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# ORDER EMBEDDING OF A MATRIX ORDERED SPACE

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**Abstract**

We characterize certain properties in a matrix ordered space in order to embed it in a $C^*$-algebra. Let such spaces be called $C^*$-ordered operator spaces. We show that for every self-adjoint operator space there exists a matrix order (on it) to make it a $C^*$-ordered operator space. However, the operator space dual of a (nontrivial) $C^*$-ordered operator space cannot be embedded in any $C^*$-algebra.

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## 1. The characterization theorem

In this short communication, we determine a set of necessary and sufficient conditions on a matrix ordered space so that it can be order embedded in some $C^*$-algebra. (Some related results can be found in [6, 12].) Let us call such spaces $C^*$-*ordered operator spaces*. We have been able to show that on any self-adjoint operator space there exists a matrix order (which may be trivial) such that the space turns out to be a $C^*$-ordered operator space. Interestingly, however, we have proved that the operator space dual of a (nontrivial) $C^*$-ordered operator space is not a $C^*$-ordered operator space. In particular, the operator space dual of an operator system cannot be order embedded in a $C^*$-algebra. This improves a result due to Blecher and Neal [1]. At the end of this paper, we discuss a class of examples of $C^*$-ordered operator spaces.

We begin by recalling some definitions. Let $V$ be a complex vector space. For $m, n \in \mathbf{N}$, $M_{m,n}(V)$ denotes the set of all $m \times n$ matrices with entries from $V$. For $m = n$, we write, $M_{m,n}(V) = M_n(V)$. When $V = \mathbf{C}$, we write $M_{m,n}(V) = M_{m,n}$.

DEFINITION 1.1. An $L^\infty$-matricially normed space (that is, an abstract operator space [10]), denoted by $(V, \{\| \ \|_n\})$, is a complex vector space $V$ together with a sequence of norms $\| \ \|_n$ (called a matrix norm on $V$) such that:

(i) $(M_n(V), \| \ \|_n)$ is a normed linear space for all $n$;
(ii) $\|v \oplus w\|_{n+m} = \max\{\|v\|_n, \|w\|_m\}$; and
(iii) $\|\alpha v \beta\|_n \leq \|\alpha\| \|v\|_n \|\beta\|$ for all $v \in M_n(V)$, $w \in M_m(V)$, $\alpha, \beta \in M_n$ and $n \in \mathbf{N}$.

DEFINITION 1.2. A $*$-vector space is complex vector space $V$ together with an involution $*$. A matrix ordered space is a $*$-vector space $V$ together with a cone $M_n(V)^+$ in $M_n(V)_{sa}$ for all $n \in \mathbf{N}$ and with the following property: if $v \in M_n(V)^+$ and $\gamma \in M_{n,m}$ then $\gamma^* v \gamma \in M_m(V)^+$ for any $n, m \in \mathbf{N}$. It is denoted by $(V, \{M_n(V)^+\})$.

DEFINITION 1.3. An $L^\infty$-matricially $*$-normed space is an $L^\infty$-matricially normed space $(V, \{\| \ \|_n\})$, such that $V$ is a $*$-vector space and that for all $v \in M_n(V)$ we have $\|v^*\|_n = \|v\|_n$.

DEFINITION 1.4. Let $V$ and $W$ be complex vector spaces. Every linear map $\phi : V \to W$ induces a sequence $\{\phi_n\}$ where $\phi_n([v_{ij}]) = [\phi(v_{ij})]$. Let $(V, \{\| \ \|_n\})$ and $(W, \{\| \ \|_n\})$ be $L^\infty$-matricially normed spaces. Then a linear map $\phi : V \to W$ is called completely bounded if $\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbf{N}\} < \infty$ and $\phi$ is called a complete isometry if $\phi_n$ is an isometry for all $n$. Let $(V, \{M_n(V)^+\})$ and $(W, \{M_n(W)^+\})$ be matrix ordered spaces and let $\phi : V \to W$ be a self-adjoint linear map. We say that $\phi$ is completely positive if $\phi_n$ is positive for all $n$, and that $\phi$ is a complete order isomorphism if it is a linear isomorphism and both $\phi$ and $\phi^{-1}$ are completely positive on their domains.

DEFINITION 1.5. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. We say that $V^+$ is *proper* if $V^+ \cap (-V^+) = \{0\}$.

It is shown in [2] that if $V^+$ is proper, then so is $M_n(V)^+$ for all $n$. In the first result we extract some necessary conditions on a matrix ordered space so that it may be embedded in a $C^*$-algebra. We also prove that these conditions are sufficient.

PROPOSITION 1.6. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. Assume that $\phi : V \to A$ is a linear complete order isomorphism for some $C^*$-algebra $A$. For each $n \in \mathbf{N}$ define*

$$\|v\|_n = \|\phi_n(v)\|$$

*for all $v \in M_n(V)$. Then:*

(1) *$(V, \{\| \ \|_n\})$ is an (abstract) operator space;*
(2) *$\|v^*\|_n = \|v\|_n$ for all $v \in M_n(V)$, $n \in \mathbf{N}$. (In other words, $V$ is an $L^\infty$-matricially $*$-normed space.)*

*Put $Q_n(V) = \{f : M_n(V) \to \mathbf{C} \mid f \geq 0$ and $\|f\| \leq 1\}$ for all $n \in \mathbf{N}$. Then:*

(3) *$\|v\| = \sup\{|f(v)| : f \in Q_n(V)\}$ for all $v \in M_n(V)_{sa}$, $n \in \mathbf{N}$;*
(4) *for $n \in \mathbf{N}$ and $v \in M_n(V)_{sa}$ we have $v \in M_n(V)^+$ if and only if $f(v) \geq 0$ for all $f \in Q_n(V)$;*
(5) *$V^+$ (and therefore $M_n(V)^+$, for all $n$) is proper.*

PROOF. (1) By definition, $\phi$ becomes a complete isometry so that $V$ may be treated as a subspace of $A$.

(2) For any $v \in M_n(V)$,

$$\|v^*\|_n = \|\phi_n(v^*)\| = \|\phi_n(v)^*\| = \|\phi_n(v)\| = \|v\|_n.$$

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(3) We know that for all $a \in M_n(A)_{sa}$,

$$\|a\| = \sup\{|g(a)| : g \in Q_n(A)\}.$$

Also, for $g \in Q_n(A)$, we have $g \circ \phi_n \in Q_n(V)$. Thus

$$\|v\|_n = \|\phi_n(v)\| = \sup\{|g(\phi_n(v))| : g \in Q_n(A)\} \leq \sup\{|f(v)| : f \in Q_n(V)\}.$$

The other part is obvious.

(4) First, let $v \in M_n(V)_{sa}$ be such that $f(v) \geq 0$ for all $f \in Q_n(V)$. Then as in (3) we have that $g \circ \phi_n(v) \geq 0$ for all $g \in Q_n(A)$. It follows that $\phi_n(v) \in M_n(A)^+$. Since $\phi$ is a complete order isomorphism, we may conclude that $v \in M_n(V)^+$. Now the converse is trivial. Finally, as $\phi$ is a (complete) order isomorphism, (5) also holds. □

We shall also prove the converse of this result. For this purpose, we shall require the following improvement on a result due to Effros and Ruan [3].

THEOREM 1.7. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. Assume that $\{\|\ \|_n\}$ is a matrix norm on $V$ such that it is an $L^\infty$-matricially $*$-normed space. Fix $n \in \mathbf{N}$ and let $f : M_n(V) \to \mathbf{C}$ be a linear self-adjoint contraction. Then there exist a linear, self-adjoint, complete contraction $\phi : V \to M_n$ and a norm-one $n^2 \times 1$ matrix $\delta$ such that*

$$f(v) = \delta^* \phi_n \delta.$$

*If in addition, $f$ is positive, then $\phi$ is completely positive too.*

PROOF. The techniques used in the proof are essentially adapted from [3]. However, for completeness, we include the main points of the proof. It is divided into several steps.

*Step I.* Consider the $C^*$-algebra $M_n$ and let $S$ be its state space. Let $C(S)$ denote the space of all real-valued, continuous functions on $S$. For $\alpha \in M_{m,n}$ and $v \in M_m(V)_{sa}$ with $\|v\|_m = 1$, we define $\psi_v^\alpha \in C(S)$ given by

$$\psi_v^\alpha(p) = p(\alpha^*\alpha) - f(\alpha^* v \alpha)$$

for all $p \in S$. Put

$$\Psi = \{\psi_v^\alpha : \alpha \in M_{m,n} \text{ and } v \in M_m(V)_{sa} \text{ with } \|v\|_m = 1\}.$$

Then $\psi_v^\alpha + \psi_w^\beta = \psi_{v \oplus w}^{\left[\begin{smallmatrix}\alpha \\ \beta\end{smallmatrix}\right]}$ and $|\lambda|^2 \psi_v^\alpha = \psi_v^{\lambda\alpha}$ for all $\alpha \in M_{m,n}$, $\beta \in M_{p,n}$, $v \in M_m(V)_{sa}$, $w \in M_p(V)_{sa}$ with $\|v\|_m = 1$, $\|w\|_p = 1$ and $\lambda \in \mathbf{C}$. Since $\{\|\ \|_n\}$ satisfies the $L^\infty$-condition, we see that $\|v \oplus w\|_{m+p} = 1$. Thus $\Psi$ is a cone. Let $\Gamma$ denote the cone of all strictly negative functions in $C(S)$. Then $\operatorname{int}(\Gamma) \neq \emptyset$ and $\Psi \cap \Gamma = \emptyset$. Thus by the geometric form of the Hahn–Banach theorem, there exists a nonzero Radon measure $\mu$ on $S$ such that $\mu|\Psi \geq 0$ and $\mu|\Gamma \leq 0$. It follows that $\mu$ is a positive measure

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and we may assume that it is a probability measure. Then $p_0 = \int_S p\, d\mu(p) \in S$. Since $\mu|\Psi \geq 0$, for any $\alpha \in M_{m,n}$ and $v \in M_m(V)_{\mathrm{sa}}$ with $\|v\|_m = 1$ we have

$$0 \leq \int_S \Psi_v^\alpha(p)\, d\mu(p) = \Psi_v^\alpha(p_0) = p_0(\alpha^*\alpha) - f(\alpha^* v \alpha).$$

In other words,

$$f(\alpha^* v \alpha) \leq p(\alpha^*\alpha)\|v\|_m,$$

for all $\alpha \in M_{m,n}$ and $v \in M_m(V)_{\mathrm{sa}}$. Now using standard techniques (see, for example, [3]), we may conclude that

$$f(\alpha^* v \beta) \leq [p(\alpha^*\alpha)p(\beta^*\beta)]^{1/2}\|v\|_m,$$

for all $\alpha, \beta \in M_{m,n}$ and $v \in M_m(V)$.

*Step II*. Let $\{\varepsilon_{ij} : 1 \leq i, j \leq n\}$ be the matrix units of $M_n$. Put $p_0(\varepsilon_{ij}) = \alpha_{ji}$ and set $A_0 = [\alpha_{ij}] \in M_n$. It follows, from [5, Exercise 4.6.18], that:

(1) $A_0 \in M_n^+$;
(2) $\mathrm{tr}(A_0) = 1$; and
(3) $p_0(B) = \mathrm{tr}(A_0 B)$ for all $B \in M_n$.

Let $A$ be the positive square root of $A_0$. Consider the closed subspace $K = A(\mathbf{C}^n)$ of $\mathbf{C}^n$. For a fixed $v \in V$, define $\hat{v} : K \times K \to \mathbf{C}$ given by

$$\hat{v}(A(\alpha), A(\beta)) = f(\alpha v \beta^*)$$

for all $\alpha, \beta \in \mathbf{C}^n$ (identified with $M_{n,1}$). Then $\hat{v}$ is a contractive sesquilinear form on $K$, for $\|A(\alpha)\|^2 = p_0(\alpha\alpha^*)$ by (3). Thus there exists a unique contractive linear map $T_v : K \to K$ such that

$$\langle T_v A(\alpha), A(\beta)\rangle = f(\alpha v \beta^*).$$

Let $P$ be the range projection of $A$. Then $\phi(v) = T_v P$ may be identified in $M_n$ and we may conclude that $v \mapsto \phi(v)$ defines a self-adjoint linear map $\phi : V \to M_n$. Let $\{\varepsilon_i : 1 \leq i \leq n\}$ be the matrix units of $M_{n,1}$. Set $\delta = (A(\varepsilon_i)) \in (\mathbf{C}^n)^n$ (identified with $M_{n^2,1}$). Then

$$\|\delta\|^2 = \sum_{i=1}^n \|A(\varepsilon_i)\|^2 = \mathrm{tr}(A_0) = 1$$

using (2). Since for $v = [v_{ij}] \in M_n(V)$ we have $v = \sum_{i,j=1}^n \varepsilon_i v_{ij} \varepsilon_j^*$, we obtain that

$$f(v) = \sum_{i,j=1}^n \langle \phi(v_{ij}) A(\varepsilon_i), A(\varepsilon_j)\rangle = \delta^* \phi_n(v)\delta.$$

Now the rest of the proof is routine. □

THEOREM 1.8. *Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. Assume that $\{\| \, \|_n\}$ is a matrix norm on $V$ and that conditions (1)–(5) of Proposition 1.6 hold in $V$. Then there exist a $C^*$-algebra $A$ and a linear, completely isometric, complete order isomorphism $\Phi : V \to A$.*

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PROOF. Let us write $CQ_n(V)$ for the set of all completely contractive completely positive maps $\phi : V \to M_n$. Then $CQ_n(V)$ is nonempty. Write $M_n^\phi$ for $M_n$ for all $\phi \in CQ_n(V)$ and put $A_n = \bigoplus M_n^\phi$ where $\phi$ runs over $CQ_n(V)$ for all $n$. Define $\Phi^{(n)} : V \to A_{2n}$ given by

$$\Phi^{(n)}(v) = (\phi(v))_{\phi \in CQ_{2n}(V)}.$$

Then $\Phi^{(n)}$ is a well-defined completely contractive completely positive map. We show that $(\Phi^{(n)})_n$ is an order isomorphism (onto its range). Let $v \in M_n(V)_{sa}$ be such that $(\Phi^{(n)})_n(v) \geq 0$. Then $\phi_n(v) \in M_n(M_{2n})^+$ for all $\phi \in CQ_{2n}(V)$. Let $f \in Q_n(V)$. Then by Theorem 1.7, there exist $\phi \in CQ_n(V)(\subset CQ_{2n}(V))$ and $\delta \in M_{n^2,1}$ such that

$$f(v) = \delta^* \phi_n(v)\delta \geq 0.$$

Thus by condition (4), $v \in M_n(V)^+$. Next, let $v \in M_n(V)_{sa}$ be such that $(\Phi^{(n)})_n(v) = 0$. Then as above, $\pm v \in M_n(V)^+$ so that by condition (5), $v = 0$. Thus $(\Phi^{(n)})_n$ is an order isomorphism for all $n$. Now set

$$A = \bigoplus \{A_{2n} : n \in \mathbf{N}\} \quad \text{(the } C^*\text{-direct sum)}.$$

Define $\Phi : V \to A$ given by $\Phi(v) = (\Phi^{(n)}(v))$, for all $v \in V$. Then $\Phi$ is a linear completely contractive complete order isomorphism. We further show that $\Phi$ is a complete isometry. Let $v \in M_n(V)$. Then

$$\begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \in M_{2n}(V)_{sa}.$$

Thus by condition (4) for given $\epsilon > 0$, there is an $f \in Q_{2n}$ such that

$$\|v\|_n - \epsilon = \left\| \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\|_{2n} - \epsilon < \left\langle f, \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\rangle.$$

By Theorem 1.7 there exist $\phi \in CQ_{2n}(V)$ and $\delta \in M_{(2n)^2,1}$ such that

$$\left| \left\langle f, \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right\rangle \right| = \delta^* \phi_{2n}\left( \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right)\delta.$$

Since

$$\left| \delta^* \phi_{2n}\left( \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right)\delta \right| \leq \left\| \phi_{2n}\left( \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \right) \right\| = \|\phi_n(v)\| \leq \|v\|_n$$

and since $\epsilon > 0$ is arbitrary, we conclude that $(\phi)_n$ is an isometry for all $n$. Now the result is immediate. □

DEFINITION 1.9. An operator space considered in Theorem 1.8 will be called a *$C^*$-ordered operator space*.

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Now we shall take another approach to examine $C^*$-ordered operator spaces. To begin with, we state the following improvement on a characterization theorem due to Effros and Ruan [3]. A proof may be extracted from the proofs of Theorems 1.7 and 1.8.

PROPOSITION 1.10. *Let $V$ be an $L^\infty$-matricially $*$-normed space. Then there exist a $C^*$-algebra $A$ and a completely isometric, self-adjoint, linear isomorphism $\phi : V \to A$.*

THEOREM 1.11. *Let $V$ be an $L^\infty$-matricially $*$-normed space. Then there exists a matrix order structure $\{M_n(V)^+\}$ on it so that it is a $C^*$-ordered operator space.*

PROOF. By Proposition 1.10, there exist a $C^*$-algebra $A$ and a completely isometric, self-adjoint, linear isomorphism $\phi : V \to A$. For each natural number $n$, set

$$M_n(V)^+ = \{v \in M_n(V)_{\mathrm{sa}} : f \circ \phi_n(v) \geq 0 \text{ for all } f \in Q_n(A)\}.$$

It is routine to check that $\{M_n(V)^+\}$ is a matrix order on $V$ and that $V^+$ is proper. Moreover, since $A$ is a $C^*$-algebra, we also have

$$\|v\|_n = \|\phi_n(v)\| = \sup\{|f \circ \phi_n(v)| : f \in Q_n(A)\}.$$

Now, by construction, $f \circ \phi_n \in Q_n(V)$ for all $f \in Q_n(A)$ so that $\phi$ has the required properties to complete the proof. □

## 2. Order embedding and operator space duality

In this section we show that, in general, operator space duality is not suitable for $C^*$-ordered operator spaces. At the end, we describe a class of examples of $C^*$-operator spaces.

PROPOSITION 2.1. *Let $V$ be a $C^*$-ordered operator space. Then for any $n \in \mathbf{N}$ and $u, v, w \in M_n(V)_{\mathrm{sa}}$, with $u \leq v \leq w$,*

$$\|v\|_n \leq \max\{\|u\|_n, \|w\|_n\}.$$

*In particular, given $n \in \mathbf{N}$ and $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_{2n}(V)^+$ for some $v \in M_n(V)$ and $u_1, u_2 \in M_n(V)^+$ we have*

$$\|v\|_n \leq \max\{\|u_1\|_n, \|u_2\|_n\}.$$

PROOF. Let $u, v, w \in M_n(V)_{\mathrm{sa}}$, with $u \leq v \leq w$ for some $n \in \mathbf{N}$. Then given $f \in Q_n(V)$ we have $f(u) \leq f(v) \leq f(w)$. Thus by the definition, $-\|u\|_n \leq |f(v)| \leq \|w\|_n$ for all $f \in Q_n(V)$. Now it follows that

$$\|v\|_n \leq \max\{\|u\|_n, \|w\|_n\}.$$

Next, let $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_{2n}(V)^+$ for some $v \in M_n(V)$ and $u_1, u_2 \in M_n(V)^+$. Then

$$-\begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \leq \begin{bmatrix} 0 & v \\ v^* & 0 \end{bmatrix} \leq \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix} \in M_{2n}(V)^+.$$

Now by the first part, the result is immediate, for $V$ is an operator space. □

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THEOREM 2.2. *Let $V$ be a nonzero $C^*$-ordered operator space. If the operator space dual $V'$ of $V$ is also a $C^*$-ordered operator space, then $V \cong \mathbf{C}$.*

PROOF. Let $f$ be a bounded self-adjoint linear functional on $V$. Since $V$ is $C^*$-ordered, by Proposition 2.1 above and [4, Theorem 3.6.2], there are bounded positive linear functionals $g_1$ and $g_2$ on $V$ such that $f = g_1 - g_2$ with $\|g_1\| + \|g_2\| \leq \|f\|$. Then $-g_2 \leq f \leq g_1$. Thus as $V'$ is also $C^*$-ordered, by Proposition 2.1, we get that $\|f\| \leq \max\{\|g_1\|, \|g_2\|\}$. Therefore, $\|g_1\| + \|g_2\| \leq \max\{\|g_1\|, \|g_2\|\}$. It follows that either $g_1 = 0$ or $g_2 = 0$. In other words, $(V')_{sa} = (V')^+ \cup (-(V')^+)$. Thus for any $f, g \in (V')_{sa}$, either $f \leq g$ or $g \leq f$. Consider

$$Q(V) = \{f \in (V')^+ : \|f\| \leq 1\}.$$

Then $Q(V)$ is nonempty, weak*-compact and convex. Let $e_1$ and $e_2$ be any two nonzero extreme points of $Q(V)$. Then as above, these are comparable in $(V')_{sa}$. For definiteness, we may assume that $e_1 \leq e_2$. If $e_1 \neq e_2$, then

$$e_2 = \tfrac{1}{2}(e_2 - e_1) + \tfrac{1}{2}(e_2 + e_1)$$

is a proper convex combination in $Q(V)$. Since $e_2$ is an extreme point of $\mathbf{C}$, we have either $\frac{1}{2}(e_2 - e_1) = 0$ or $\frac{1}{2}(e_2 + e_1) = 0$. Since $e_1$ and $e_2$ are nonzero, we must have $e_1 = e_2$. In other words, $Q(V)$ has a unique nonzero extreme point, say $e_0$. Since 0 is also an extreme point of $Q(V)$, for any $f \in Q(V)$ we get, by the Krien–Milman theorem, that $f = ke_0$ for some $k \in [0, 1]$. Then $\|e_0\| = 1$. Now it is immediate that $V \cong \mathbf{C}$. □

The following result due to Blecher and Neal [1] is a special case of the above result.

COROLLARY 2.3. *The operator space dual of a nonscalar $C^*$-algebra cannot be order embedded in any $C^*$-algebra.*

REMARK 2.4. It will not be hard to show that the operator space dual of an $L^\infty$-matricially $*$-normed space is again an $L^\infty$-matricially $*$-normed space. Thus the operator space duality seems to have a problem with the relation with the matrix norm and the matrix order. However, we are not in a position to comment on this at this moment.

At the end we record that matrix order unit spaces (operator systems) are $C^*$-ordered operator spaces. More generally, every approximate matrix order unit space is a $C^*$-ordered operator space. The latter class includes the class of operator systems and that of $C^*$-algebras (unital or nonunital). These classes possess a structure richer than that of $C^*$-ordered operator spaces. We explain this as follows.

DEFINITION 2.5. We say that $V^+$ is *generating* if given $v \in V$ there exist $v_0, v_1, v_2, v_3 \in V^+$ such that $v = \sum_{k=0}^{3} i^k v_k$, where $i^2 = -1$.

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It is proved in [7, Proposition 1.8] that $V^+$ is generating if and only if given $v \in V$ there are $u_1, u_2 \in V^+$ such that $\begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+$ and that in this case $M_n(V)^+$ is generating for all $n$. In this case, we say that $(V, \{M_n(V)^+\})$ is a positively generated matrix ordered space.

DEFINITION 2.6. Let $(V, \{M_n(V)^+\})$ be a positively generated matrix ordered space. A norm $\| \ \|$ on $V$ will be called a Riesz norm if for all $v \in V$,

$$\|v\| = \left\{\max(\|u_1\|, \|u_2\|) : u_1, u_2 \in V^+ \text{ and } \begin{bmatrix} u_1 & v \\ v^* & u_2 \end{bmatrix} \in M_2(V)^+\right\}.$$

DEFINITION 2.7. An $L^\infty$-matricially Riesz normed space (matrix regular operator space [11]) is a positively generated matrix ordered space $(V, \{M_n(V)^+\})$ together with a matrix norm $\{\| \ \|_n\}$ such that $\| \ \|_n$ is a Riesz norm on $M_n(V)$ and $M_n(V)^+$ is norm closed for all $n$ and that $(V, \{\| \ \|_n\})$ is an $L^\infty$-matricially normed space. It is denoted by $(V, \{\| \ \|_n\}, \{M_n(V)^+\})$. An $L^\infty$-matricially Riesz normed space is called a $C^*$-matricially Riesz normed space if it is also a $C^*$-ordered operator space.

It follows, from Proposition 1.6 and Theorem 1.8, that an $L^\infty$-matricially Riesz normed space can be order embedded in a $C^*$-algebra if and only if it is a $C^*$-matricially Riesz normed space. Schreiner [11] proved that the operator space dual of an $L^\infty$-matricially Riesz normed space is again an $L^\infty$-matricially Riesz normed space. It follows, from Theorem 2.2, that, every $L^\infty$-matricially Riesz normed space is not a $C^*$-matricially Riesz normed space. However, the spaces we define below are $C^*$-ordered operator spaces.

DEFINITION 2.8. Let $(V, \{M_n(V)^+\})$ be a matrix ordered space. An increasing net $\{e_\lambda\}$ in $V^+$ is called an *approximate order unit* for $V$ if for each $v \in V$ there is a $k > 0$ such that

$$\begin{bmatrix} ke_\lambda & v \\ v^* & ke_\lambda \end{bmatrix} \in M_2(V)^+ \quad \text{for some } \lambda.$$

In this case $\{e^n_\lambda\}$ acts as an approximate order unit for $M_n(V)$ for all $n$, where $e^n_\lambda = e_\lambda \oplus \cdots \oplus e_\lambda$. Moreover, $\{e_\lambda\}$ determines a matrix Riesz seminorm $\{\| \ \|_n\}$ on $V$. We call $(V, \{e_\lambda\})$ an *approximate matrix order unit space* if $(V, \{\| \ \|_n\}, \{M_n(V)^+\})$ is an $L^\infty$-matricially Riesz normed space.

When $e_\lambda = e$ for all $\lambda$ we drop the term 'approximate' in the above notions. For example, $(V, e)$ denotes a matrix order unit space. For details, refer to [9].

Let $V$ be an approximate matrix order unit space. It follows, from Theorem 1.8 and [8, Proposition 1.20], that

$$M_n(V')_{\text{sa}} = \text{co}(Q_n(V) \cup (-Q_n(V)))$$

for all $n$. Thus we may conclude with the following result.

PROPOSITION 2.9. *An approximate matrix order unit space is a $C^*$-matricially Riesz normed space.*

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