\circ)$ has a natural structure of a complex affine variety. For any element $m \in M$, we denote by b_m the evaluation $\chi \mapsto \chi(m)$.

Now, let σ be an irreducible cuspidal representation of M, and set $M^\sigma = \{m \in M : {}^m\sigma \cong \sigma\}$. Then M/M^σ is a finite abelian group and, abusing notation, we let $\mathcal{A}$ denote the ring of regular functions on the quotient variety $X(M/M^\circ)/X(M/M^\sigma)$. Since M^σ/M° is once again a free $\mathbb{Z}$-module (of the same rank as M/M°), we have $\mathcal{A} \cong \mathbb{C}[M^\sigma/M^\circ]$, by $m \mapsto b_m$. Furthermore, letting σ_0 denote an arbitrary irreducible constituent of $\sigma|_{M^\circ}$, we have a canonical isomorphism $\mathcal{A} \cong \mathrm{End}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M \sigma_0)$. Indeed, this follows from a simple application of Mackey theory. We refer the reader to [17, Sections 1.17 and 4] for additional details.

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2.4 The Hecke algebra of a Bernstein component

If π is an irreducible representation of G, there is a Levi subgroup M of G and an irreducible cuspidal representation σ of M such that π is (isomorphic to) a subquotient of $i_P^G(\sigma)$. Here, P is a parabolic subgroup of G with a Levi component M. The pair (M,σ) is determined by π up to conjugacy; we call (M,σ) the cuspidal support of π.

We say that the two pairs (M_1,σ_1) and (M_2,σ_2) as above are inertially equivalent if there exist an element $g \in G$ and an unramified character χ of M_2 such that

$$^{g}\sigma_1 = \sigma_2 \otimes \chi.$$

This is an equivalence relation on the set of all pairs (M,σ). Given an equivalence class $[(M,\sigma)]$, we denote by $\mathrm{Rep}_{(M,\sigma)}(G)$ the full subcategory of $\mathrm{Rep}(G)$ defined by the requirement that all irreducible subquotients of every object in $\mathrm{Rep}_{(M,\sigma)}(G)$ be supported within the inertial class $[(M,\sigma)]$. A classic result of Bernstein then shows that the category $\mathrm{Rep}(G)$ decomposes as a direct product

$$\mathrm{Rep}(G) = \prod_{[(M,\sigma)]} \mathrm{Rep}_{(M,\sigma)}(G)$$

taken over the set of all inertial equivalence classes. We refer to $\mathrm{Rep}_{(M,\sigma)}(G)$ as the Bernstein component attached to the pair (M,σ). For a detailed discussion of the above results, see [3] or [4].

For each Bernstein component $\mathrm{Rep}_{(M,\sigma)}(G)$, one can construct a projective generator $\Gamma_{(M,\sigma)}$ by setting

$$\Gamma_{(M,\sigma)} = i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0)).$$

Here, σ_0 is any irreducible component of the (semisimple) restriction $\sigma|_{M^\circ}$. We now obtain a functor from the category $\mathrm{Rep}_{(M,\sigma)}(G)$ to the category of right $\mathrm{End}_G(\Gamma_{(M,\sigma)})$-modules given by

$$\pi \mapsto \mathrm{Hom}(\Gamma_{(M,\sigma)},\pi).$$

The fact that $\Gamma_{(M,\sigma)}$ is a projective generator implies that this is an equivalence of categories. This is [4, Lemma 22]; a detailed proof of this fact is also given in [22, Theorem 1.5.3.1].

Given a Bernstein component attached to $\mathfrak{s} = (M,\sigma)$, we use $\mathcal{H}_{\mathfrak{s}}$ to denote $\mathrm{End}_G(\Gamma_{\mathfrak{s}})$ and refer to it as the Hecke algebra attached to the component $\mathfrak{s}$. Furthermore, for any $\pi \in \mathrm{Rep}(G)$, we let $\pi_{\mathfrak{s}}$ denote the corresponding $\mathcal{H}_{\mathfrak{s}}$-module $\mathrm{Hom}(\Gamma_{\mathfrak{s}},\pi)$.

Although we do not use it here, we point out that there is another highly useful approach to analyzing Bernstein components, based on the theory of types developed by Bushnell and Kutzko [8]. One can show that the Hecke algebra used by Bushnell and Kutzko is in fact isomorphic to the algebra $\mathcal{H}_{\mathfrak{s}}$ introduced above; we prove this fact in Appendix A. Therefore—for the purposes of this paper—the two approaches are equivalent.

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2.5 Cuspidal representations

Here, we briefly recall some facts and introduce notation related to cuspidal representations of classical groups.

Let ρ and τ be irreducible unitarizable cuspidal representations of $\mathrm{GL}_k(F)$ and G_{n_0}, respectively. We consider the representation $\nu^\alpha\rho \rtimes \tau$, where $\alpha \in \mathbb{R}$. Here, and throughout the paper, we use ν to denote the unramified character $|\det|$ of the general linear group. If ρ is not self-dual, the above representation never reduces. If ρ is self-dual, then there exists a unique $\alpha \geq 0$ such that $\nu^\alpha\rho \rtimes \tau$ is reducible; we denote it by α_ρ.

The number α_ρ has a natural description in terms of Langlands parameters. Let ϕ be the L-parameter of τ. Then ϕ decomposes into a direct sum of irreducible representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$. We view ρ as a representation of W_F; we say that it is of the same type as ϕ if the corresponding W_F-representation factors through a group of the same type (orthogonal/symplectic) as ϕ. Letting S_a denote the (unique) irreducible algebraic a-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$, we now set

$$a_\rho = \max\{a : \rho \otimes S_a \text{ appears in } \phi\}.$$

If the above set is empty, we let

$$a_\rho = \begin{cases} -1, & \text{if } \rho \text{ is of the same type as } \phi, \\ 0, & \text{otherwise.} \end{cases}$$

With this description of a_ρ, we have $\alpha_\rho = \frac{a_\rho + 1}{2}$.

2.6 The structure of the Hecke algebra

We retain the notation ρ, τ, and G_n from the previous subsection, and consider the cuspidal component $\mathfrak{s}$ attached to the representation

$$\underbrace{\rho \otimes \cdots \otimes \rho}_{n \text{ times}} \otimes \tau$$

of the Levi subgroup $M = \mathrm{GL}_k(F) \times \cdots \times \mathrm{GL}_k(F) \times G_{n_0}$ in G_N, where $N = nk + n_0$. In the rest of the paper, we restrict our attention to cuspidal components of the above form. This does not present a significant loss of generality, since the Hecke algebra of a general cuspidal component is the product of algebras corresponding to the components described above. To simplify notation, we set $\mathcal{H} = \mathcal{H}_\mathfrak{s}$.

The structure of the Hecke algebra $\mathcal{H}$ has been completely described by Heiermann [16, 17]. In his work, Heiermann shows that $\mathcal{H}$ is a Hecke algebra with parameters (the type of the algebra and the parameters depending on the specifics of the given case). When the component in question is of the form described above, we have three distinct cases, which we now summarize. For basic definitions and results on Hecke algebras with parameters, we refer to the work of Lusztig [19].

In what follows, we let t denote the order of the (finite) group $\{\chi \in X^{\mathrm{nr}}(\mathrm{GL}_k(F)) : \rho \otimes \chi \cong \rho\}$. In all three cases, the commutative algebra $\mathcal{A}$ (see Section 2.3) is a subalgebra of $\mathcal{H}$. In the present setting, the rank of the free module M^σ/M° is equal

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to n. We can thus identify $\mathcal{A} \cong \mathbb{C}[M^\sigma/M^\circ]$ with the algebra of Laurent polynomials $\mathbb{C}[X_1^\pm, \dots, X_n^\pm]$. We fix this isomorphism explicitly: For $i = 1, \dots, n$, let h_i be the element of M which is equal to $\mathrm{diag}(\varpi, 1, \dots, 1)$ on the ith GL factor, and equal to the identity elsewhere. Then $X_i = b_{h_i}^t$. The three cases are:

(i) No representation of the form $\rho \otimes \chi$ with $\chi \in X^{\mathrm{nr}}(\mathrm{GL}_k(F))$ is self-dual.

In this case, the algebra $\mathcal{H}$ is described by an affine Coxeter diagram of type $\tilde{A}_{n-1}$ with equal parameters t. In other words, it is isomorphic to the algebra $\mathcal{H}_n$ described in [11]: There are elements $T_1, \dots, T_{n-1}$ which satisfy the quadratic relation

$$(T_i + 1)(T_i - q^t) = 0, \quad i = 1, \dots, n-1$$

and commutation relations

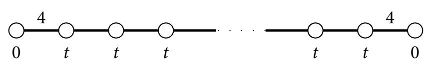
$$T_i f - f^{s_i} T_i = (q^t - 1)\frac{f - f^{s_i}}{1 - X_{i+1}/X_i}, \quad i = 1, \dots, n-1,$$

where f^{s_i} is obtained from $f \in \mathcal{A}$ by swapping X_i and X_{i+1}.

In the two remaining cases, there is an unramified character χ such that $\rho \otimes \chi$ is self-dual. Without loss of generality, we may assume that ρ is self-dual. Then, up to isomorphism, there is a unique representation of the form $\rho \otimes \chi \not\cong \rho$ which is also self-dual; we denote it by ρ^-. We set $\alpha = \alpha_\rho$ and $\beta = \alpha_{\rho^-}$ (see Section 2.5 for notation). Since the situation is symmetric, we may (and will) assume that $\alpha \geq \beta$. The description of $\mathcal{H}$ now involves two additional operators T_0 and T_n (see Remark 2.1). We have the following two cases:

(ii) $\alpha = \beta = 0$.

In this case, $\mathcal{H}$ is described by an affine Coxeter diagram of type $\tilde{C}_n$:



The nodes correspond to operators $T_0, \dots, T_n$ which satisfy the quadratic relations

$$T_0^2 = 1, \quad T_n^2 = 1, \quad (T_i + 1)(T_i - q^t) = 0 \quad \text{for } i = 1, \dots, n-1,$$

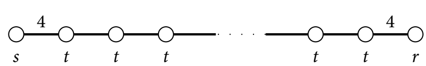
and the braid relations as prescribed by the diagram. The commutation relations for T_i, $i = 1, \dots, n-1$, are the same as in Case (i), whereas T_n satisfies

$$f T_n - T_n f^\vee = 0$$

with $f^\vee(X_1, \dots, X_{n-1}, X_n) = f(X_1, \dots, X_{n-1}, 1/X_n)$.

(iii) $\alpha > 0$.

In this case, $\mathcal{H}$ is described by an affine Coxeter diagram of type $\tilde{C}_n$:



Here, $s = t(\alpha - \beta)$ and $r = t(\alpha + \beta)$. Again, the nodes correspond to operators $T_0, \dots, T_n$ which satisfy quadratic relations analogous to those in Case (ii), along

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with the braid relations. The commutation relations for T_i, $i = 1, \dots, n-1$, are the same as in Case (i), whereas T_n satisfies

$$f\,T_n - T_n f^\vee = \left((q^r - 1) + \frac{1}{X_n}(\sqrt{q}^{r+s} - \sqrt{q}^{r-s})\right)\frac{f - f^\vee}{1 - 1/X_n^2}.$$

Cases (i)–(iii) correspond to Cases (I)–(III) listed in [16, Section 3.1]. The above results are collected in Section 3.4 of [16]. We take a moment to explain the situation in the even orthogonal case. Papers [16, 17] do not treat the full orthogonal group; rather, they contain results about the special orthogonal group $\mathrm{SO}(2n)$. In the special orthogonal case, there is a nontrivial R-group (see [14]) which complicates the structure of the Hecke algebra; this was ultimately worked out by Heiermann in [18]. Because of this, we choose to work with $\mathrm{O}(2n)$ instead. This is indeed justified: Annex A of [18] shows that the results of [16, 17] generalize to the full orthogonal case.

A detailed construction of the operators T_i (starting from standard intertwining operators) is the subject matter of [17]; we do not need the details here, except in a special case discussed in the final part of Section 3.2. To facilitate the comparison of the above summary to the works of Heiermann [16–18], we point out the ways in which our summary deviates from them.

Remark 2.1 (a) The explicit isomorphism $\mathbb{C}[M^\sigma/M^\circ] \cong \mathbb{C}[X_1^\pm, \dots, X_n^\pm]$ we use is different than the one used in [16]; there, Heiermann sets $X_i = b^t_{h_i h_{i+1}^{-1}}$ for $i = 1, \dots, n-1$ (and, in Case (ii), $X_n = b^t_{h_{n-1}h_n}$).

(b) The operator T_0 which appears in Cases (ii) and (iii) above is not needed to describe $\mathcal{H}$, and is therefore not used in [16, 17]. To be precise, the Hecke algebra is generated over $\mathcal{A}$ by the operators $T_1, \dots, T_n$ and determined by the quadratic and braid relations they satisfy, along with the commutation relations listed above. Each of the operators $T_1, \dots, T_n$ corresponds to a simple reflection in the Weyl group, whereas the operator T_0 corresponds to the reflection given by the (in this case, unique) minimal element of the root system (see [19, Section 1.4]). In fact, we define T_0 by setting

$$T_0 = \sqrt{q}^{s+2t(n-1)+r} X_1 T_w^{-1},$$

where $T_w = T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1$ (see [19, Sections 2.8 and 3.3]). We use T_0 out of convenience, as it allows a more symmetric description of certain $\mathcal{H}$-modules.

(c) The description of $\mathcal{H}$ in Case (ii) differs from the one given in [16], which views T_n as the nontrivial element of the R-group. However, one can verify that the description we use is equivalent. With our description, (ii) can be viewed as a special case of (iii) (with $r = s = 0$); however, since our results in (ii) require additional analysis, we still state the two cases separately.

2.7 Generic representations

We recall only the most basic facts here; a general reference is, e.g., [24].

Assume that G is split, and let U is be a maximal unipotent subgroup of G. Fix a nondegenerate character ψ of U. Recall that a character of U is said to be nondegenerate if it is nontrivial on every root subgroup corresponding to a simple

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root. We say that a representation (π, V) of G is ψ-generic if there exists a so-called Whittaker functional—that is, a linear functional $L : V \to \mathbb{C}$ such that

$$L(\pi(u)v) = \psi(u)L(v), \quad \forall u \in U, v \in V.$$

The key fact we use throughout is that the space of Whittaker functionals is at most one-dimensional. However, this fact does not hold for the disconnected $\mathrm{O}(2n, F)$, and we need to adjust the definition of Whittaker character as follows. In this case, the Levi factor of the normalizer of U in $\mathrm{O}(2n, F)$ is $\mathrm{GL}_1(F) \times \cdots \times \mathrm{GL}_1(F) \times \mathrm{O}(2, F)$ and there exists $\alpha \in \mathrm{O}(2, F)\backslash \mathrm{SO}(2, F)$ normalizing ψ. Observe that the order of α is 2. We extend ψ to a character $\tilde{\psi}$ of $\tilde{U} = U \rtimes \langle \alpha \rangle$ by $\tilde{\psi}(\alpha) = 1$. With this extension, the space of Whittaker functionals for any irreducible representation of $\mathrm{O}(2n, F)$ is at most one-dimensional.

Now, let $P = MN$ be a parabolic subgroup of G. If σ is an irreducible generic representation of M, then one can construct a Whittaker functional on $i_P^G \sigma$ (see [24, Proposition 3.1] and equation (3.11)); in other words, the induced representation is ψ-generic as well. We use this fact later, in Section 3.2.

3 The Gelfand–Graev representation

Continuing with split G, let U be a maximal unipotent subgroup of G and fix a nondegenerate character $\psi : U \to \mathbb{C}^\times$. The Gelfand–Graev representation of G is the compactly induced representation $\mathrm{c}-\mathrm{ind}_U^G(\psi)$. However, if $G = \mathrm{O}(2n, F)$, the pair (U, ψ) is replaced by the pair $(\tilde{U}, \tilde{\psi})$ in this definition. With this modification for $\mathrm{O}(2n, F)$ in mind, the Gelfand–Graev representation is the "universal" ψ-generic representation: Every ψ-generic representation of G appears as a quotient (with multiplicity one).

From this point on, we assume that the cuspidal representation τ—used to define the Bernstein component $\mathfrak{s}$ in Section 2.6—is generic. We let Π denote $(\mathrm{c}-\mathrm{ind}_U^G(\psi))$ viewed as an $\mathcal{H}$-module. Our goal is to determine the structure of Π.

We begin by investigating the structure of Π as an $\mathcal{A}$-module. We point out that the proof of the following proposition applies, without modification, to any split reductive p-adic group.

Proposition 3.1 *As $\mathcal{A}$-modules, we have $\Pi \cong \mathcal{A}$.*

Proof The $\mathcal{H}$-module Π is given by $\mathrm{Hom}_G(\Gamma_{\mathfrak{s}}, \mathrm{c}-\mathrm{ind}_U^G(\psi))$, where $\Gamma_{\mathfrak{s}} = i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0))$. Recall that σ_0 was taken to be an arbitrary irreducible constituent of $\sigma|_{M^\circ}$. However, having now fixed the Whittaker datum for M (and thus for M°), there exists a unique irreducible summand of $\sigma|_{M^\circ}$ which is ψ-generic. Thus, from now on, we assume that σ_0 is this unique ψ-generic constituent of $\sigma|_{M^\circ}$.

To view Π as an $\mathcal{A} = \mathrm{End}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0))$-module, we use the Bernstein version of Frobenius reciprocity:

$$\begin{aligned}\Pi &= \mathrm{Hom}_G(i_P^G(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0)), \mathrm{c}-\mathrm{ind}_U^G(\psi))\\ &= \mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0), r_{\overline{N}}(\mathrm{c}-\mathrm{ind}_U^G(\psi)));\end{aligned}$$

here, $r_{\overline{N}}$ denotes the Jacquet functor with respect to $\overline{P} = M\overline{N}$, the parabolic opposite to P.

We now use the fact that $r_{\overline{N}}(\mathrm{c}-\mathrm{ind}_U^G(\psi))$ is isomorphic to the Gelfand–Graev representation of M, $\mathrm{c}-\mathrm{ind}_{U\cap M}^M(\psi)$ (see [6, Section 2.2]). Furthermore, with the above choice of σ_0, the representation $\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0)$ is precisely the sum of all maximal $({}^m\sigma_0)$-isotypic components of $\mathrm{c}-\mathrm{ind}_{U\cap M}^M(\psi)$, where ${}^m\sigma_0$ ranges over the set of all M-conjugates of σ_0. Indeed, $\mathrm{c}-\mathrm{ind}_{U\cap M}^M(\psi)$ is itself induced from the Gelfand–Graev representation of M°, $\mathrm{c}-\mathrm{ind}_{U\cap M}^{M^\circ}(\psi)$. Since σ_0 appears with multiplicity one, and no other m-conjugate of σ_0 is generic, we have $\mathrm{c}-\mathrm{ind}_{U\cap M}^{M^\circ}(\psi) \cong \sigma_0 \oplus \sigma_0^\perp$, where $\sigma_0^\perp$ is a representation which contains no M-conjugate of σ_0. Inducing to M, we get $\mathrm{c}-\mathrm{ind}_{U\cap M}^M(\psi) = \mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0) \oplus \mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0^\perp)$, which proves the above claim about isotypic components. Thus, viewed as an $\mathcal{A}$-module, Π is isomorphic to

$$\begin{aligned}
&\mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0), r_{\overline{N}}(\mathrm{c}-\mathrm{ind}_U^G(\psi)))\\
&\quad = \mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0), \mathrm{c}-\mathrm{ind}_{U\cap M}^M(\psi))\\
&\quad = \mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0), \mathrm{c}-\mathrm{ind}_{M^\circ}^M(\sigma_0)) = \mathcal{A}. \qquad \blacksquare
\end{aligned}$$

Remark 3.2 We point out that the above differs from the proof of the analogous statement in [11]. It is shown there that any $\mathcal{H}$-module Π which is

(i) projective;
(ii) finitely generated; and which satisfies
(iii) $\dim \mathrm{Hom}_{\mathcal{H}}(\Pi, \pi) \leq 1$ for any principal series representation π

is isomorphic to $\mathcal{A}$ when viewed as an $\mathcal{A}$-module (see [11, Lemmas 2.2 and 2.3]). The Gelfand–Graev representation can be shown to satisfy properties (i)–(iii): Property (i) is provided by Corollary 8.6 of [11]; (ii) is proved in [6], and (iii) follows from the multiplicity one property of generic representations. In Section 4, we present another useful application of the above approach to proving that an $\mathcal{H}$-module is isomorphic to $\mathcal{A}$.

Proposition 3.1 suggests the following approach to determine the $\mathcal{H}$-module structure of Π: First, we find all possible $\mathcal{H}$-module structures on $\mathcal{A}$. After that, we need to only determine which one of those structures describes Π. In the following subsection, we compute the possible $\mathcal{H}$-structures on $\mathcal{A}$.

3.1 $\mathcal{H}$-module structures on $\mathcal{A}$

In order to treat the case of general Bernstein components—and not just those described in Section 2.6—we work in a slightly more general setting in this section. We thus investigate the possible $\mathcal{H}$-module structures on $\mathcal{A}$ (where $\mathcal{H}$ is generated by $T_0, \dots, T_n$ over $\mathcal{A}$), but we assume that $\mathcal{A} = \mathcal{A}'[X_1^\pm, \dots, X_n^\pm]$, where $\mathcal{A}'$ is an integral domain containing $\mathbb{C}$ as a subring. For Bernstein components described in Section 2.6, we have $\mathcal{A}' = \mathbb{C}$; in general, $\mathcal{A}'$ itself is a (Laurent) polynomial ring over $\mathbb{C}$.

First, assume that we are in Case (i) (see Section 2.6). Then the situation is precisely the one treated in [11], and the possible $\mathcal{H}$-module structures on $\mathcal{A}$ are determined in Section 2.2 there. We have the following.

Proposition 3.3 (Case (i)) *Let Π be an $\mathcal{H}$-module which is isomorphic to $\mathcal{A}$ as an $\mathcal{A}$-module. Then $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_{S_n}} \varepsilon$, where ε is a one-dimensional representation of $\mathcal{H}_{S_n}$.*

Here, $\mathcal{H}_{S_n}$ denotes the finite-dimensional algebra generated by $T_1, \dots, T_{n-1}$; we have $\mathcal{H} = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}_{S_n}$. Furthermore, $\mathcal{H}_{S_n}$ has precisely two one-dimensional representations:

$$\varepsilon_{-1} : T_i \mapsto -1 \quad \text{for } i = 1, \dots, n-1; \quad \text{and}$$
$$\varepsilon_{q^t} : T_i \mapsto q^t \quad \text{for } i = 1, \dots, n-1.$$

We now treat Cases (ii) and (iii) simultaneously. Recall that, in these cases, the algebra $\mathcal{H}$ is described by an affine Coxeter diagram of type $\tilde{C}_n$. We let $\mathcal{H}_0$ and $\mathcal{H}_n$ denote the algebras obtained by removing the vertices which correspond to T_0 and T_n, respectively. In other words, $\mathcal{H}_0$ is generated by $T_1, \dots, T_n$ as an $\mathcal{A}$-algebra, whereas $\mathcal{H}_n$ is generated by $T_0, \dots, T_{n-1}$. Note that we have $\mathcal{H} = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}_n = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}_0$. We now prove the following result.

Proposition 3.4 (Cases (ii) and (iii)) *Let Π be an $\mathcal{H}$-module which is isomorphic to $\mathcal{A}$ as an $\mathcal{A}$-module. Then*

$$\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_0 \quad \textit{or} \quad \Pi \cong \mathcal{H} \otimes_{\mathcal{H}_n} \varepsilon_n.$$

Here, ε_0 (resp. ε_n) is a one-dimensional representation of $\mathcal{H}_0$ (resp. $\mathcal{H}_n$).

Proof We first restrict our attention to the subalgebra generated by $T_1, \dots, T_{n-1}$, which is contained in both $\mathcal{H}_0$ and $\mathcal{H}_n$. This is precisely the algebra $\mathcal{H}_{S_n}$ discussed in [11]. The possible $\mathcal{H}_{S_n}$-structures on $\mathcal{A}$ are determined in Section 2.2 there. To summarize the relevant results, there exists an invertible element $g_0 \in \mathcal{A}$ on which the operators $T_1, \dots, T_n$ act by the same scalar, either q^t or -1.

We now determine how T_0 and T_n act on g_0. Since g_0 is invertible, we have $T_n g_0 = f g_0$ for some $f \in \mathcal{A}$. Recall that T_n satisfies the quadratic relation

$$T_n^2 = (q^r - 1)T_n + q^r$$

as well as the commutation relation

$$T_n f - f^\vee T_n = \left((q^r - 1) + \frac{1}{X_n}(\sqrt{q}^{r+s} - \sqrt{q}^{r-s})\right) \frac{f - f^\vee}{1 - 1/X_n^2}.$$

Here, and throughout the proof, we let $r = s = 0$ if we are considering Case (ii). Recall that $f^\vee$ denotes the function $f^\vee(X_1, \dots, X_n) = f(X_1, \dots, X_{n-1}, \frac{1}{X_n})$. Using the above and comparing the two sides of $T_n^2 g_0 = (q^r - 1)T_n g_0 + q^r g_0$, we get

$$f f^\vee = (q^r - 1)\frac{X_n f^\vee - \frac{1}{X_n} f}{X_n - \frac{1}{X_n}} - (\sqrt{q}^{r+s} - \sqrt{q}^{r-s})\frac{f - f^\vee}{X_n - \frac{1}{X_n}} + q^r.$$

To simplify notation, we now set $b = q^r - 1$ and $c = (\sqrt{q}^{r+s} - \sqrt{q}^{r-s})$. We also temporarily drop the index n, writing X instead of X_n. Clearing out the denominators, we rearrange the above equation into

$$(*) \qquad (X^2 - 1) f f^\vee = b(X^2 f^\vee - f) - c(Xf - Xf^\vee) + q^r(X^2 - 1).$$

Our first goal is to find the possible solutions $f \in \mathcal{A}$ of this equation.

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Lemma 3.5 *The above equation has the following solutions:*

(i) $f = b + cX^{-1} + bX^{-2} + \cdots + cX^{1-2d} + q^t X^{-2d}$, $\quad d \in \mathbb{Z}_{>0}$

(ii) $f = b + cX^{-1} + bX^{-2} + \cdots + cX^{1-2d} - X^{-2d}$, $\quad d \in \mathbb{Z}_{>0}$

(iii) $f = b + cX^{-1} + bX^{-2} + \cdots + bX^{-2d} \pm \sqrt{q}^{r\pm s} X^{-2d-1}$, $\quad d \in \mathbb{Z}_{\geq 0}$

(iv) $f = \mp\sqrt{q}^{r\pm s} X^{2d+1} - bX^{2d} - cX^{2d-1} - \cdots - cX$, $\quad d \in \mathbb{Z}_{\geq 0}$

(v) $f = -q^r X^{2d} - cX^{2d-1} - \cdots - bX^2 - cX$, $\quad d \in \mathbb{Z}_{>0}$

(vi) $f = X^{2d} - cX^{2d-1} - \cdots - bX^2 - cX$, $\quad d \in \mathbb{Z}_{>0}$

along with the constant solutions $f = q^t$ and $f = -1$. ■

Proof Each $f \in \mathcal{A}$ can be written as

$$(\dagger) \qquad f = a_k X^k + a_{k-1}X^{k-1} + \cdots + a_0 + a_{-1}X^{-1} + \cdots + a_{-l}X^{-l}$$

for some functions $a_{-l}, \ldots, a_k \in \mathcal{A}'[X_1^{\pm}, \ldots, X_{n-1}^{\pm}]$, with $a_k, a_{-l} \neq 0$. We write $\operatorname{maxdeg}(f)$ for k and $\operatorname{mindeg}(f)$ for $-l$. Now, let f be a solution of (*). We begin our analysis of (*) by solving some special cases. We claim the following:

$$(3.1) \qquad \begin{array}{llll} \text{If} & f = a_0, & \text{then} & a_0 = q^r \text{ or } a_0 = -1. \\ \text{If} & f = a_1 X, & \text{then} & a_1 = \mp\sqrt{q}^{r\pm s}. \\ \text{If} & f = a_0 + a_{-1}X^{-1} \text{ and } a_{-1} \neq 0, & \text{then} & a_0 = b \text{ and } a_{-1} = \pm\sqrt{q}^{r\pm s}. \end{array}$$

To verify this, we first look at solutions $f = a_0$. In this case, the equation (*) reduces to $a_0^2 = ba_0 + q^r$. This equation has two constant solutions, $a_0 = -1$ and $a_0 = q^r$. These are also the only solutions, since $\mathcal{A}$ has no zero divisors. When $f(X) = a_1 X$, the equation becomes $a_1^2 + a_1 c - q^r = 0$. Again, the only two solutions of this equation are the constant ones: $a_1 = \mp\sqrt{q}^{r\pm s}$. Finally, when $f = a_0 + a_{-1}X^{-1}$, the equation reduces to the following system:

$$a_1 b = a_1 a_0 \quad \text{and} \quad a_0^2 + a_{-1}^2 = a_0 b + a_{-1}c + q^r.$$

Since we are assuming that $a_1 \neq 0$, the first equation gives us $a_0 = b$, and then the second becomes $a_{-1}^2 - ca_{-1} - q^r = 0$. Again, we have two solutions: $a_{-1} = \pm\sqrt{q}^{r\pm s}$.

Next, when f is a solution of (*) given by (†), we observe

$$(3.2) \qquad k \text{ and } l \text{ cannot both be positive.}$$

Indeed, let LHS and RHS denote the left-hand side and the right-hand side of (*), respectively. We then have $\operatorname{maxdeg}(\text{LHS}) = k + l + 2$, whereas $\operatorname{maxdeg}(\text{RHS}) \leq \max\{l + 2, k + 1, 2\}$. Therefore, equality of degrees cannot be achieved unless $k \leq 0$ or $l \leq 0$. In fact, the same argument gives us a slightly stronger statement in one case:

$$(3.3) \qquad \text{If } k > 0, \text{ then } a_0 = 0.$$

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Finally, we make use of the following fact, which is readily verified by direct computation:

(3.4) For any positive integer d, f is a solution of (*) if and only if $X^{2d} f - R_d$ is also a solution.

Here, $R_d = \dfrac{bX^2 + cX}{X^2 - 1}(X^{2d} - 1) = bX^{2d} + cX^{2d-1} + \cdots + bX^2 + cX$.

We are now ready to find all the solutions. By (3.2), any solution of f contains either only positive powers of X, or only nonpositive. We therefore consider two separate cases.

Case A: f has only nonpositive powers, i.e., $f = a_0 + a_{-1}X^{-1} + \cdots + a_{-l}X^{-l}$.

Let $d = \lfloor l/2 \rfloor$. We use (3.4) and look at another solution, $g = X^{2d} f - R_d$.

We first assume that $l = 2d$ is even. In this case, g only has nonnegative powers of X, but it has a nonzero constant term, a_{-l}. Therefore, (3.3) shows that the coefficients next to the positive powers must be zero: $a_0 - b = a_{-1} - c = \cdots = a_{-l+1} - c = 0$. Now, (3.1) shows that there are only two possibilities for the constant term: $a_{-l} = q^t$ or $a_{-l} = -1$. We thus get two solutions:

$$f = b + cX^{-1} + bX^{-2} + \cdots + cX^{1-2d} + q^t X^{-2d} \quad \text{and}$$
$$f = b + cX^{-1} + bX^{-2} + \cdots + cX^{1-2d} - X^{-2d}.$$

Next, assume that $l = 2d + 1$ is odd. Now, g has a nonzero coefficient (i.e., a_{-l}) next to X^{-1}, so by (3.2) the coefficients next to positive powers must be equal to 0. This gives us $a_0 = b, a_{-1} = c, \ldots, a_{2-l} = c$. Furthermore, g is thus of the form $a_{1-l} + a_{-l}X^{-1}$, so we can read off the coefficients a_{1-l} and a_{-l} from (3.1). We thus arrive at two more solutions:

$$f = b + cX^{-1} + bX^{-2} + \cdots + bX^{-2d} \pm \sqrt{q}^{r \pm s} X^{-2d-1}.$$

Case B: f only has positive powers, i.e., $f = a_k X^k + \cdots + a_1 X$.

This time, we set $d = \lfloor k/2 \rfloor$ and use (3.4) to obtain the solution $g = \dfrac{1}{X^{2d}}(f + R_d)$.

First, assume that $k = 2d + 1$ is odd. Then g has a nonzero coefficient (i.e., a_k) next to X, so (3.2) and (3.3) imply that all the lower coefficients are zero. This immediately gives us $a_1 = -c, a_2 = -b, \ldots, a_{2d} = -b$. Furthermore, we have $g = a_k X$, so (3.1) shows that we have two possibilities for a_k. We therefore get two solutions:

$$f = \mp \sqrt{q}^{r \pm s} X^{2d+1} - bX^{2d} - cX^{2d-1} - \cdots - cX.$$

Finally, assume that $k = 2d$ is even. First, if $k > 2$, consider another solution $g' = X^{2-2d}(f + R_{2d-2})$. Now, g' has a nonzero coefficient (i.e., a_k) next to X^2, so the coefficient next to nonpositive powers of X have to be 0 by (3.2) and (3.3). This gives us $a_1 = -c, a_2 = -b, \ldots, a_{2d-2} = -b$. In particular, this shows that $g = (a_k + b) + (a_{k-1} + c)X^{-1}$. Since $a_k + b \neq b$ (i.e., $a_k \neq 0$), (3.1) shows that we have only two possibilities:

$$a_{k-1} + c = 0 \quad \text{and} \quad a_k + b \in \{q^r, -1\}.$$

In other words, $a_{k-1} = -c$ and $a_k \in \{-q^r, 1\}$. We thus get the remaining solutions:

$$f = -q^r X^{2d} - cX^{2d-1} - \cdots - bX^2 - cX \quad \text{and} \quad f = X^{2d} - cX^{2d-1} - \cdots - bX^2 - cX.$$

■

We continue the proof of Proposition 3.4. We have just proved that $T_n g_0 = f g_0$ where $f \in \mathcal{A}$ is one of the elements listed in Lemma 3.5. First, assume that f is one of the constant solutions, i.e., $f = -1$ or $f = q^r$. Then g_0 is an invertible element of $\mathcal{A}$ on which $T_1, \ldots, T_{n-1}, T_n$ all act as scalars. In other words, we have a one-dimensional representation ε_0 of the algebra $\mathcal{H}_0$. Since $\mathcal{H} = \mathcal{A} \otimes_{\mathbb{C}} \mathcal{H}_0$, it follows that the corresponding $\mathcal{H}$-module structure on $\mathcal{A}$ is isomorphic to

$$\mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_0.$$

Now, if f is of type (i) or (ii) listed in the statement of Lemma 3.5, set

$$g_1 = (X_1 X_2 \cdot \cdots \cdot X_n)^{-d} g_0.$$

Since $(X_1 X_2 \cdot \cdots \cdot X_n)^{-d}$ commutes with $T_1, \ldots, T_{n-1}$, g_1 is still an eigenvector for each of these operators. We claim that g_1 is an eigenvector for T_n as well. Indeed, using the appropriate commutation relation and the fact that T_n commutes with $X_1, \ldots, X_{n-1}$, we get

$$\begin{aligned}
T_n g_1 &= (X_1 X_2 \cdot \cdots \cdot X_{n-1})^{-d} \cdot T_n X_n^{-d} g_0 \\
&= (X_1 X_2 \cdot \cdots \cdot X_{n-1})^{-d} \left(X_n^d T_n + \frac{bX_n + c}{X_n^2 - 1} (X_n^{-d} - X_n^d) \right) g_0 \\
&= (X_1 X_2 \cdot \cdots \cdot X_{n-1})^{-d} \left(X_n^d f + \frac{bX_n + c}{X_n^2 - 1} (X_n^{-d} - X_n^d) \right) g_0 \\
&= (X_1 X_2 \cdot \cdots \cdot X_{n-1})^{-d} \left(X_n^{2d} f - \frac{bX_n + c}{X_n^2 - 1} (X_n^{2d} - 1) \right) g_0.
\end{aligned}$$

Simplifying the expression in the parentheses, we obtain λX_n^{-d}, so that $T_n g_1 = \lambda g_1$, where $\lambda = q^t$ (resp. -1) when f is of type (i) (resp. (ii)). We have thus once more found a common eigenvector for $T_1, \ldots, T_{n-1}, T_n$. Again, we deduce that the corresponding $\mathcal{H}$-module structure is isomorphic to $\mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_0$, where ε_0 is a one-dimensional representation of $\mathcal{H}_0$.

When f is of type (v) or (vi), we use the same argument and arrive at the same conclusion. The only difference in this case is that we have to set $g_1 = (X_1 X_2 \cdot \cdots \cdot X_n)^d g_0$ in order to obtain a common eigenvector for $T_1, \ldots, T_{n-1}, T_n$.

In the remaining cases—that is, when f is of type (iii) or (iv)—we cannot find such an eigenvector, but we claim that we can find an invertible $g_1 \in A$ which is a common eigenvector for $T_0, T_1, \ldots, T_{n-1}$. Just like in the previous cases, this will imply that the $\mathcal{H}$-structure on $\mathcal{A}$ is isomorphic to $\mathcal{H} \otimes_{\mathcal{H}_n} \varepsilon_n$ for some one-dimensional representation ε_n of $\mathcal{H}_n$.

If $T_n g_0 = f g_0$ with f of type (iii), we set $g_1 = (X_1 X_2 \cdot \cdots \cdot X_n)^{-d} g_0$. If f is of type (iv), let $g_1 = (X_1 X_2 \cdot \cdots \cdot X_n)^{d+1} g_0$. In both cases, g_1 is an eigenvector for $T_1, \ldots, T_{n-1}$ and a computation analogous to the one we carried out in for Cases (i) and (ii) shows

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that we have

$$T_n g_1 = (b \pm \sqrt{q}^{r\pm s} X_n^{-1}) g_1.$$

The following lemma then shows that g_1 is also an eigenvector for T_0 and thus concludes the proof of Proposition 3.4.

Lemma 3.6 *Let g be an invertible element of $\mathcal{A}$, which is an eigenvector for $T_1, \dots, T_{n-1}$, such that $T_n g = (b \pm \sqrt{q}^{r\pm s} X_n^{-1}) g$. Then g is also an eigenvector for T_0.*

Proof Recall that $T_0 = \sqrt{q}^{s+2(n-1)t+r} X_1 T_w^{-1}$, with $T_w = T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1$. In both cases, all the operators $T_1, \dots, T_{n-1}$ act by the same scalar $\lambda \in \{-1, q^t\}$. We therefore have

$$T_0 g = \sqrt{q}^{s+2(n-1)t+r} \lambda^{-(n-1)} X_1 T_1^{-1} \cdot \dots \cdot T_{n-1}^{-1} T_n^{-1} g.$$

We now recall that $T_n^{-1} = \frac{1}{q^r}(T_n - b)$; this follows from the quadratic relation for T_n. Therefore, by the assumption in the statement of the lemma, $T_n^{-1} g = \pm \sqrt{q}^{\pm s - r} X_n^{-1}$. Thus,

$$T_0 g = \mu \cdot \lambda^{-(n-1)} \cdot \sqrt{q}^{2(n-1)t} X_1 T_1^{-1} \cdot \dots \cdot T_{n-1}^{-1} X_n^{-1} g, \tag{3.5}$$

with $\mu \in \{-1, q^s\}$. Finally, it remains to notice that, for every $i = 1, \dots, n-1$, we have

$$T_i^{-1} X_{i+1}^{-1} = \frac{1}{q^t} X_i^{-1} T_i. \tag{3.6}$$

Indeed, from the quadratic relation, we have $T_i^{-1} = \frac{1}{q^t}(T_i - (q^t - 1))$. Combining this with the commutation relation for T_i, we get (3.6). Successively applying (3.6) to (3.5) (and taking into account that each T_i acts on g by λ), we get

$$T_0 g = \mu g,$$

which we needed to prove. Notice that the possible eigenvalues are precisely the zeros of $(x - q^s)(x+1) = 0$, the quadratic equation satisfied by T_0. ∎

The above lemma shows that, in Cases (iii) and (iv), we have an invertible element $g_1 \in \mathcal{A}$ which is a common eigenvector for $T_0, T_1, \dots, T_n$. Consequently, the $\mathcal{H}$-module structure on $\mathcal{A}$ is given by $\mathcal{H} \otimes_{\mathcal{H}_n} \varepsilon_n$ for some one-dimensional representation ε_n of $\mathcal{H}_n$. This concludes the proof of Proposition 3.4.

In view of Proposition 3.4, there are eight candidates for the $\mathcal{H}$-structure (four, if $n = 1$): First, we may take the tensor product over $\mathcal{H}_0$ or $\mathcal{H}_n$; after that, there are four one-dimensional representations of $\mathcal{H}_0$ (resp. $\mathcal{H}_n$) to choose from. To verify this, note that the braid relations imply that—in any one-dimensional representation—the operators $T_1, \dots, T_{n-1}$ act by the same scalar, which has to be a zero of the quadratic relation satisfied by T_i: $(x - q^t)(x+1) = 0$. We therefore have two possibilities for the action of the operators T_i, and two additional possibilities (again, the zeros of the quadratic relation) for T_n (resp. T_0). For example, the one-dimensional representations of $\mathcal{H}_0$

are given by

$$\begin{aligned} &\varepsilon_{-1,-1} : \{T_n \mapsto -1, T_i \mapsto -1\}, && \varepsilon_{q^r,-1} : \{T_n \mapsto q^r, T_i \mapsto -1\}, \\ &\varepsilon_{-1,q^t} : \{T_n \mapsto -1, T_i \mapsto q^t\}, && \varepsilon_{q^r,q^t} : \{T_n \mapsto q^r, T_i \mapsto q^t\}. \end{aligned}$$

Corollary 3.7 *General case. Let Π be an $\mathcal{H}$-module which is isomorphic to $\mathcal{A}$ as an $\mathcal{A}$-module. Then there exists a finite subalgebra $\mathcal{H}_W \cong \mathbb{C}[W]$, where W is a finite group, such that $\mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_W$, and*

$$\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_W} \varepsilon,$$

where ε is a one-dimensional representation of $\mathcal{H}_W$.

Proof Recall that $\mathcal{H}$ is a tensor product of Hecke algebras each of which is isomorphic to the Iwahori Hecke algebra of GL_n or an algebra of type $\tilde{C}_n$ with unequal parameters. Propositions 3.3 and 3.4 deal with these two cases, with additional flexibility that allows $\mathcal{A} = \mathcal{A}'[X_1^{\pm}, \dots, X_n^{\pm}]$, where $\mathcal{A}' = \mathbb{C}[Y_1^{\pm}, \dots, Y_m^{\pm}]$. Thus, the corollary follows by repeated application of these two propositions. ∎

3.2 The Gelfand–Graev module

To complete the analysis of the Gelfand–Graev representation, we need to determine which of the $\mathcal{H}$-module structures from the previous section is isomorphic to $\Pi = (\mathrm{c}-\mathrm{ind}_U^G(\psi))_{\mathfrak{s}}$. We consider Cases (i)–(iii) separately.

Case (i). Let δ be the unique irreducible subrepresentation of $\rho\nu^{\frac{n-1}{2}} \times \rho\nu^{\frac{n-3}{2}} \times \cdots \times \rho\nu^{\frac{1-n}{2}}$. Then $\pi = \delta \rtimes \tau$ is an irreducible generic representation. The corresponding $\mathcal{H}$-module is one-dimensional: By the Bernstein version of Frobenius reciprocity, we have

(3.7)
$$\begin{aligned} \mathrm{Hom}_G(\Gamma_{\mathfrak{s}}, \pi) = \mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\rho \otimes \cdots \otimes \rho \otimes \tau), \nu^{\frac{1-n}{2}}\rho \otimes \cdots \otimes \nu^{\frac{n-1}{2}}\rho \otimes \tau \\ \oplus \nu^{\frac{1-n}{2}}\rho^\vee \otimes \cdots \otimes \nu^{\frac{n-1}{2}}\rho^\vee \otimes \tau). \end{aligned}$$

Since $\rho^\vee$ is not an unramified twist of ρ in this case, the above Hom-space is only one-dimensional. By Proposition 3.3, $\mathrm{Hom}_G(\Gamma_{\mathfrak{s}}, \Pi)$ is isomorphic to either $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_{S_n}} \varepsilon_{-1}$ or $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_{S_n}} \varepsilon_{q^t}$. To determine which, we need only look at the action of $\mathcal{H}$ on the one-dimensional module π. We now need to examine the definition of the operators T_i, $i = 1, \dots, n-1$. In [17], T_i is defined in Section 5.2 by the formula

$$T_i = R_i + (q^t - 1)\frac{X_i/X_{i+1}}{X_i/X_{i+1} - 1}. \tag{3.8}$$

The intertwining operator R_i has a pole at 0, and a zero at the point of reducibility (see [17, Section 1.8]). Since $\nu^{\frac{3-n}{2}-i}\rho \times \nu^{\frac{3-n}{2}-i+1}\rho$ reduces, the operator R_i acts by 0 in this case. It therefore remains to determine the action of X_i/X_{i+1}. Equation (3.7) shows that it suffices to determine the action of X_i/X_{i+1} on

$$\mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\rho \otimes \cdots \otimes \rho \otimes \tau), \nu^{\frac{1-n}{2}}\rho \otimes \cdots \otimes \nu^{\frac{n-1}{2}}\rho \otimes \tau).$$

Recalling the definition of X_i (Section 2.6), we immediately see that X_i/X_{i+1} acts by

$$\frac{(|\varpi|^{\frac{3-n}{2}-i})^t}{(|\varpi|^{\frac{3-n}{2}-i+1})^t} = \frac{q^{t(\frac{n-3}{2}+i)}}{q^{t(\frac{n-3}{2}+i-1)}} = q^t.$$

This implies that T_i also acts by $(q^t-1)\dfrac{q^t}{q^t-1} = q^t$. Since π is a quotient of Π, we conclude that we must have $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_{S_n}} \varepsilon_{q^t}$.

Case (iii). In this situation, the $\mathfrak{s}$-component of the Gelfand–Graev representation has two irreducible generic representations whose $\mathcal{H}$-module is one-dimensional. These are the two (generalized) Steinberg representations: π and π', which are the unique irreducible subrepresentations of

$$\nu^{\alpha+n-1}\rho \times \cdots \times \nu^{\alpha}\rho \rtimes \tau \quad \text{and} \quad \nu^{\beta+n-1}\rho^- \times \cdots \times \nu^{\beta}\rho^- \rtimes \tau,$$

respectively. Recall that α (resp. β) is the unique positive real number such that $\nu^\alpha\rho \rtimes \tau$ (resp. $\nu^\beta\rho^- \rtimes \tau$) reduces (see Section 2.6). We now compare the action of the operators $T_0, \dots, T_n$ on these two representations—that is, on $\mathrm{Hom}_G(\Gamma_{\mathfrak{s}}, \pi)$ and $\mathrm{Hom}_G(\Gamma_{\mathfrak{s}}, \pi^-)$, where $\Gamma_{\mathfrak{s}}$ is the projective generator defined in Section 2.4.

We start by analyzing the action on π. We first focus on T_i, $i = 1, \dots, n-1$. Again, T_i is defined by (3.8), and once more, the operator R_i acts by 0. By the Bernstein version of Frobenius reciprocity, we have

$$\mathrm{Hom}_G(\Gamma_{\mathfrak{s}}, \pi) = \mathrm{Hom}_M(\mathrm{c}-\mathrm{ind}_{M^\circ}^M(\rho \otimes \cdots \otimes \rho \otimes \tau), \nu^{-\alpha-n+1}\rho \otimes \cdots \otimes \nu^{-\alpha}\rho \otimes \tau).$$

We immediately see that X_i/X_{i+1} acts by

$$\frac{(|\varpi|^{-\alpha-n+i})^t}{(|\varpi|^{-\alpha-n+i+1})^t} = \frac{q^{t(\alpha+n-i)}}{q^{t(\alpha+n-i-1)}} = q^t.$$

Again, this shows that T_i acts by $(q^t-1)\dfrac{q^t}{q^t-1} = q^t$. For T_n, we have a similar formula:

$$T_n = R_n + (q^r-1)\frac{X_n\left(X_n - \dfrac{q^{t\beta}-q^{t\alpha}}{q^r-1}\right)}{X_n^2-1}. \tag{3.9}$$

Once more, R_n acts by 0, and X_n acts by $(|\varpi|^{-\alpha})^t = q^{t\alpha}$. Recalling that $r = t(\alpha+\beta)$, we see that T_n acts by q^r. Finally, since

$$T_0 = \sqrt{q}^{r+2t(n-1)+s} X_1 T_1^{-1} \cdot \dots \cdot T_{n-1}^{-1} T_n^{-1} T_{n-1}^{-1} \cdot \dots \cdot T_1^{-1},$$

and since X_1 acts by $q^{(\alpha+n-1)t}$, we see that T_0 acts by $\dfrac{\sqrt{q}^{r+2t(n-1)+s}}{q^{2t(n-1)} \cdot q^r} q^{(\alpha+n-1)t} = q^s$.

We do the same with π^-. Again, X_i/X_{i+1} acts by q^t, which shows that T_i acts by q^t as well. This time X_n acts by $-q^{t\beta}$: Recall that $\rho^- = \chi_0 \otimes \rho$ with $X_n(\chi_0) = -1$, so $X_n(\chi_0\nu^{-\beta}) = -q^{t\beta}$. Repeating the above calculations, we now see that T_n acts by q^r, whereas T_0 acts by -1.

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The above analysis allows us to single out the $\mathcal{H}$-module structure on Π. Since T_0 does not act by the same scalar on π and π^-, we deduce that $\Pi = \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon$ for some one-dimensional representation ε of $\mathcal{H}_0$. Now, since every T_i ($i = 1, \ldots, n-1$) acts by q^t and T_n acts by q^r, we deduce that $\Pi = \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_{q^r, q^t}$ (see the end of Section 3.1 for notation).

Case (ii) The first part of our analysis remains the same as in Case (iii). The representation

$$\nu^{n-1}\rho \times \nu^{n-2}\rho \times \cdots \times \rho \rtimes \tau$$

has two irreducible subrepresentations (both of which are in discrete series when $n > 1$, and tempered when $n = 1$), only one of which is generic. Denote the generic subrepresentation by π. Let π^- denote the generic representation resulting from an analogous construction, when ρ is replaced by ρ^-. Again, the $\mathcal{H}$-modules corresponding to π and π^- are one-dimensional, and the same calculations we used in Case (iii) show that the operators T_i, $i = 1, \ldots, n-1$, act by q^t. This leaves us four possible $\mathcal{H}$ structures to consider

(3.10)

$$\begin{aligned}
&\mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_0, \quad \text{with} \quad \varepsilon_0(T_n) = \pm 1 \quad (\text{and } \varepsilon_0(T_i) = q^t, i = 1, \ldots, n-1); \quad \text{and}\\
&\mathcal{H} \otimes_{\mathcal{H}_n} \varepsilon_n, \quad \text{with} \quad \varepsilon_n(T_0) = \pm 1 \quad (\text{and } \varepsilon_n(T_i) = q^t, i = 1, \ldots, n-1).
\end{aligned}$$

So far, we have been able to view Case (ii) as a special instance of Case (iii) which occurs when $r = s = 0$. However, to obtain an explicit description of the Gelfand–Graev module, we need more information than we used above in Case (iii). The reason is that the standard intertwining operator $\chi\rho \rtimes \tau \to \chi^{-1}\rho^\vee \rtimes \tau$ no longer has a pole when $X_n(\chi) = \pm 1$. In Case (iii), the operator R_n (see formula (3.9))—which is constructed from the standard intertwining operator—vanishes at the point of reducibility, and the action of T_n is determined by the action of the function

$$(q^r - 1)\frac{X_n\left(X_n - \dfrac{q^{t\beta} - q^{t\alpha}}{q^r - 1}\right)}{X_n^2 - 1}$$

used to remove the poles of R_n. In this case, however, R_n no longer vanishes and is regular at the point of reducibility; consequently, the above function does not appear in the construction and we have $T_n = R_n$. We know that this operator acts by 1 or -1 on the $\mathcal{H}$-modules π and π^-, but we still have a certain amount of freedom in our choices. Indeed, as one verifies easily, the operator $T_n' = (-1)^e X_n^f T_n$ (where $e \in \{0, 1\}$ and $f \in \mathbb{Z}$) satisfies the same relations as T_n. Therefore, we obtain the same Hecke algebra if we replace T_n by T_n', but the action of T_n' on π obviously differs from the action of T_n.

In fact, we know that X_n acts on π by 1, and on π^- by -1. Therefore, X_n^2 acts by 1 on both, so replacing T_n by $X_n^2 T_n$ does not affect our description of the Gelfand–Graev module. We thus have four choices that affect the description ($e = 0$ or 1; f even or odd), and as we vary the four choices, the description of the Gelfand–Graev module varies through all the four possibilities described in (3.10).

This discussion shows that—to determine the action explicitly—we need to specify the choices appearing in the construction of the operator R_n. We now explain one possible normalization using Whittaker models. To be concrete, we now focus on $G = \mathrm{SO}(2N+1)$; the same approach is possible when G is symplectic or even orthogonal. We also specialize our discussion to the case $n = 1$ to simplify notation (thus, the cuspidal representation which defines the component is $\rho \otimes \tau$); the general case is analogous and follows from this one. We thus drop the subscripts and write T, X instead of T_n, X_n.

We fix a nondegenerate character ψ of the unipotent radical U of $G = \mathrm{SO}(2N+1)$. Let V_ρ denote the space of the representation ρ, and let λ be a ψ-Whittaker functional on V_ρ: $\lambda(\rho(u)v) = \psi(u)\lambda(v)$, for $v \in V_\rho$. Notice that λ is then also a ψ-Whittaker functional for $\rho \otimes \chi$ for any unramified character $\chi \in \mathrm{GL}_k(F)$: We have

$$\lambda((\chi \otimes \rho)(u)v) = \chi(u)\psi(u)\lambda(v) = \psi(u)\lambda(v),$$

since $\det u = 1$ and thus $u \in \ker \chi$. Abusing notation, we also let λ denote the ψ-Whittaker functional of $\rho \otimes \tau$ (or $\chi\rho \otimes \tau$ for any unramified χ, as we have just shown). Following Proposition 3.1 of [24], we now form a ψ-Whittaker functional Λ_χ on the space of $i_P^G(\chi\rho \otimes \tau)$ by setting

$$\Lambda_\chi(f) = \int_N \lambda\left(f(wn)\right)\psi(n)^{-1}dn, \tag{3.11}$$

where w is a representative of the nontrivial element of the Weyl group; in our case, we take w to be the block antidiagonal matrix

$$\begin{pmatrix} & & I_k \\ & I_{2(N-k)+1} & \\ I_k & & \end{pmatrix}.$$

Since π and π^- are generic, it suffices to determine the action of T on their respective Whittaker functionals if we want to determine how T acts on the $\mathcal{H}$-modules $\mathrm{Hom}_G(\Gamma_\mathfrak{s}, \pi)$ and $\mathrm{Hom}_G(\Gamma_\mathfrak{s}, \pi^-)$.

For any unramified character χ, we have the specialization map $\mathrm{sp}_\chi : \Gamma_\mathfrak{s} \mapsto i_P^G(\chi\rho \otimes \tau)$ (cf. [17, Section 3.1]). The unique (up to scalar multiple) element of $\mathrm{Hom}_G(\Gamma_\mathfrak{s}, \pi)$ factors through $\mathrm{sp}_1 : \Gamma_\mathfrak{s} \to i_P^G(\rho \otimes \tau)$; similarly, any element of $\mathrm{Hom}_G(\Gamma_\mathfrak{s}, \pi^-)$ factors through sp_{χ_0} (recall that $\rho^- = \chi_0 \otimes \rho$). Notice that Λ_1 and Λ_{χ_0} are the Whittaker models of π and π^-, respectively.

To determine the action of T on Λ_χ (for any χ), we must compare $\Lambda_\chi \mathrm{sp}_\chi$ and $\Lambda_\chi \circ \mathrm{sp}_\chi \circ T$. The operator T is defined by the following property:

$$\mathrm{sp}_\chi T = \varphi \circ J(\chi^{-1}) \circ \mathrm{sp}_{\chi^{-1}}$$

(cf. [17, Sections 3.1 and 3.2]). Here, $J(\chi^{-1})$ denotes the standard intertwining operator $i_P^G(\chi^{-1}\rho \otimes \tau) \to i_P^G(\chi\rho^\vee \otimes \tau)$. To explain φ, recall that ρ is assumed to be self-dual. Therefore, we can fix an isomorphism $\varphi : \rho^\vee \mapsto \rho$ and induce to an isomorphism $i_P^G(\chi\rho^\vee \otimes \tau) \to i_P^G(\chi\rho \otimes \tau)$ for any unramified χ, which we again denote by φ by abuse of notation.

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Let $\Lambda_\chi^\vee$ denote the Whittaker functional on $i_P^G(\chi\rho^\vee \otimes \tau)$ obtained using (3.11) from a fixed Whittaker functional $\lambda^\vee$ for $\rho^\vee$. By the uniqueness of Whittaker functionals, $\Lambda_\chi \circ \varphi = c \cdot \Lambda_\chi^\vee$ for some constant c. Furthermore, since φ is induced from an isomorphism $\varphi : \rho^\vee \mapsto \rho$, it follows immediately that c does not depend on χ. Therefore, we have

$$\Lambda_\chi \circ \mathrm{sp}_\chi \circ T = c \cdot \Lambda_\chi^\vee \circ J(\chi^{-1}) \circ \mathrm{sp}_{\chi^{-1}}.$$

Note that there is a natural way to normalize φ in such a way that $c = 1$. We denote by g^τ the transpose of an element $g \in GL_k(F)$ with respect to the antidiagonal (and with $g^{-\tau}$ its inverse). One can then define a new representation ρ_1 by $\rho_1(g) = \rho(g^{-\tau})$. This representation is isomorphic to the contragredient of ρ; the advantage is that it acts on V_ρ, the space of ρ. Furthermore, for any diagonal matrix (i.e., an element of the maximal torus) $t \in GL_k(F)$, we may conjugate ρ_1 to get $\rho_2(g) = {}^t\rho_1(g) = \rho_1(t^{-1}gt)$. Then $\rho_2 \cong \rho_1$, and with a suitable choice of t, ρ_2 becomes ψ-generic with the same Whittaker functional λ. For example, assume that ψ is given by

$$\psi(u) = \psi_0(u_{1,2} + \cdots + u_{k-1,k}),$$

where ψ_0 is a nontrivial additive character of F, and u is an upper-triangular unipotent matrix with entries $u_{1,2}, \ldots, u_{k-1,k}$ above the main diagonal. Then one checks immediately that $t = \mathrm{diag}(1, -1, \ldots, (-1)^{k-1})$ gives

$$\lambda(\rho_2(u)v) = \psi(u)\lambda(v)$$

for any $v \in V_\rho$. In short, we may assume that $\Lambda_\chi \circ \mathrm{sp}_\chi \circ T = \Lambda_\chi^\vee \circ J(\chi^{-1}) \circ \mathrm{sp}_{\chi^{-1}}$.

This leads to the second choice we have to make in the construction of T: that of the normalization of the intertwining operator J. Here, we choose the standard normalization introduced by Shahidi (cf. [24, Theorem 3.1]). Under this assumption, we have

$$\Lambda_\chi^\vee \circ J(\chi^{-1}) = \Lambda_{\chi^{-1}}$$

for every unramified character χ. Thus,

$$\Lambda_\chi \circ \mathrm{sp}_\chi \circ T = \Lambda_{\chi^{-1}} \circ \mathrm{sp}_{\chi^{-1}}.$$

With this, we are ready to compare the action of T on π and π^-. For π, we specialize at $\chi = 1$; this gives us

$$\Lambda_1 \circ \mathrm{sp}_1 \circ T = \Lambda_1 \circ \mathrm{sp}_1,$$

i.e., T acts trivially.

For π^-, we specialize at χ_0. We notice that $\chi_0^{-1} = \chi_0\eta$ for some character η such that $\eta \circ \rho \cong \rho$. This shows that $\mathrm{sp}_{\chi^{-1}} = \phi_\eta \circ \mathrm{sp}_{\chi_0}$, where ϕ_η is the isomorphism $\rho \mapsto \eta \otimes \rho$ defined in [17, Section 1.17] (again, we induce to $\phi_\eta : i_P^G(\rho \otimes \tau) \to i_P^G(\eta\rho \otimes \tau)$ and abuse the notation). Finally, using the uniqueness of Whittaker functionals again, we see that $\Lambda_{\chi\eta} \circ \phi_\eta = d \cdot \Lambda_\chi$ for some constant d which does not depend on χ. We can normalize ϕ_η so that $d = 1$; then we have

$$\Lambda_{\chi_0} \circ \mathrm{sp}_{\chi_0} \circ T = \Lambda_{\chi_0^{-1}} \circ \mathrm{sp}_{\chi_0^{-1}} = \Lambda_{\chi_0\eta} \circ \phi_\eta \circ \mathrm{sp}_{\chi_0} = \Lambda_{\chi_0} \circ \mathrm{sp}_{\chi_0}.$$

Therefore, T acts trivially on π^- as well.

To summarize, if we use Shahidi's normalization of the standard intertwining operator, and normalize φ as we did above, it follows that T acts trivially on both π and π^-. This implies that the Gelfand–Graev module is isomorphic to

$$\mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon_0$$

(see (3.10)), where $\varepsilon_0(T_n) = 1$. Note that this is analogous to our results in Case (iii), because T_n again acts by q^r, only this time $r = 0$.

This completes our analysis of the structure of $\mathcal{H}$. We conclude the section by providing an alternative proof for the following result of [6].

Corollary 3.8 *We have*

$$\mathrm{End}_{\mathcal{H}}(\Pi) \cong Z(\mathcal{H}),$$

the center of $\mathcal{H}$.

Proof Obviously, $Z(\mathcal{H})$ is contained in $\mathrm{End}_{\mathcal{H}}(\Pi)$, so we need to prove that any element of $\mathrm{End}_{\mathcal{H}}(\Pi)$ is given by a multiplication with an element $f \in Z(\mathcal{H})$. We prove the corollary in Case (iii); the proof in Cases (i) and (ii) is analogous.

We start by recalling that $Z(\mathcal{H}) = \mathcal{A}^W$, the Weyl group invariants of $\mathcal{A}$. Note that any element of $\mathrm{End}_{\mathcal{H}}(\Pi)$ can be viewed as an element of $\mathcal{A}$. Indeed, let $f \in \mathrm{End}_{\mathcal{H}}(\Pi)$. We have $\mathrm{End}_{\mathcal{H}}(\Pi) \subseteq \mathrm{End}_{\mathcal{A}}(\Pi)$, but we know that $\Pi = \mathcal{A}$ as an $\mathcal{A}$-module. Therefore, $f \in \mathrm{End}_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}$. Thus, it remains to prove that f is invariant under the action of the Weyl group.

It suffices to prove that f is invariant under the set of simple reflections which generate the Weyl group. In other words, we need to prove that

$$f^\vee = f \quad \text{and} \quad f^{s_i} = f, \quad i = 1, \ldots, n-1,$$

using the notation of Section 2.6. This follows immediately from what we now know about the structure of Π as an $\mathcal{H}$-module: $\Pi = \mathcal{H} \otimes_{\mathcal{H}_0} \varepsilon$. In other words, we have shown that there exists an element $g \in \mathcal{A} \cong \Pi$ (constructed in Section 3.1) on which the elements $T_1, \ldots, T_{n-1}$ and T_n act by scalar multiplication with q^t, and q^r, respectively.

We now look at the commutation relation

$$T_n f - f^\vee T_n = \left((q^r - 1) + \frac{1}{X_n}(\sqrt{q}^{r+s} - \sqrt{q}^{r-s})\right) \frac{f - f^\vee}{1 - 1/X_n^2}$$

satisfied by T_n and f. Applying this to g (recall that $T_n g = q^r g$), and using the fact that f is in $\mathrm{Hom}_{\mathcal{H}}(\Pi)$ (so that $T_n f g = f T_n g$), we get

$$(f - f^\vee) q^r \cdot g = \left((q^r - 1) + \frac{1}{X_n}(\sqrt{q}^{r+s} - \sqrt{q}^{r-s})\right) \frac{f - f^\vee}{1 - 1/X_n^2} \cdot g.$$

This is an equality in $\mathcal{A}$. Since $q^r \neq \left((q^r - 1) + \frac{1}{X_n}(\sqrt{q}^{r+s} - \sqrt{q}^{r-s})\right) \frac{1}{1-1/X_n^2}$ and $g \neq 0$, it follows that $f - f^\vee$ must be 0. Therefore, $f = f^\vee$. We get $f = f^{s_i}$ in the same way, using the commutation relations satisfied by the operators T_i. This proves the corollary. ∎

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4 An application to the GGP restriction problem

The theory of Bernstein–Zelevinsky derivatives implies that the restriction of an irreducible supercuspidal representation σ of $\mathrm{GL}(n+1)$ to $\mathrm{GL}(n)$ is isomorphic to the Gelfand–Graev representation of $\mathrm{GL}(n)$. Thus, given a maximal ideal $\mathcal{J}$ in the Bernstein center of $\mathrm{GL}(n)$, there exists only one irreducible $\mathrm{GL}(n)$-quotient of σ annihilated by $\mathcal{J}$. The goal of this short section is to show that a similar statement holds when restricting a cuspidal representation of an orthogonal group $\mathrm{O}(m+1)$ to a subgroup $\mathrm{O}(m) \subset \mathrm{O}(m+1)$.

Lemma 4.1 *Let σ be an irreducible representation of $\mathrm{O}(m+1)$. For any inertial data $\mathfrak{s}$ of $\mathrm{O}(m)$, let $\sigma[\mathfrak{s}]$ be the corresponding Bernstein summand of σ. We have:*

- *If σ is supercuspidal, then it is a projective $\mathrm{O}(m)$-module.*
- $\dim \mathrm{Hom}_{\mathrm{O}(m)}(\sigma, \pi) \le 1$ *for any irreducible representation π of $\mathrm{O}(m)$.*
- *$\sigma[\mathfrak{s}]$ is a finitely generated $\mathrm{O}(m)$-module.*

The same conclusions hold if we replace orthogonal by special orthogonal groups.

Proof The first statement is an observation: σ is a direct summand of $C_c^\infty(\mathrm{O}(m+1))$ (the space of locally constant and compactly supported functions on $\mathrm{O}(m+1)$) and $C_c^\infty(\mathrm{O}(m+1))$ stays projective after restriction to $\mathrm{O}(m)$. The second is the multiplicity one theorem [2]. For the third, observe that we have a surjection

$$C_c^\infty(\mathrm{O}(m+1)) \to \sigma^\vee \boxtimes \sigma.$$

By Theorem A of [1] or Remark 5.1.7 of [23], the Bernstein components of $C_c^\infty(\mathrm{O}(m+1))$, considered as an $\mathrm{O}(m+1) \times \mathrm{O}(m)$-module, are finitely generated. The third bullet now follows at once. ■

The following is the main result of this section.

Proposition 4.2 *Let σ be an irreducible supercuspidal representation of $\mathrm{O}(m+1)$. Let $\mathfrak{s}$ be inertial data for a subgroup $\mathrm{O}(m) \subset \mathrm{O}(m+1)$ such that $\sigma[\mathfrak{s}] \neq 0$. Let $\mathcal{Z}$ be the center of the Bernstein component corresponding $\mathfrak{s}$. The block $\sigma[\mathfrak{s}]$ is indecomposable, and for every maximal ideal $\mathcal{J}$ in $\mathcal{Z}$, there exists unique irreducible representation π of $\mathrm{O}(m)$ annihilated by $\mathcal{J}$ such that $\mathrm{Hom}_{\mathrm{O}(m)}(\sigma, \pi) \cong \mathbb{C}$.*

Proof Assume that m is even. Let $\Gamma_\mathfrak{s}$ be the projective generator associated with the inertial data $\mathfrak{s}$, and let $\mathcal{H}$ be the algebra of endomorphisms of $\Gamma_\mathfrak{s}$. Since $\sigma[\mathfrak{s}] \neq 0$, combining the above lemma and Remark 3.2, one concludes that

$$\mathrm{Hom}_{\mathrm{O}(m)}(\Gamma_\mathfrak{s}, \sigma) \cong \mathcal{A}$$

as $\mathcal{A}$-modules, proving indecomposability of the block, and then by Corollary 3.7

$$\mathrm{Hom}_{\mathrm{O}(m)}(\Gamma_\mathfrak{s}, \sigma) \cong \mathcal{H} \otimes_{\mathcal{H}_W} \varepsilon$$

for some finite subalgebra $\mathcal{H}_W \cong \mathbb{C}[W]$, where W is a finite group, such that $\mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_W$ and $\mathcal{A}^W$ is the center $\mathcal{Z}$ of $\mathcal{H}$, that is, the center of the Bernstein component corresponding to $\mathfrak{s}$.

Now, recall that all irreducible representations annihilated by $\mathcal{J}$ are subquotients of a single principal series representation

$$\mathcal{H} \otimes_{\mathcal{A}} \chi \cong \mathcal{H}_W,$$

where χ is a character of $\mathcal{A}$. Observe that the principal series is isomorphic to $\mathcal{H}_W \cong \mathbb{C}[W]$ as an $\mathcal{H}_W$-module. Since the one-dimensional type ε appears with multiplicity one in $\mathcal{H}_W$, there exists unique irreducible representation annihilated by $\mathcal{J}$ containing the type ε. But precisely, these representations are irreducible quotients of $\mathcal{H} \otimes_{\mathcal{H}_W} \varepsilon$, by the Frobenius reciprocity.

Now, assume that m is odd. In this case, we shall derive the result working with $\mathrm{SO}(m)$ and its Hecke algebras. Observe that $\mathrm{O}(m) = \mathrm{SO}(m) \times \{\pm 1_m\}$, so representations of $\mathrm{O}(m)$ and $\mathrm{SO}(m)$ are easy to relate. Let $\mathfrak{s}_0$ be the restriction to $\mathrm{SO}(m)$ of the inertial data $\mathfrak{s}$. On the other hand, the inertial data $\mathfrak{s}_0$ give a pair of inertial data $\mathfrak{s}^{\pm}$ of $\mathrm{O}(m)$ by specifying how -1_m acts. Let σ_0 be the restriction of σ to $\mathrm{SO}(m+1)$. We have two cases. Assume that σ_0 is irreducible. Then we can apply the above lemma to special orthogonal groups to prove the proposition for special orthogonal groups, that is, $\sigma_0[\mathfrak{s}_0]$ is indecomposable and has an explicit $\mathcal{H}$-structure by Corollary 3.7. Now, observe that $-1_m \in \mathrm{O}(m)$ naturally acts on $\sigma_0[\mathfrak{s}_0]$. By indecomposability of $\sigma_0[\mathfrak{s}_0]$, -1_m has to act by the same scalar on whole block. Therefore, either $\sigma[\mathfrak{s}^+] \neq 0$ or $\sigma[\mathfrak{s}^-] \neq 0$ and proposition holds in this case, for whichever of this two blocks is nontrivial. Now, assume that σ_0 is reducible. Then $\sigma \otimes \det \cong \sigma$. Decompose $\sigma = \sigma^+ \oplus \sigma^-$ where -1_m acts by 1 and -1 on the two summands. Since $\sigma \otimes \det \cong \sigma$, it follows that σ^+ and σ^- are isomorphic multiplicity-free, projective $\mathrm{SO}(m)$-modules. Now, arguing as before, it follows that the proposition holds for both components $\sigma[\mathfrak{s}^+] \neq 0$ and $\sigma[\mathfrak{s}^-] \neq 0$. ∎

Note that the above result is compatible with Gan–Gross–Prasad conjectures [13], and it sheds some light on the restriction problem beyond tempered representations. Of course, the above proposition holds for any σ that is projective as an $\mathrm{O}(m)$-module. It would be interesting to classify irreducible σ that are projective when restricted to $\mathrm{O}(m)$. Projectivity of restriction from $\mathrm{GL}(n+1)$ to $\mathrm{GL}(n)$ was studied in [12], and a complete classification of irreducible representations of $\mathrm{GL}(n+1)$ that are projective as $\mathrm{GL}(n)$-modules was obtained in [9].

Appendix A An isomorphism of projective generators

Let G be a reductive group, and let $\mathfrak{s}$ be an inertial class of cuspidal data (M,σ), where M is a Levi subgroup of G. Now, recall the Bushnell–Kutzko theory of types [8]: Any such $\mathfrak{s}$ is expected to have a type (J,λ), where J is a compact subgroup of G, and λ is an irreducible representation of J such that $\mathrm{c}-\mathrm{ind}_J^G \lambda$ is a projective generator for $\mathrm{Rep}_{\mathfrak{s}}(G)$. One is interested in the structure of the Hecke algebra $\mathcal{H}(G,\lambda) = \mathrm{End}_G(\mathrm{c}-\mathrm{ind}_J^G \lambda)$. In this section, we show that, under certain conditions (when (J,λ) exists), the Hecke algebra $\mathcal{H}(G,\lambda) = \mathrm{End}_G(\mathrm{c}-\mathrm{ind}_J^G \lambda)$ is isomorphic to the algebra $\mathcal{H}_{\mathfrak{s}} = \mathrm{End}_G(\Gamma_{\mathfrak{s}})$ constructed in Section 2.4. More precisely, we have the following:

Theorem A.1 *Assume that G is a classical group and that the residue characteristic of F is different from 2. Let $\mathfrak{s} = [(M,\sigma)]$ be an inertial equivalence class*

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in G. There exists an $\mathfrak{s}$-type (J,λ) such that the generators $\Gamma_{\mathfrak{s}}$ and $\mathrm{c-ind}_J^G\lambda$ are isomorphic.

We break up the main part of the proof into three auxiliary results (see Lemmas A.2–A.4) which hold for arbitrary reductive p-adic groups. We use the theory of covers developed by Bushnell and Kutzko. Any inertial equivalence class $\mathfrak{s} = [(M,\sigma)]$ in G also determines a (cuspidal) inertial equivalence class $\mathfrak{s}_M = [(M,\sigma)]$ in M. Let (J,λ) be a type for $\mathfrak{s}$, and let (J_M,λ_M) be a type for $\mathfrak{s}_M$. We say that the (J,λ) is a cover of the type (J_M,λ_M) if J decomposes with respect to M (in particular, $J_M = J\cap M$ and $\lambda_M = \lambda|_M$) and the equivalence of categories $\mathrm{Rep}_{\mathfrak{s}}(G) \to \mathcal{H}(G,\lambda)$-Mod commutes with parabolic induction and the Jacquet functor in the appropriate sense (see Definition 8.1 and paragraph 5 of Introduction of [8]). We then have the following.

Lemma A.2 (Theorem 7.9(iii) of [8]) *Let P be any parabolic subgroup with Levi factor M. For any smooth representation $V \in \mathrm{Rep}(G)$, the Jacquet functor with respect to P induces an isomorphism*

$$V^{\lambda} = (V_N)^{\lambda_M}.$$

Here, V^{λ} denotes the λ-isotype of V, i.e., the sum of all G-invariant subspaces of V isomorphic to λ.

We use this to reduce the proof of Theorem A.1 to the case of cuspidal components.

Lemma A.3 *Let (J,λ) be a type for $\mathfrak{s} = [(M,\pi)]$ in G, and let (J_M,λ_M) be a type for $\mathfrak{s}_M = [(M,\pi)]$ in M. Assume that (J,λ) is a cover of (J_M,λ_M).*

If the Bernstein generator $\Gamma_{\mathfrak{s}_M}$ is isomorphic to the Bushnell–Kutzko generator $\mathrm{c-ind}_{J_M}^M\lambda_M$ for the cuspidal component $\mathrm{Rep}_{\mathfrak{s}_M}(M)$, then we also have an isomorphism of generators for the component $\mathrm{Rep}_{\mathfrak{s}}(G)$.

Proof Lemma A.2 shows that we have

$$\mathrm{Res}_{J_M}^J((\mathrm{Res}_J^G V)^{\lambda}) = (\mathrm{Res}_{J_M}^M r_N(V))^{\lambda_M}$$

for any G-module V. Here, Res_H^G denotes the restriction functor from G to H, and r_N denotes the Jacquet functor with respect to $P = MN$. In other words, we get the following isomorphism of functors $\mathrm{Rep}(G) \to \mathrm{Rep}(J_M)$:

$$(*) \qquad \mathrm{Res}_{J_M}^J \circ (\lambda\text{-iso}) \circ \mathrm{Res}_J^G = (\lambda_M\text{-iso}) \circ \mathrm{Res}_{J_M}^M \circ r_N,$$

where we have used λ-iso (resp. λ_M-iso) to denote taking the λ- (resp. λ_M-) isotype.

All of the above functors have left adjoints:

- $\mathrm{c-ind}_{J_M}^J$ and $\mathrm{c-ind}_{J_M}^M$ for $\mathrm{Res}_{J_M}^J$ and $\mathrm{Res}_{J_M}^M$, respectively;
- $i_{\overline{P}}^G$ for r_N (this is the Bernstein form of Frobenius reciprocity; here, $\overline{P} = M\overline{N}$ is the parabolic subgroup opposite to P);
- λ-iso and λ_M-iso are self-adjoint, because we are working with (necessarily semisimple) representations of compact groups J and J_M.

Since adjoints are unique (up to equivalence), taking the adjoint of (*), we get

$$\mathrm{c-ind}_J^G \circ (\lambda\text{-iso}) \circ \mathrm{c-ind}_{J_M}^J = i_{\overline{P}}^G \circ \mathrm{c-ind}_{J_M}^M \circ (\lambda_M\text{-iso}).$$

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We now apply both sides of the above equality to λ_M. On the right-hand side, we get $i_{\overline{P}}^{G}(\mathrm{c}-\mathrm{ind}_{J_M}^{M}\lambda_M)$. By the assumptions from the statement of the lemma, we have $\mathrm{c}-\mathrm{ind}_{J_M}^{M}\lambda_M=\Gamma_{\mathfrak{s}_M}$; therefore, $i_{\overline{P}}^{G}(\mathrm{c}-\mathrm{ind}_{J_M}^{M}\lambda_M)$ is exactly the Bernstein generator $i_{\overline{P}}^{G}(\Gamma_{\mathfrak{s}_M})=\Gamma_{\mathfrak{s}}$. Here, we used the fact that the construction of the Bernstein generator does not depend on the choice of parabolic P (we choose $\overline{P}$) with fixed Levi M (cf. [4, Proposition 35]).

On the left-hand side, we get $\mathrm{c}-\mathrm{ind}_J^G((\mathrm{c}-\mathrm{ind}_{J_M}^{J}\lambda_M)^{\lambda})$. However, Frobenius reciprocity gives us $\dim\mathrm{Hom}_J(\mathrm{c}-\mathrm{ind}_{J_M}^{J}\lambda_M,\lambda)=\dim\mathrm{Hom}_{J_M}(\lambda_M,\lambda|_M)=1$, which follows from $\lambda|_M=\lambda_M$. Therefore, $(\mathrm{c}-\mathrm{ind}_{J_M}^{J}\lambda_M)^{\lambda}=\lambda$, and the left-hand side becomes $\mathrm{c}-\mathrm{ind}_J^G(\lambda)$, i.e., the Bushnell–Kutzko generator. Thus,

$$\mathrm{c}-\mathrm{ind}_J^G(\lambda)\cong\Gamma_{\mathfrak{s}},$$

as claimed. ∎

The above lemma allows us to focus on cuspidal components of the form $\mathfrak{s}_M=[(M,\sigma)]$ in M. If we want to prove the isomorphism of generators in general, it remains to prove that the generators of the cuspidal components are isomorphic. In other words, we would like to show that

$$\mathrm{c}-\mathrm{ind}_{M^\circ}^{M}\sigma_0=\mathrm{c}-\mathrm{ind}_{J_M}^{M}\lambda_M,$$

where σ_0 is an (any) irreducible constituent of $\sigma|_{M^\circ}$. We shall accomplish this under the following assumptions. Assume that

$$\sigma=\mathrm{c}-\mathrm{ind}_{\tilde{J}_M}^{M}\tilde{\lambda}_M,$$

where (see (5.5) in [8]):

- $\tilde{J}_M$ is compact modulo center subgroup of M such that $J_M=\tilde{J}_M\cap M^\circ$,
- the restriction of $\tilde{\lambda}_M$ to J_M is λ_M,
- any $x\in M$ which intertwines the representation λ_M belongs to $\tilde{J}_M$.

Lemma A.4 *Let $\sigma=\mathrm{c}-\mathrm{ind}_{\tilde{J}_M}^{M}\tilde{\lambda}_M$ be a cuspidal representation of M where the pair $(\tilde{J}_M,\tilde{\lambda}_M)$ satisfies the above three bullets. Then $\sigma_0=\mathrm{c}-\mathrm{ind}_{J_M}^{M^\circ}\lambda_M$ is an irreducible M°-summand of σ, and we have a canonical isomorphism (provided by induction in stages)*

$$\mathrm{c}-\mathrm{ind}_{M^\circ}^{M}\sigma_0\cong\mathrm{c}-\mathrm{ind}_{J_M}^{M}\lambda_M.$$

Proof Using Frobenius reciprocity and Mackey theory (provided by [28, Section 5.5] in this setting), we get

$$\begin{aligned}\mathrm{Hom}_{M^\circ}(\sigma_0,\sigma_0)&\cong\mathrm{Hom}_{M^\circ}\left(\mathrm{c}-\mathrm{ind}_{J_M}^{M^\circ}\lambda_M,\mathrm{c}-\mathrm{ind}_{J_M}^{M^\circ}\lambda_M\right)\\&\cong\mathrm{Hom}_{J_M}\left(\lambda_M,\bigoplus_x\mathrm{c}-\mathrm{ind}_{J_M\cap J_M^x}^{J}\mathrm{Res}_{J_M\cap J_M^x}^{J_M^x}\lambda_M^x\right),\end{aligned}$$

where the sum is taken over a set of double coset representatives in $J_M \backslash M^\circ / J_M$. Fixing one such x, we see that

$$\mathrm{Hom}_{J_M}(\lambda_M, \mathrm{c-ind}^J_{J_M \cap J^x_M} \mathrm{Res}^{J^x_M}_{J_M \cap J^x_M} \lambda^x) \cong \mathrm{Hom}_{J_M \cap J^x_M}(\lambda_M, \lambda^x_M)$$

(here, we are using Frobenius reciprocity for a compact group, so that restriction is also a left adjoint for c – ind). Since only $x \in \tilde{J}_M$ intertwine λ_M, and $J_M = \tilde{J}_M \cap M^\circ$, we have

$$\mathrm{Hom}_{M^\circ}(\sigma_0, \sigma_0) \cong \mathrm{Hom}_{J_M}(\lambda_M, \lambda_M) = \mathbb{C},$$

which we needed to prove. ∎

Finally, we may put together the above results.

Proof According to (5.5) in [8], if M is a general linear group over a division algebra, then the conditions of the above lemma are satisfied for every irreducible cuspidal representation σ of M. Clearly, if the conditions are satisfied for (M_1, σ_1) and (M_2, σ_2), then they are satisfied for $M = M_1 \times M_2$ and $\sigma = \sigma_1 \otimes \sigma_2$. Recall that a Levi subgroup in a classical group is a product of general linear groups and a smaller classical group. By a result of Stevens [26], irreducible cuspidal representations of classical groups are induced from open compact subgroups if F has odd residue characteristic. Thus, in these cases, for every irreducible cuspidal representation σ of M, there exists a type $(\tilde{J}_M, \tilde{\lambda}_M)$ satisfying the three bullets above, and Lemma A.4 applies. Moreover, by [7, 27] (for general linear groups) and [21] (for classical groups), G admits a type (J, λ) which is a cover of type (J_M, λ_M), so we can apply Lemma A.3 to obtain an isomorphism of generators for $\mathrm{Rep}_{\mathfrak{s}}(G)$.

This completes the proof of Theorem A.1. ∎

We remark that Theorem A.1 holds beyond classical groups, provided that the conditions of two lemmas are satisfied. For exceptional G_2 examples, see [5].

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