Ergod. Th. & Dynam. Sys., (2024), **44**, 727–748
doi:10.1017/etds.2023.27

Multiply connected wandering domains of meromorphic functions: the pursuit of uniform internal dynamics

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(*Received* 30 *March* 2022 *and accepted in revised form* 29 *January* 2023)

Abstract. Recently, Benini *et al* showed that, in simply connected wandering domains of entire functions, all pairs of orbits behave in the same way relative to the hyperbolic metric, thus giving us our first insight into the general internal dynamics of such domains. The author proved in a recent paper [G. R. Ferreira. Multiply connected wandering domains of meromorphic functions: internal dynamics and connectivity. *J. Lond. Math. Soc.* (2) **106** (2022), 1897–1919] that the same is not true for multiply connected wandering domains, a natural question is how inhomogeneous multiply connected wandering domains can be. We give an answer to this question, in that we show that uniform dynamics inside an open subset of the domain generalizes to the whole wandering domain. As an application of this result, we construct the first example of a meromorphic function with a semi-contracting infinitely connected wandering domain.

Key words: holomorphic dynamics, wandering domains, hyperbolic geometry
2020 Mathematics Subject Classification: 37F10 (Primary); 30D05 (Secondary)

1. *Introduction*

Let $f : \mathbb{C} \to \widehat{\mathbb{C}}$ be a meromorphic function, where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. The study of its iterates, undertaken first by Fatou and Julia in the 1920s, comprises the area of *complex dynamics*, a field of research that has been increasingly active since the latter half of the twentieth century. The *Fatou set* of *f*, defined as

$$F(f) := \{z \in \mathbb{C} : (f^n)_{n\in\mathbb{N}} \text{ is defined and normal in a neighbourhood of } z\},$$

is known to be the set of 'regular' dynamics, and its connected components, called *Fatou components*, are mapped into one another by *f*. So, if $U \subset F(f)$ is a Fatou component of *f*, $f^n(U)$ is, for all $n \in \mathbb{N}$, contained in some Fatou component U_n of *f*. This separates Fatou components into two kinds: those for which there exist $n > m \geq 0$ such that $U_n = U_m$, called *(pre)periodic* components; and those for which all U_n are distinct, called *wandering domains*.

The internal dynamics of preperiodic components has been studied for a century, going back to Fatou and Cremer (see, for instance, [8]). It is well known, for instance, that a periodic Fatou component falls into one of five dynamically distinct types, each one having a distinct topological model. Studying the internal dynamics of wandering domains, on the other hand, is a far more recent undertaking. The first steps were taken by Bergweiler, Rippon, and Stallard in 2013 [9], who described the behaviour of multiply connected wandering domains of entire functions (recall that the *connectivity* of a domain $\Omega \subset \widehat{\mathbb{C}}$ is its number of complementary components). Their methods made extensive use of the geometric properties of multiply connected wandering domains of entire functions, but more recent work examined the internal dynamics of wandering domains in terms of the hyperbolic metric (which is always an available tool when talking about Fatou components) for both simply [7] and multiply connected [15] wandering domains.

This is the point of view that we adopt in this work as well. Denoting the hyperbolic metric in the hyperbolic domain $\Omega \subset \widehat{\mathbb{C}}$ by d_Ω (see §2 for the relevant definitions), the starting point is the following question: given a meromorphic function f with a wandering domain U and points $z, w \in U$, what happens to

$$d_{U_n}(f^n(z), f^n(w)) \quad \text{as } n \to +\infty?$$

A central result of [7] is that, if U and all its iterates are simply connected, then the answer is *qualitatively* independent of our particular choice of z and w; in other words, all pairs of orbits behave in the same way. On the other hand, it was shown in [15] that if U is multiply connected the answer may depend on the chosen pair of points, and, in particular, that all possible long-term behaviours can co-exist in the same domain.

An immediate question, then, is how complicated this co-existence can be. The observed cases in [15] are all 'well behaved', in the sense that there are dynamically defined, smooth laminations of U that determine how the iterates of each pair of points behave. In particular, every non-empty open subset of U contains distinct pairs of points exhibiting all the behaviours present in U. Here, we show that (in some sense) this uniformity is a general feature of wandering domains of meromorphic functions.

THEOREM 1.1. *Let U be a wandering domain of the meromorphic function f. Suppose that there exist a point $z_0 \in U$ and a neighbourhood $V \subset U$ of z_0 such that one of the following properties holds for every $w \in V$:*

(a) $d_{U_n}(f^n(w), f^n(z_0)) \to 0$ *(we say that V is contracting relative to z_0);*
(b) $d_{U_n}(f^n(w), f^n(z_0))$ *decreases to a limit $c(z_0, w) > 0$ without ever reaching it, except for a discrete (in V) set of points for which $f^k(w) = f^k(z_0)$ for some $k \in \mathbb{N}$ (we say that V is semi-contracting relative to z_0);*
(c) *there exists $N \in \mathbb{N}$ (uniform over compact subsets of V) such that $d_{U_n}(f^n(w), f^n(z_0)) = c(z_0, w) > 0$ for $n \geq N$ (we say that V is locally eventually isometric relative to z_0).*

Then the same property holds for every $w \in U$.

Remark. Benini *et al*'s results [7, Theorem A] for simply connected wandering domains generalize to meromorphic functions as long as all iterates of the wandering domains are simply connected, and so the conclusion of Theorem 1.1 always holds for such domains. Theorem 1.1 is also vacuous for multiply connected wandering domains with finite eventual connectivity, which have their internal dynamics dictated by their eventual connectivity (see [15, Theorem 1.1]), and so it is of most interest for infinitely connected wandering domains of meromorphic functions.

Theorem 1.1 shows that, even in the multiply connected setting, the internal dynamics of wandering domains exhibits 'uniformity': if something happens relative to a base point in a non-empty open set, then it happens in all of U.

As an application of Theorem 1.1, we construct a new type of example: a meromorphic function with an infinitely connected semi-contracting wandering domain. A simply connected example of such a domain, the first of its kind, was constructed by Benini *et al.* using approximation theory; [15, Theorem 1.1] tells us that, if a semi-contracting orbit of multiply connected wandering domains is to be found, it must be infinitely connected. With that in mind, in §4 we modify [7, Example 2] via quasiconformal surgery to prove the following result.

THEOREM 1.2. *There exists a meromorphic function f with an infinitely connected wandering domain V and a non-empty open subset $V' \subset V$ such that, for $z_0 \in V'$, V is semi-contracting relative to z_0.*

In [15], the author introduced *bimodal* and *trimodal* wandering domains, where different long-term qualitative behaviours of the hyperbolic metric co-exist. In contrast, we say that a wandering domain is *unimodal* if the limiting behaviour of the sequence $(d_{U_n}(f^n(z), f^n(w)))_{n\in\mathbb{N}}$ is independent of the choice of z and w in U. Given that examples of contracting and eventually isometric examples of multiply connected wandering domains already exist (see, for instance, [29, Example 1] and [23, Theorem (iii)]), Theorem 1.2 completes the proof of the existence of all possible unimodal behaviours in such wandering domains. In particular, we see that the possible internal dynamics of multiply connected wandering domains (and, in particular, infinitely connected ones) includes all possible internal dynamics that exists for simply connected wandering domains.

2. *Preliminaries and notation*

We use this section to establish some notation and terminology regarding the hyperbolic metric; we refer the reader to [4] and [20, Ch. 3] for a more detailed treatment of the subject. We start with the following definition.

Definition. A continuous function $f : X \to Y$, where X and Y are topological spaces, is called an *unbranched covering map* if:

(i) f is surjective;

(ii) every $y \in Y$ has a neighbourhood $U_y \subset Y$ such that $f^{-1}(U_y)$ is a union of disjoint open sets $V_y \subset X$ and $f : V_y \to U_y$ is a homeomorphism for each V_y.

Furthermore, if X is simply connected, f is called a universal covering map and X is called a universal covering space of Y.

With that in mind, we recall Koebe's uniformization theorem (see [14, Ch. 10] for a modern proof).

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LEMMA 2.1. (Uniformization theorem) *Let X be a Riemann surface. Then there exists a holomorphic map* $f : \tilde{X} \to X$ *that is a universal covering map, and* $\tilde{X}$ *is exactly one of* $\mathbb{C}$, $\widehat{\mathbb{C}}$, *or* $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, *the unit disc. Furthermore, given any* $p \in X$, *f can be chosen so that* $f(0) = p$.

In keeping with the name 'uniformization theorem', we will sometimes call the universal covering maps given by Lemma 2.1 *uniformizing maps*. We are interested in the case where the universal covering space of X is $\mathbb{D}$, which happens, for instance, if X is a domain on $\widehat{\mathbb{C}}$ such that $\widehat{\mathbb{C}} \setminus X$ contains at least three points (see [**4**, Theorem 10.2]). In this case, X is called a *hyperbolic surface* (or, if $X \subset \widehat{\mathbb{C}}$, a *hyperbolic domain*), for it admits a *hyperbolic metric*, that is, a complete conformal metric of constant curvature -1.

Given a hyperbolic surface X, we will use ρ_X and d_X to denote its *hyperbolic density* and *hyperbolic distance*, respectively. Since we are following [**4**, **20**], the hyperbolic density of the unit disc is

$$\rho_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2},$$

giving us constant curvature -1 as promised. A curve $\gamma : [0, 1] \to X$ has *hyperbolic length*

$$\ell_X(\gamma) := \int_\gamma \rho_X(z)\, |dz|;$$

it is a consequence of the completeness of the hyperbolic metric that any two points $w, z \in X$ can be joined by a smooth curve $\gamma \subset X$ such that

$$d_X(w, z) = \ell_X(\gamma) = \min\{\ell_X(\gamma') : \gamma' \subset X \text{ joins } w \text{ and } z\},$$

and this γ is called a *hyperbolic geodesic*.

Now, let $f : X \to Y$ be a holomorphic map between hyperbolic surfaces. Its *hyperbolic distortion* at $z \in X$ is given by

$$\|Df(z)\|_X^Y := \lim_{w \to z} \frac{d_Y(f(w), f(z))}{d_X(w, z)} = \frac{\rho_Y(f(z))|f'(z)|}{\rho_X(z)}.$$

This notation refers to the fact that the hyperbolic distortion is also the norm of the differential of f at z, viewed as a linear map from T_zX to $T_{f(z)}Y$, with the metrics on the tangent spaces induced by the respective hyperbolic metrics; see [**20**, §3.3]. The Schwarz–Pick lemma [**4**, Theorem 10.5] can now be expressed as follows.

LEMMA 2.2. (Schwarz–Pick lemma) *Let* $f : X \to Y$ *be a holomorphic map between hyperbolic Riemann surfaces. Then, for all* $z \in X$,

$$\|Df(z)\|_X^Y \le 1, \tag{1}$$

with equality if and only if X is an unbranched covering map. Additionally,

$$d_Y(f(z), f(w)) \le d_X(z, w) \tag{2}$$

for any distinct z and w in X, with equality if and only if f is biholomorphic.

A function for which equality holds in (1) is called a *local hyperbolic isometry (or, if the metric is clear from the context, a local isometry)*: it preserves hyperbolic distances in small neighbourhoods around each point. A function satisfying equality in (2) is simply called a *hyperbolic isometry*.

An immediate consequence of Lemma 2.2 and the chain rule is that hyperbolic distortion is *locally conformally invariant*. More precisely, we have (see, for instance, [**4**, Theorem 10.5], [**20**, Proposition 3.3.4] or [**21**, Theorem 7.3.1]) the following lemma.

LEMMA 2.3. *Let $f : X \to Y$ be a holomorphic map between hyperbolic Riemann surfaces, let $z_0 \in X$, let $\varphi : \mathbb{D} \to X$ be a uniformizing map such that $\varphi(0) = z_0$, and let $\psi : \mathbb{D} \to Y$ be a uniformizing map such that $\psi(0) = f(z_0)$. Then there exists a holomorphic function $\tilde{f} : \mathbb{D} \to \mathbb{D}$ such that $\tilde{f}(0) = 0$ and $\psi \circ \tilde{f} = f \circ \varphi$. Furthermore, $\tilde{f}$ is unique and satisfies*

$$|\tilde{f}'(0)| = \|Df(\varphi(0))\|_X^Y. \tag{3}$$

A function $\tilde{f}$ given by Lemma 2.3 is called a *lift* of *f*.

Another important aspect of hyperbolic geometry that is relevant to us are the hyperbolic discs. In the hyperbolic surface *X*, these are the open balls

$$B_X(p, r) := \{z \in X : d_X(p, z) < r\},$$

where $p \in X$ is the centre and $r > 0$ is the radius. They are not necessarily topologically equivalent to Euclidean open balls: an example would be to take the punctured disc $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$, any point $p \in \mathbb{D}^*$, and a sufficiently large $r > 0$; then, $B_{\mathbb{D}^*}(p, r)$ surrounds the origin, and is therefore doubly connected. Thus, given $p \in X$, we must look at the *injectivity radius* of *X* at *p*: the largest value of $r > 0$ for which $B_X(p, r)$ is actually isometric to $B_{\mathbb{D}}(0, r)$. Another way of saying this is that the injectivity radius at *p* is the largest $r > 0$ for which, taking a uniformizing map $\varphi : \mathbb{D} \to X$ with $\varphi(0) = p$, the restriction of φ to $B_{\mathbb{D}}(0, r)$ is injective (hence the name injectivity radius); see [**6**, p. 166].

3. *Uniformity in long-term behaviours of the hyperbolic metric*

In this section, we prove Theorem 1.1. We divide it into three cases, each one considering a different kind of 'unimodality'.

Before doing so, however, we take some time to discuss the function $u : U \to [0, +\infty)$ defined as

$$u(z) := \lim_{n\to+\infty} d_{U_n}(f^n(z), f^n(z_0)),$$

where we have taken some base point $z_0 \in U$. It is the limit of the sequence $(u_n)_{n\in\mathbb{N}}$, where

$$u_n(z) := d_{U_n}(f^n(z), f^n(z_0)) \quad \text{for } z \in U,$$

which by the Schwarz–Pick lemma (Lemma 2.2) satisfies

$$u_1 \geq u_2 \geq \cdots \geq u. \tag{4}$$

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The following lemma gives us an inkling of how the convergence of the hyperbolic distances to their final values happens.

LEMMA 3.1. *The convergence $u_n \to u$ is locally uniform. In particular, u is continuous.*

Proof. We will apply the Arzelà–Ascoli theorem [1, Theorem 14].

First, for any $z \in U$, it is clear from the Schwarz–Pick lemma that $(u_n(z))_{n\in\mathbb{N}}$ is contained in the compact set $[0, d_U(z, z_0)]$. Second, we must show that the sequence $(u_n)_{n\in\mathbb{N}}$ is also equicontinuous on compact subsets of U; to this end, we apply the reverse triangle inequality to obtain

$$d_{U_n}(f^n(z), f^n(w)) \geq |d_{U_n}(f^n(z), f^n(z_0)) - d_{U_n}(f^n(w), f^n(z_0))| = |u_n(z) - u_n(w)| \tag{5}$$

for every $z, w \in U$. Now, taking any $\epsilon > 0$ and $K \subset U$ a compact set, we can choose $\delta > 0$ such that

$$|z - w| < \delta \Rightarrow d_U(z, w) < \epsilon \quad \text{for any } z, w \in K,$$

by the fact that ρ_U is bounded above and below on K. Thus, by (5) and the Schwarz–Pick lemma, we get for all $n \in \mathbb{N}$ that

$$|u_n(z) - u_n(w)| \leq d_{U_n}(f^n(z), f^n(w)) \leq d_U(z, w) < \epsilon$$

whenever $|z - w| < \delta$ and $z, w \in K$, and therefore $(u_n)_{n\in\mathbb{N}}$ is equicontinuous on K. It follows that the Arzelà–Ascoli theorem [1, Theorem 14] applies: there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $u_{n_k} \to u$ locally uniformly. Then (4) implies that locally uniform convergence happens for the whole sequence $(u_n)_{n\in\mathbb{N}}$.

Finally, the locally uniform convergence of u_n implies that u is continuous. □

The continuity of u will be especially important for us as we discuss how the behaviour of orbits in small neighbourhoods 'globalizes' to the whole of U.

3.1. *Proof of Theorem 1.1(a): the contracting case.*

What we prove in this subsection is actually stronger than what Theorem 1.1(a) claims: we will show that, if U contains a non-empty open set V such that

$$d_{U_n}(f^n(z), f^n(w)) \to 0 \text{ as } n \to +\infty \quad \text{for every } z, w \in V,$$

then the same is true for every $z, w \in U$. In order to do this, we will use results of Benini *et al* relating to non-autonomous dynamics in the unit disc (more specifically, [7, §2]). As preparation for that, we prove the following lemma, which tells us how to associate the dynamics of a wandering domain to a composition of inner functions. Recall that an *inner function* is a holomorphic function $g : \mathbb{D} \to \mathbb{D}$ such that the radial limit

$$g(e^{i\theta}) := \lim_{r \nearrow 1} g(re^{i\theta})$$

exists and satisfies $|g(e^{i\theta})| = 1$ for Lebesgue almost every $\theta \in [0, 2\pi)$.

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LEMMA 3.2. *Let $f:\mathbb{C}\to\widehat{\mathbb{C}}$ be a meromorphic function with a multiply connected wandering domain U, and let z_0 be a point in U. Given uniformizing maps $\varphi_n:\mathbb{D}\to U_n$ such that $\varphi_n(0)=f^n(z_0)$, there exists a unique sequence of inner functions $g_n:\mathbb{D}\to\mathbb{D}$ such that, for all $n\geq 0$,*

(i) *g_n fixes the origin, and*

(ii) *$f|_{U_n}$ lifts to g_n, that is, $\varphi_{n+1}\circ g_n=f\circ\varphi_n$.*

Proof. The existence, analyticity, and uniqueness of the functions g_n are guaranteed by Lemma 2.3. It remains to show that they are inner functions.

Since $\varphi_n(\mathbb{D})=U_n$ and U is a wandering domain, it is clear that $f\circ\varphi_n(\mathbb{D})$ is not dense in $\widehat{\mathbb{C}}$ (indeed, it omits every other U_m, $m\neq n+1$); thus, we can (if necessary) compose $f\circ\varphi_n$ with a Möbius transformation m_n to obtain a *bounded* map $m_n\circ f\circ\varphi_n:\mathbb{D}\to\mathbb{C}$. By Fatou's theorem [18, Lemma 6.9], the radial limit $f\circ\varphi_n(e^{i\theta})$ exists and is finite for θ outside a set $E_n\subset[0,2\pi)$ of Lebesgue measure zero. Furthermore, also by Fatou's theorem, there exists a set E'_n of measure zero such that $g_n(e^{i\theta})$ exists for $\theta\in[0,2\pi)\setminus E'_n$. Now, for every $n\geq 0$, the set $F_n:=E_n\cup E'_n$ has measure zero, and for $\theta\in[0,2\pi)\setminus F_n$ we have

$$\lim_{r\nearrow 1}\varphi_{n+1}\circ g_n(re^{i\theta})=\lim_{r\nearrow 1} f\circ\varphi_n(re^{i\theta}),$$

where both limits exist and are finite. Since φ_n, being a covering map, has no asymptotic values in U_n and f is continuous, the right-hand side converges to a point in ∂U_{n+1}. Thus, on the left-hand side, we must have $|g_n(re^{i\theta})|\to 1$ as $r\nearrow 1$ by continuity of φ_{n+1} and the open mapping theorem. It follows that the g_n are inner functions. □

An immediate consequence of Lemma 3.2 is that, defining $G_n=g_n\circ g_{n-1}\circ\cdots\circ g_0$, we have

$$\varphi_n\circ G_{n-1}=f^n\circ\varphi_0 \quad \text{for every } n\geq 1. \tag{6}$$

However, since the domains U_n are multiply connected, the limiting behaviours of G_n and $(f^n)|_U$ relative to the hyperbolic metric are *not* necessarily the same, except in one particular case.

LEMMA 3.3. *With the notation and definitions above, if $G_n(w)\to 0$ as $n\to+\infty$ for all $w\in\mathbb{D}$, then U is a contracting wandering domain.*

Proof. Let $w\in\mathbb{D}$; since $G_n(w)\to 0$ as $n\to+\infty$, we know that $d_{\mathbb{D}}(0,G_n(w))\to 0$. By (6), we have

$$d_{U_n}(f^n(z_0),f^n\circ\varphi_0(w))=d_{U_n}(\varphi_n(0),\varphi_n\circ G_{n-1}(w)) \quad \text{for } w\in\mathbb{D},$$

and applying the Schwarz–Pick lemma to the right-hand side yields

$$d_{U_n}(f^n(z_0),f^n\circ\varphi_0(w))\leq d_{\mathbb{D}}(0,G_{n-1}(w)),$$

which goes to zero as claimed. □

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For a sequence $(G_n)_{n\in\mathbb{N}}$ as above, [7, Theorem 2.1] tells us that $d_{\mathbb{D}}(0, G_n(w)) \to 0$ for $w \in \mathbb{D}$ if and only if

$$\sum_{n\geq 0}(1 - |g_n'(0)|) = +\infty,$$

or equivalently, if none of the $g_n'(0)$ equal zero, $G_n'(0) \to 0$. Since f^n lifts to G_n, the local conformal invariance of hyperbolic distortion (recall (3)) implies that this is equivalent to saying that a sufficient condition for U to be a contracting wandering domain is

$$\lim_{n\to+\infty} \|Df^n(z_0)\|_U^{U_n} = 0.$$

Now our course of action is to choose a $z_0 \in V$, the contracting neighbourhood inside U, and show that the condition above holds. To this end, we prove the following hyperbolic version of Landau's theorem. Unlike in Landau's original theorem, the function here is not normalized (if it were, it would be a hyperbolic isometry!), and thus there is dependence on the derivative of f.

LEMMA 3.4. *Let $f : \Omega_1 \to \Omega_2$ be a holomorphic map between hyperbolic domains, let $r > 0$, and let $z_0 \in \Omega_1$. If $0 < \|Df(z_0)\|_{\Omega_1}^{\Omega_2} < 1$, then $f(B_{\Omega_1}(z_0, r))$ contains a hyperbolic disc of radius*

$$2Lr^*\|Df(z_0)\|_{\Omega_1}^{\Omega_2},$$

where $r^* = \tanh(r/2)$ and $L \in (0.5, 0.544)$ is Landau's constant.

Proof. First, we take uniformizing maps $\varphi : \mathbb{D} \to \Omega_1$ and $\psi : \mathbb{D} \to \Omega_2$ with $\varphi(0) = z_0$ and $\psi(0) = f(z_0)$. Then, by Lemma 2.3, f admits a lift $F : \mathbb{D} \to \mathbb{D}$ (that is, F satisfies $\psi \circ F = f \circ \varphi$), which can be chosen to satisfy $F(0) = 0$, and by (3) we also have $|F'(0)| = \|Df(z_0)\|_{\Omega_1}^{\Omega_2}$. The ball $B_{\mathbb{D}}(0, r)$ is contained (by the Schwarz–Pick lemma) in $\varphi^{-1}(B_{\Omega_1}(z_0, r))$, and its image under F contains (by Landau's theorem [24]; see [27] for a discussion of the bounds given here) a disc of Euclidean radius $Lr^*|F'(0)| = Lr^*\|Df(z_0)\|_{\Omega_1}^{\Omega_2}$, where we have set $r^* = \tanh(r/2)$ so that $d_{\mathbb{D}}(0, r^*) = r$. Since we do not know where the centre w_0 of this disc is, we cannot accurately calculate its hyperbolic size. Nevertheless, we do know that, for any w on its boundary,

$$d_{\mathbb{D}}(w_0, w) = \int_\gamma \rho_{\mathbb{D}}(s)\, |ds| \geq 2\int_\gamma |ds| \geq 2|w - w_0| = 2Lr^*\|Df(z_0)\|_{\Omega_1}^{\Omega_2},$$

where $\gamma \subset \mathbb{D}$ is a hyperbolic geodesic connecting w_0 to w. Hence, $F(B_{\mathbb{D}}(0, r))$ contains a disc of hyperbolic radius $2Lr^*\|Df(z_0)\|_{\Omega_1}^{\Omega_2}$.

In order to transfer this knowledge from $\mathbb{D}$ to Ω_2, we will use tools from [3] (see also [21, Ch. 10]). More specifically, for a subdomain D of Ω_2, let $R(D, \Omega_2)$ denote the hyperbolic radius of the largest hyperbolic disc contained in D; this is the hyperbolic Bloch constant, and Beardon *et al* proved in [3, Lemma 4.2] that it is invariant under unbranched covering maps. More specifically, let B^* denote the component of $\psi^{-1}(f(B_{\Omega_1}(z_0, r)))$

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containing the origin. Then, since $f \circ \varphi = \psi \circ F$, $F(0) = 0$, and B^* is connected, we have $B^* \supset F(B_{\mathbb{D}}(0, r))$, and thus

$$R(f(B_{\Omega_1}(z_0, r)), \Omega_2) = R(B^*, \mathbb{D}) \geq R(F(B_{\mathbb{D}}(0, r)), \mathbb{D}).$$

Since we know that $F(B_{\mathbb{D}}(0, r))$ contains a disc of hyperbolic radius $2Lr^* \|Df(z_0)\|_{\Omega_1}^{\Omega_2}$, we have

$$R(F(B_{\mathbb{D}}(0, r)), \mathbb{D}) \geq 2Lr^* \|Df(z_0)\|_{\Omega_1}^{\Omega_2}$$

and we are done. □

With these results in hand, we can finally prove what we intended. Notice that, by the triangle inequality, the hypotheses of Theorem 1.1(a) imply those of Theorem 3.5.

THEOREM 3.5. *Let U be a wandering domain of a meromorphic function f. If U contains a non-empty open set V such that*

$$d_{U_n}(f^n(z), f^n(w)) \to 0$$

for every z and w in V, then U is contracting.

Proof. Take any $z_0 \in V$, and take uniformizing maps $\varphi_n : \mathbb{D} \to U_n$ such that $\varphi_n(0) = f^n(z_0)$. Let g_n be the lifts of f given by Lemma 3.2. We want to show that

$$\sum_{n \geq 0} (1 - |g_n'(0)|) = +\infty, \tag{7}$$

whence our theorem will follow by Lemma 3.3 and [7, Theorem 2.1]. We can assume that none of the $g_n'(0)$ are zero; indeed, if there are infinitely many such g_n, then (7) is trivially true, and if there are only finitely many such g_n we pass to U_N, $f^N(z_0)$, and $f^N(V)$ for some sufficiently large *N*. Thus (recall that $G_n = g_n \circ \cdots \circ g_0$), as anticipated on page 8, (7) is equivalent to $|G_n'(0)| \to 0$ or, by (3), to $\|Df^n(z_0)\|_U^{U_n} \to 0$.

Assume now that this is not the case; notice that, again by the Schwarz–Pick lemma, the sequence $(\|Df^n(z_0)\|_U^{U_n})_{n \in \mathbb{N}}$ is decreasing, and so if it gets arbitrarily close to zero on a subsequence then it is in fact tending to zero. In other words, there must exist some constant $c > 0$ such that $\|Df^n(z_0)\|_U^{U_n} > c$ for all $n \geq 0$. Choose some $r > 0$ such that $K := \overline{B_U(z_0, r)} \subset V$; then, by Lemma 3.1, we have $\text{diam}_{U_n}(f^n(K)) \to 0$, while by Lemma 3.4 $f^n(K)$ always contains a hyperbolic ball of radius $2Lc \cdot \tanh(r/2)$. This is clearly a contradiction; we are done. □

3.2. *Proof of Theorem 1.1(b): the semi-contracting case.*

In this subsection, we want to show that if there exist some point $z_0 \in U$ and an open neighbourhood V of z_0 such that $u_n(z) \searrow u(z) > 0$ without ever reaching it for $z \in V$ outside of a discrete set (u_n and u as defined at the beginning of §3, with z_0 as base point), then the same holds for every $z \in U$ except those for which $f^n(z) = f^n(z_0)$ for some $n \in \mathbb{N}$.

We will divide the proof into two cases. We prove first that no point $z \in U$ can be contracting relative to z_0, and then that no point can be eventually isometric relative to z_0.

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For the first case, we will revisit the function u; in particular, we notice that it is 'f-invariant' in the sense that, if $u^* : U_1 \to [0, +\infty)$ is defined taking as a base point $f(z_0)$, then $u^*(f(z)) = u(z)$. By an abuse of notation, we will keep referring to the functions defined on U_n with base points $f^n(z_0)$ as u and hope it will not lead to confusion. Now what we want to show is that $u(z) = 0$ if and only if $f^n(z) = f^n(z_0)$ for some n.

The 'if' is trivial. For the 'only if' part, let $z \in U$ be such that $f^n(z) \neq f^n(z_0)$ for all n, and assume that $u(z) = 0$. Then, since U is semi-contracting in V, we can take a closed curve $\gamma \subset V$ that surrounds z_0 and avoids the zeros of u, which are discrete in V by hypothesis; by Lemma 3.1, $u|_\gamma$ achieves a minimum $c > 0$. Additionally, V is in the Fatou set, meaning that $(f^n)|_V$ is holomorphic for every $n \in \mathbb{N}$, and so by the argument principle $f^n(\gamma)$ surrounds $B_{U_n}(f^n(z_0), c)$ for all $n \in \mathbb{N}$. Therefore, since we have that $d_{U_n}(f^n(z), f^n(z_0)) \to 0$, there must be $N_1 \in \mathbb{N}$ such that $f^{N_1}(\gamma)$ surrounds $f^{N_1}(z)$, and thus (again by the argument principle) there exists $w \in V$ surrounded by γ such that $f^{N_1}(w) = f^{N_1}(z)$. Now, by f-invariance of u, we have $u(w) = u(z) = 0$, and so by the definition of V there exists $N_2 \in \mathbb{N}$ for which $f^{N_2}(w) = f^{N_2}(z_0)$. Finally, we see that $f^{N_1+N_2}(z) = f^{N_1+N_2}(z_0)$, which is a contradiction since we assumed that $f^n(z) \neq f^n(z_0)$ for all $n \in \mathbb{N}$. It follows that $u(z) > 0$.

Let us now exclude the possibility of eventually isometric points. We show that if there exists $z \in U$ such that

$$d_{U_n}(f^n(z), f^n(z_0)) = c(z, z_0) > 0 \quad \text{for } n \geq N, \tag{8}$$

then $f : U_n \to U_{n+1}$ is a local hyperbolic isometry (equivalently, an unbranched covering map) for $n \geq N$. Indeed, let $\gamma \subset U_n$ be a hyperbolic geodesic joining $f^n(z)$ to $f^n(z_0)$. Then, by the Schwarz–Pick lemma,

$$\ell_{U_{n+1}}(f \circ \gamma) \leq \ell_{U_n}(\gamma),$$

while by the definition of the hyperbolic distance and (8) we have

$$\ell_{U_{n+1}}(f \circ \gamma) \geq d_{U_{n+1}}(f^{n+1}(z), f^{n+1}(z_0)) = d_{U_n}(f^n(z), f^n(z_0)) = \ell_{U_n}(\gamma).$$

Hence, $\ell_{U_{n+1}}(f \circ \gamma) = \ell_{U_n}(\gamma)$, or, equivalently,

$$\int_\gamma \rho_{U_n}(s)\, |ds| = \int_{f \circ \gamma} \rho_{U_{n+1}}(s')\, |ds'|.$$

By a change of variables, this becomes

$$\int_\gamma \rho_{U_n}(s)\, |ds| = \int_\gamma \rho_{U_{n+1}}(f(s))|f'(s)|\, |ds|,$$

and since $0 \leq \rho_{U_{n+1}}(f(z))|f'(s)| \leq \rho_{U_n}(s)$ by the Schwarz–Pick lemma, we are forced to conclude that $\rho_{U_{n+1}}(f(s))|f'(s)| = \rho_{U_n}(s)$ for every $s \in \gamma$. From the equality case of the Schwarz–Pick lemma, we deduce that $f : U_n \to U_{n+1}$ is a local hyperbolic isometry.

From now on, we will exchange U for U_N, z_0 for $f^N(z_0)$, and V for $f^N(V)$ for some sufficiently large N, and work as if f maps one wandering domain onto the next one locally isometrically for every $n \in \mathbb{N}$. We can do this because of the 'f-invariance' of u: if $V \subset U$

is semi-contracting relative to $z_0 \in V$, then $f^N(V) \subset U_N$ is semi-contracting relative to $f^N(z_0) \in f^N(V)$.

We will see that V being semi-contracting relative to $z_0 \in V$ is incompatible with $f : U_n \to U_{n+1}$ being a local hyperbolic isometry. Indeed, let w be any point in V such that $f^n(w) \neq f^n(z_0)$ for all n and $\overline{B_U(z_0, d_U(z_0, w))} \subset V$. Since V is semi-contracting, $u_n(w) = d_{U_n}(f^n(w), f^n(z_0))$ forms a non-increasing sequence that is also not eventually constant, that is, there is no N such that $u_n(w)$ is constant for $n \geq N$. However, as $f : U_n \to U_{n+1}$ is a local hyperbolic isometry for all $n \geq 0$, we can produce a sequence $(w_n)_{n\in\mathbb{N}}$ in $U \setminus \{w\}$ such that $f^n(w) = f^n(w_n)$ and $d_{U_n}(f^n(w), f^n(z_0)) = d_U(w_n, z_0)$ for all $n \in \mathbb{N}$ as follows.

Taking distance-minimizing geodesics $\gamma_n \subset U_n$ joining $f^n(z_0)$ to $f^n(w)$, we apply the path lifting property to $f^n : U \to U_n$ (see [**14**, Proposition 10] or [**17**, Proposition 1.30]) to obtain a curve $\tilde{\gamma}_n \subset U$ joining z_0 to a point $w_n \in U$ such that $f^n(w_n) = f^n(w)$. Since $f^n : U \to U_n$ is a local hyperbolic isometry, it preserves curve length, and thus

$$d_{U_n}(f^n(z_0), f^n(w)) = \ell_{U_n}(\gamma_n) = \ell_U(\tilde{\gamma}_n) \geq d_U(z_0, w).$$

On the other hand, by the Schwarz–Pick lemma,

$$d_U(z_0, w_n) \geq d_{U_n}(f^n(z_0), f^n(w)),$$

and therefore $d_U(z_0, w_n) = d_{U_n}(f^n(z_0), f^n(w))$ for all $n \in \mathbb{N}$. Notice, though, that $(d_{U_n}(f^n(z_0), f^n(w)))_{n\in\mathbb{N}}$ is, by hypothesis, a non-constant non-increasing sequence, so that $w_n \neq w$ for all large enough n.

By the same token, the sequence $(w_n)_{n\in\mathbb{N}}$ is confined to the annulus $\{z \in U : u(w) \leq d_U(z, z_0) \leq d_U(w, z_0)\} \subset V$, and hence has an accumulation point $w^* \in V$. Since u is f-invariant, every w_n satisfies $u(w_n) = u(w)$, and thus by the continuity of u we have $u(w^*) = u(w)$. But it is also the case by the definition of the points w_n that $u(w^*) = u(w) = \lim_{n\to+\infty} d_U(w_n, z_0)$, and so by continuity of the hyperbolic distance we have

$$u(w^*) = d_U(w^*, z_0).$$

It follows from the definition of u that w^* is an eventually isometric point relative to z_0 lying in V, which is a contradiction.

Together, these two arguments show that every point in U is semi-contracting relative to z_0.

3.3. *Proof of Theorem 1.1(c): the eventually isometric case.* Here, we assume that U contains a non-empty open set V and a point $z_0 \in V$ such that $d_{U_n}(f^n(z), f^n(z_0)) = c(z, z_0) > 0$ for every $z \in V$ (except for at most countably many points for which $f^k(z) = f^k(z_0)$ for some $k \in \mathbb{N}$) and all sufficiently large n (say, $n \geq N$, with N locally uniform over compact subsets of V). We want to show that the same holds for every $w \in U$ relative to $z_0 \in V$. Perhaps surprisingly, this is the most delicate case we will deal with here; it needs certain machinery that we now take the time to introduce.

We consider the set $\mathcal{H}_2$ of hyperbolic surfaces, identifying any two surfaces that are isometric. In other words, $\mathcal{H}_2$ is a space of equivalence classes of hyperbolic surfaces. We can 'refine' it a little by considering *marked* hyperbolic surfaces: pairs (S, p) where

$S \in \mathcal{H}_2$ and $p \in S$ is a base point. The space of all such pairs, again up to isometry equivalence, is denoted $\mathcal{H}_2^*$; we will now imbue it with a topology.

Definition. Let $((S_n, p_n))_{n\in\mathbb{N}}$ be a sequence of marked hyperbolic surfaces. We say that the sequence converges to the marked surface $(S^*, p^*) \in \mathcal{H}_2^*$ *in the sense of Gromov* if, for every $r > 0$, there exists a sequence of smooth orientation-preserving diffeomorphisms $\phi_n : U_n \to \phi_n(U_n) \subset S_n$ such that the following assertions hold.

(i) Each $U_n \subset S^*$ is a neighbourhood of p^* containing $B_{S^*}(p^*, r)$.

(ii) For all $n \in \mathbb{N}$, $\phi_n(p^*) = p_n$.

(iii) Each ϕ_n is K_n-bi-Lipschitz relative to the hyperbolic metrics of S^* and S_n, and $K_n \to 1$ as $n \to +\infty$.

Despite its (relatively) intuitive definition, this topology, called the *geometric topology*, offers little insight into its own properties. Fortunately, there is an equivalent way of defining it via Kleinian groups, originally due to Claude Chabauty, which we will not state here. The following result was proved using this alternative definition (see [**6**, Theorem E.1.10], [**12**, Corollary I.3.1.7], or [**25**, Proposition 7.8]).

LEMMA 3.6. *Let $((S_n, p_n))_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_2^*$ such that the injectivity radius of S_n at p_n is at least some $r > 0$ for all $n \in \mathbb{N}$. Then there exists a subsequence $((S_{n_k}, p_{n_k}))_{k\in\mathbb{N}}$ converging in the sense of Gromov to a marked hyperbolic surface (S^*, p^*).*

Notice that Lemma 3.6 makes no claim about the limit surface (or, to be more precise, the equivalence class of limit surfaces). The nature of the limit surface can, in fact, be extremely counter-intuitive, but since appreciating the intricacies of the geometric topology is not our aim here, Lemma 3.6 will suffice.

Convergence in the sense of Gromov can be though of as a 'locally uniform convergence' of the surfaces' geometry around the base point. Thus, Lemma 3.6 is, in a certain sense, a 'normal families' criterion for $\mathcal{H}_2$. We will also need more conventional results on normal families, such as the following restatement of the Arzelà–Ascoli theorem for Lipschitz functions; see [**5**, Theorem 7.1].

LEMMA 3.7. *Let X and Y be Riemann surfaces with complete conformal metrics d_X and d_Y, and suppose that $\{f_\alpha : X \to Y\}_{\alpha\in A}$ is a family of locally uniformly d_X-to-d_Y Lipschitz functions. Then $\{f_\alpha\}_{\alpha\in A}$ is a normal family if and only if there exists $x \in X$ such that $\{f_\alpha(x) : \alpha \in A\}$ is relatively compact in Y.*

With this in mind, let us begin our study of the eventually isometric case in Theorem 1.1.

Let $U, f, V \subset U$ and $z_0 \in V$ be as in the statement of Theorem 1.1(c). We can exclude the possibility of contracting points in U (relative to z_0) by an argument similar to the one used in §3.2: the iterates of such a point $w \in U$ would eventually intersect $f^n(V)$, and the f-invariance of u would imply that $u(w) > 0$, contradicting our choice of w.

Now, suppose that there exists a point $w \in U \setminus V$ that is semi-contracting relative to z_0. The same argument as in §3.2 shows that $f : U_n \to U_{n+1}$ is a local hyperbolic isometry for $n \geq N$ (N here being chosen according to some compact subset of V), but the rest of the argument does not carry through—there is nothing unexpected about finding an eventually isometric point relative to z_0 in V; think, for instance, of the 'annulus model' described in [15, §2]. Instead, we will proceed with a 'normal families' argument. As before, we swap U for U_N, z_0 for $f^N(z_0)$, and V for $f^N(V)$ while keeping the same notation.

First, we take a universal covering map $\varphi_0 : \mathbb{D} \to U$ with $\varphi_0(0) = z_0$, and build the functions $\psi_n : \mathbb{D} \to U_n$ given by $\psi_n(z) = f^n \circ \varphi_0(z)$; since f^n is a local hyperbolic isometry, these are all universal covering maps with $\psi_n(0) = f^n(z_0)$. Furthermore, the hypothesis that V is locally eventually isometric implies that, if r_0 is such that $\overline{B_U(z_0, r_0)} \subset V$ is an embedded disc (and N was chosen accordingly), then f^n maps $B_U(z_0, r_0)$ isometrically onto $f^n(B_U(z_0, r_0))$ for all n. In particular, the sets $f^n(B_U(z_0, r_0))$ will all be embedded discs of hyperbolic radius r_0. Therefore, the injectivity radii at $f^n(z_0)$ of the hyperbolic surfaces U_n are uniformly bounded below by r_0. Thus, by Lemma 3.6, the sequence of marked hyperbolic surfaces $((U_n, f^n(z_0)))_{n\in\mathbb{N}}$ admits a subsequence $((U_{n_k}, f^{n_k}(z_0)))_{k\in\mathbb{N}}$ converging in the sense of Gromov to a marked hyperbolic surface (U^*, z^*) (say).

It follows from the definition, then, that we can take some $r > 10 d_U(z_0, w)$ (where $w \in U \setminus V$ is assumed to be semi-contracting relative to z_0) and find diffeomorphisms $\phi_k : \overline{B_{U^*}(z^*, r)} \to \phi_k(\overline{B_{U^*}(z^*, r)}) \subset U_{n_k}$ such that $\phi_k(z^*) = f^{n_k}(z_0)$ and each ϕ_k is K_k-bi-Lipschitz relative to the hyperbolic metrics on U^* and U_{n_k}, with $K_k \to 1$ as $k \to +\infty$. Since $K_k \to 1$, we can assume without loss of generality that $\sup_k K_k < 10$, so that every ϕ_k^{-1} is defined on a hyperbolic ball of radius $d_U(z_0, w)$ around $f^{n_k}(z_0)$. In particular, the functions

$$\widehat{\psi}_k = \phi_k^{-1} \circ \psi_{n_k} : B_{\mathbb{D}}(0, d_U(z_0, w)) \to U^*$$

are all well defined and map 0 to z^*. We want to show that $\{\widehat{\psi}_k\}_{k\in\mathbb{N}}$ is a normal family in the sense of precompactness relative to locally uniform convergence (see [5], as well as [11, Theorem 1.21] and [19] for an account of this interpretation of normality) with analytic limit functions; to this end, we make two claims.

CLAIM 3.8. *Each $\widehat{\psi}_k$ is $(K_k)^2$-quasiregular.*

Proof. Being K_k-bi-Lipschitz, each ϕ_k is $(K_k)^2$-quasiconformal (see, for instance, [10, Ch. 1] or [20, Proposition 4.5.14]), and the composition of a restriction of a universal covering map (which is, of course, 1-quasiregular) with a $(K_k)^2$-quasiconformal map yields a K_k^2-quasiregular map. □

CLAIM 3.9. *The family $\{\widehat{\psi}_k\}_k$ is uniformly Lipschitz relative to the hyperbolic metric.*

Proof. For z and z' in $B_w = B_{\mathbb{D}}(0, d_U(z_0, w))$, we have, using the bi-Lipschitz constant of ϕ_k and the Schwarz–Pick lemma respectively,

$$\begin{aligned} d_{U^*}(\phi_k^{-1} \circ \psi_{n_k}(z), \phi_k^{-1} \circ \psi_{n_k}(z')) &\le K_k d_{U_{n_k}}(\psi_{n_k}(z), \psi_{n_k}(z')) \\ &\le K_k d_{\mathbb{D}}(z, z') \le K_k d_{B_w}(z, z'). \end{aligned}$$

Since $K_k \to 1$, we have that $K := \sup_k K_k < 10$ is a uniform Lipschitz constant for $\phi_k^{-1} \circ \psi_{n_k} = \widehat{\psi}_k$. □

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Now, since $\{\widehat{\psi}_k(0) : k \in \mathbb{N}\} = \{z^*\}$ is clearly a relatively compact subset of U^*, Lemma 3.7 tells us that $(\widehat{\psi}_k)_k$ is a normal family, and so admits a subsequence $(\widehat{\psi}_{k_m})_m$ converging locally uniformly to a limit function

$$\psi^* : B_{\mathbb{D}}(0, d_U(z_0, w)) \to U^*.$$

Since $K_k \to 1$ and the limit of K-quasiregular functions is K-quasiregular, it follows by Claim 3.8 that ψ^* is 1-quasiregular (see, for example, [**10**, Theorem 4.2], [**19**], [**2**, Corollary 5.5.7], or [**28**, Theorem VI.8.6]) and hence analytic by the quasiregular version of Weyl's lemma (see [**11**, Proposition 1.37]).

We want to show that ψ^* is not constant; to that end, take a point $z' \in B_{\mathbb{D}}(0, d_U(z_0, w))$ such that $\varphi_0(z') \in V \setminus \{z_0\}$. Then, by the definition of the maps $\widehat{\psi}_n$ and the bi-Lipschitz property of ϕ_k,

$$d_{U^*}(z^*, \widehat{\psi}_{k_m}(z')) \geq \frac{d_{U_n}(\psi_{k_m}(0), \psi_{k_m}(z'))}{K_{k_m}} \geq \frac{d_{U_n}(\psi_{k_m}(0), \psi_{k_m}(z'))}{10};$$

since $\psi_n = f^n \circ \varphi_0$ and f^n is isometric when restricted to V, we have that $d_{U_n}(\psi_{k_m}(0), \psi_{k_m}(z')) = d_U(z_0, \varphi_0(z')) > 0$ for all m. Thus, by making $m \to +\infty$, we see that $d_{U^*}(\psi^*(0), \psi^*(z')) > 0$, and so ψ^* is indeed non-constant.

We are now in position to make a case against the existence of w, the semi-contracting point relative to z_0. The fact that $f : U_n \to U_{n+1}$ are local hyperbolic isometries implies that, in order for $d_{U_n}(f^n(w), f^n(z_0))$ to decrease infinitely many times, we must (as in §3.2) be able to find a sequence $w_n \in U \setminus \{w\}$ such that $f^n(w_k) = f^n(w)$ for $1 \leq k \leq n$ and $d_U(w_n, z_0) = d_{U_n}(f^n(w), f^n(z_0)) < d_U(z_0, w)$. By the completeness of the hyperbolic metric in U, the sequence w_n admits an accumulation point $w^* \in B_U(z_0, d_U(z_0, w))$, and this lifts to a sequence $\tilde{w}_n \in B_{\mathbb{D}}(0, d_U(z_0, w))$ with an accumulation point $\tilde{w}^* \in B_{\mathbb{D}}(0, d_U(z_0, w))$. However, repeated application of the triangle inequality yields

$$\begin{aligned} d_{U^*}(\psi^*(\tilde{w}_{k_m}), \psi^*(\tilde{w}^*)) \leq\ & d_{U^*}(\psi^*(\tilde{w}_{k_m}), \widehat{\psi}_{k_l}(\tilde{w}_{k_m})) \\ & + d_{U^*}(\widehat{\psi}_{k_l}(\tilde{w}_{k_m}), \widehat{\psi}_{k_l}(\tilde{w}_{k_l})) + d_{U^*}(\widehat{\psi}_{k_l}(\tilde{w}_{k_l}), \psi^*(\tilde{w}^*)), \end{aligned}$$

where $m \in \mathbb{N}$ is fixed and $l \geq m$. By the definitions of $\tilde{w}_n$ and $\widehat{\psi}_k$, we have that $\widehat{\psi}_{k_l}(\tilde{w}_{k_l}) = \widehat{\psi}_{k_l}(\tilde{w}_{k_m})$, so that the middle term in the sum above vanishes. As for the other two, taking the limit $l \to +\infty$ makes them arbitrarily small, since $\widehat{\psi}_{k_l}$ converges to ψ^* locally uniformly. It follows that $\psi^*(\tilde{w}_{k_m}) = \psi^*(\tilde{w}^*)$ for all $m \in \mathbb{N}$. This is a contradiction, since ψ^* is analytic and as such $(\psi^*)^{-1}(p)$ is discrete for any $p \in U^*$.

We have completed the proof of case (c), and hence finished the proof of Theorem 1.1.

Remark. This argument can be adapted to rule out the existence of contracting points *and* to drop the dependence on the base point z_0, which takes care of the eventually isometric case in its general form. Indeed, the contradiction above can be obtained for any pair of points $z, w \in U$ such that $d_{U_n}(f^n(z), f^n(w))$ is not eventually constant; one need only choose a sufficiently large radius for the discs where the ϕ_k are to be defined.

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4. *A semi-contracting multiply connected wandering domain*

4.1. *The patient.* Benini *et al* constructed the first known examples of semi-contracting wandering domains using approximation theory (the one we are interested in is [7, Example 2(a)]). As such, we can describe its properties mostly in an asymptotic fashion; but by controlling the rate of convergence, this will suffice for our ends.

To describe the example, we shall need the following sets and functions. Let $T_n : \mathbb{C} \to \mathbb{C}$ and $b_n : \mathbb{D} \to \mathbb{D}$ be defined as

$$T_n(z) = z + 4n \quad \text{and} \quad b_n(z) = z \cdot \frac{z + a_n}{1 + a_n z},$$

where a_n is a real sequence satisfying $0 < a_n < 1$ and $a_n \nearrow 1$ fast enough that

$$\lambda := \prod_{n=1}^{+\infty} a_n > 0;$$

as our construction progresses, we will impose further restrictions on the speed with which $a_n \nearrow 1$. Define also the discs $\Delta_n' := B(4n, r_n)$ and $\Delta_n := B(4n, R_n)$, where $0 < r_n < 1 < R_n$, both sequences $(r_n)_{n\in\mathbb{N}}$ and $(R_n)_{n\in\mathbb{N}}$ tend to 1 as $n \to +\infty$, and the rate of convergence is relatively 'free'—that is, $1 - r_n$ and $R_n - 1$ have upper bounds depending only on $1 - a_n$.

With all that in place, their example consists of an entire function f with a bounded simply connected wandering domain U such that, for all $n \geq 0$, the following properties hold.

(A) $\overline{\Delta_n'} \subset U_n \subset \Delta_n$, so, in particular, U_n 'is asymptotically a disc' (recall that U_n is the Fatou component containing $f^n(U)$).

(B) $|f(z) - T_{n+1} \circ b_{n+1} \circ T_n^{-1}(z)| < \epsilon_{n+1}$ for $z \in \overline{\Delta_n}$, where $\epsilon_n < (R_n - r_n)/4$.

(C) $f^n(0) = 4n$.

(D) $\deg f|_{U_n} = \deg b_{n+1} = 2$ and $f|_{U_n}$ satisfies

$$|f(z) - z - 4| \to 0 \quad \text{locally uniformly for } z \in U_n \text{ as } n \to +\infty.$$

In particular, the critical points z_n of $f|_{U_n}$ satisfy

$$\operatorname{dist}(z_n, \partial U_n) \to 0 \quad \text{as } n \to +\infty,$$

where dist denotes Euclidean distance.

It was shown in [7, Example 2] that the wandering domain U is semi-contracting; let us take a closer look at it.

Let $B_n(z) := b_n \circ b_{n-1} \circ \cdots \circ b_1(z)$ for $n \in \mathbb{N}$. By Montel's theorem, B_n admits a subsequence converging locally uniformly to some $B : \mathbb{D} \to \overline{\mathbb{D}}$; since $b_n(0) = 0$ for all $n \in \mathbb{N}$, we have $B(0) = 0$, and thus $B : \mathbb{D} \to \mathbb{D}$. Furthermore, by the Weierstrass convergence theorem and the assumptions on a_n,

$$B'(0) = \lim_{n\to+\infty} B_n'(0) = \prod_{n=1}^{+\infty} a_n = \lambda > 0,$$

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and so B is non-constant. Additionally, since $B_n'(0) = \prod_{k=1}^{n} a_k > 0$ and, by Schwarz's lemma applied to the sequence $(B_n)_{n\in\mathbb{N}}$,

$$|B_n(z)| \searrow |B(z)| \quad \text{for all } z \in \mathbb{D}, \tag{9}$$

the limit function B is unique.

Next, we recall [7, Corollary 2.4].

LEMMA 4.1. *Let $h : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $h(0) = 0$ and $|h'(0)| = \mu$. Then, for all $w \in \mathbb{D}$,*

$$h_1(|w|) := |w| \cdot \frac{\mu - |w|}{1 - \mu|w|} \leq |h(w)| \leq |w| \cdot \frac{\mu + |w|}{1 + \mu|w|} =: h_2(|w|).$$

Elementary calculus shows that the function $h_1 : [0, 1] \to \mathbb{R}$ is a concave function with maximum

$$\frac{2\mu^2 + (2 - \mu^2)\sqrt{1 - \mu^2} - 2}{\mu^2\sqrt{1 - \mu^2}}.$$

Thus, if we choose $c \in (0, 1)$ such that

$$c < \frac{2\lambda^2 + (2 - \lambda^2)\sqrt{1 - \lambda^2} - 2}{\lambda^2\sqrt{1 - \lambda^2}},$$

then there exists a round annulus $A \subset \mathbb{D}$ centred at 0 such that $|B(z)| > c$ for $z \in A$. Now recall (9); in particular, since $(|B_n(z)|)_{n\in\mathbb{N}}$ is a non-increasing sequence for all $z \in \mathbb{D}$ and $B_n(z) = b_n \circ b_{n-1} \circ \cdots \circ b_1(z)$, we also have $|b_n(z)| > c$ for $z \in A$ for all $n \in \mathbb{N}$.

Now take some positive $c' < c$; applying Rouche's theorem to condition (B) satisfied by f, we conclude that, for all sufficiently large n, there exists a topological annulus $A_n' \subset \overline{\Delta_n'}$ such that $|f(z) - 4(n + 1)| > c'$ for all $z \in A_n'$, and A_n' surrounds the disc $\{z \in \mathbb{C} : |z - 4n| \leq c'\}$.

4.2. *The surgery.* At each U_n, we want to cut out a small disc and replace f by an appropriately rescaled version of the Joukowski map $z \mapsto z + z^{-1}$ inside of it, giving us a pole in each domain. After that, we must join f to the Joukowski map through quasiconformal interpolation in an appropriate annulus $A_n \subset U_n$ such that $\overline{A_n} \subset U_n$. Since we want the resulting quasiregular map g_0 to be quasiconformally conjugate to a meromorphic one, there are two conditions we require to hold.

(1) Ensure that the dilatations K_n of the interpolating map in A_n (respectively) satisfy

$$\prod_{n=1}^{+\infty} K_n < +\infty;$$

(2) For any $z \in A_n$, its orbit under g_0 does not intersect A_m for $m < n$.

The two main results we will use in this surgery are both due to Kisaka and Shishikura. First, we have a way to interpolate quasiconformally in round annuli [22, Lemma 6.2].

LEMMA 4.2. *Let $k \in \mathbb{N}$, $0 < R_1 < R_2$, and φ_j be analytic on a neighbourhood of $C_j := \{|z| = R_j\}$ such that $\varphi_j|_{C_j}$ winds around the origin k times ($j = 1, 2$). If there exist positive constants δ_0 and δ_1 such that*

$$\left| \log\left(\frac{\varphi_2(R_2e^{i\theta})}{R_2^k}\frac{R_1^k}{\varphi_1(R_1e^{i\theta})}\right)\right| \leq \delta_0 \tag{10}$$

and

$$\left| z\frac{d}{dz}\left(\log\frac{\varphi_j(z)}{z^k}\right)\right| \leq \delta_1, \quad z = R_je^{i\theta}, \ (j = 1, 2), \tag{11}$$

for every $\theta \in [0, 2\pi)$, and if δ_0 and δ_1 satisfy

$$C = 1 - \frac{1}{k}\left(\frac{\delta_0}{\log(R_2/R_1)} + \delta_1\right) > 0, \tag{12}$$

then there exists a quasiregular map

$$H : \{z : R_1 \leq |z| \leq R_2\} \to \mathbb{C}^*$$

without critical points that interpolates between φ_1 and φ_2 and has dilatation constant

$$K_H \leq \frac{1}{C}. \tag{13}$$

Second, we need a sufficient condition for us to conjugate the resulting quasiregular map to a meromorphic one [**22**, Theorem 3.1]. Although it was originally stated only for entire maps, it is easy to see how to deal with the presence of poles: this lemma is a particular case of the necessary and sufficient conditions given by Sullivan's straightening theorem, which is flexible enough for transcendental meromorphic maps (see [**11**, §§5.2 and 5.3]).

LEMMA 4.3. *Let $g : \mathbb{C} \to \widehat{\mathbb{C}}$ be a quasiregular map. Suppose there are (disjoint) measurable sets $E_j \subset \mathbb{C}$, $j = 1, 2, \ldots$, such that:*
- (i) *for almost every $z \in \mathbb{C}$, the g-orbit of z meets E_j at most once for every j;*
- (ii) *g is K_j-quasiregular on E_j;*
- (iii) *$K_\infty := \prod_{j\geq 1} K_j < +\infty$; and*
- (iv) *g is holomorphic (Lebesgue) almost everywhere outside $\bigcup_{j\geq 1} E_j$.*

Then there exists a K_∞-quasiconformal map ϕ ('fixing infinity') such that $f = \phi \circ g \circ \phi^{-1}$ is a meromorphic function.

We have our patient and our tools; let us begin the surgery. We refer to Figure 1 for a sketch of some of the sets and functions here and their relations to each other. First, we take the points $z_n := 4n \in U_n$, $n \geq N$, which form an orbit under f; N here is large enough that the annuli A'_n described at the end of §4.1 exist for $n \geq N$. We take some $r > 0$ such that, for $n \geq N$, the circle $C_n = \{z : |z_n - z| = r\}$ (shown by the blue inner disc in Figure 1) is surrounded by A'_n. Inside the discs $\{z : |z_n - z| \leq r\}$, we will remove f and transplant appropriately translated versions of the rescaled Joukowski map

$$J_n(z) = \frac{\mu_n r^2}{\mu_n^2 r^2 - 1}\left(\mu_n z + \frac{1}{\mu_n z}\right),$$

where the $\mu_n > 1/r$ are parameters that will give us great control over the dilatation of the interpolating maps. The strange scaling constant here requires explaining; its role is to guarantee that $J_n(C_n)$ surrounds C_{n+1}, which will help us enforce condition (2). It can

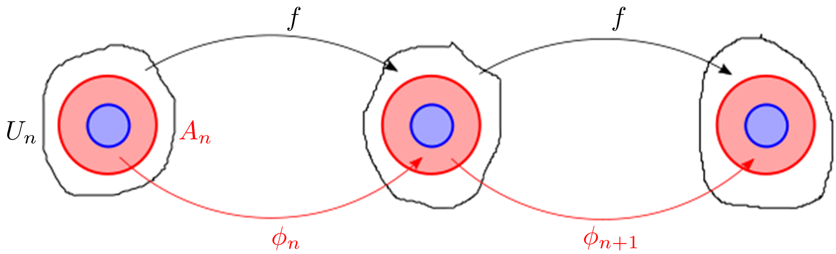

FIGURE 1. In the blue inner discs bounded by C_n, the original map f was replaced by appropriately translated versions of γ_n. To connect this with f, we interpolate on the red outer annuli A_n using the quasiconformal maps ϕ_n.

be calculated as follows: as explained in [1, pp. 94–95], the function $z \mapsto z + 1/z$ maps a circle of radius $\rho > 1$ injectively onto an ellipse with major semi-axis $\rho + 1/\rho$ and minor semi-axis $\rho - 1/\rho$. Denoting the (as yet unknown) scaling constant by λ, the condition '$J_n(C_n)$ surrounds C_{n+1}' then becomes

$$\lambda\left(\mu_n r - \frac{1}{\mu_n r}\right) = r,$$

and solving for λ yields precisely

$$\lambda = \frac{\mu_n r^2}{\mu_n^2 r^2 - 1}.$$

Next, since $f|_{U_n}$ converges locally uniformly to the translation $z \mapsto z + 4$ (in the sense outlined in property (D)), we can take some $r' > r$ with $C_n' := \{z : |z_n - z| = r'\}$ (red in Figure 1) also surrounded by A_n', and such that $|f(C_n') - z_{n+1}| > r + \delta$ for all sufficiently large n. Now, if n is (again) large enough, the only pre-image of z_{n+1} under f surrounded by C_n' is z_n itself, and hence $\text{ind}(f \circ C_n', z_{n+1}) = 1$ (the notation $\text{ind}(\gamma, \alpha)$ denotes the winding number of the curve γ around the point $\alpha \in \mathbb{C}$). We are ready to start interpolating the maps $T_{n+1} \circ J_n \circ T_n^{-1}$ on C_n and f on C_n'. For the sake of convenience, we pass to the unit disc, interpolating instead the functions J_n and $T_{n+1}^{-1} \circ f \circ T_n$ on $\{z \in \mathbb{C} : |z| = r\}$ and $\{z \in \mathbb{C} : |z| = r'\}$ (respectively).

Needless to say, we are applying Lemma 4.2 with $k = 1$. We would like first to draw attention to condition (11) in the case $j = 1$, that is, for the function J_n. The left-hand quantity takes the form

$$\left|\frac{2}{\mu_n^2 r^2 e^{2i\theta} + 1}\right|,$$

which has the upper bound (achieved for $\theta = \pi/2$)

$$\frac{2}{\mu_n^2 r^2 - 1}.$$

Clearly, taking $\mu_n \to +\infty$ gives us excellent control over how small this term is; it is through this 'trick' that we will control the dilatation introduced by the Joukowski map.

For the case $j = 2$, we have a function of the form $f(z) = b_{n+1}(z) + \epsilon_n(z)$, where $\epsilon_n \to 0$ uniformly as $n \to +\infty$ and b_n are the Blaschke products described in §4.1. The exact form of the left-hand side of condition (11) is

$$\left| z \frac{d}{dz} \left(\log \frac{b_{n+1}(z) + \epsilon_n(z)}{z} \right) \right|, \tag{14}$$

and by controlling how fast $a_n \nearrow 1$ and applying Cauchy's integral formula to ϵ_n, we can ensure that ϵ_n and its derivative tend to zero as fast as necessary to control the size of the constants in Lemma 4.2. Thus, up to a small error term tending to zero arbitrarily fast, (14) takes the form

$$\left| z \frac{b'_{n+1}(z)}{b_{n+1}(z)} - 1 \right| = \left| \frac{(1 + a_{n+1})z + a_{n+1}^2 z^2 + a_{n+1}}{(1 + a_{n+1}^2)z + a_{n+1} z^2 + a_{n+1}} - 1 \right|,$$

which again tends to zero arbitrarily fast by choosing an appropriate sequence $a_n \nearrow 1$. Finally, the left-hand side of condition (10) takes the form

$$\left| \log \left(\frac{b_{n+1}(r' e^{i\theta}) + \epsilon_n(r' e^{i\theta})}{r' e^{i\theta}} \frac{r e^{i\theta}}{J_n(r e^{i\theta})} \right) \right|$$

for $\theta \in [0, 2\pi)$, and we can use the triangle inequality to bound it by

$$\left| \log \frac{b_{n+1}(r' e^{i\theta}) + \epsilon_n(r' e^{i\theta})}{r' e^{i\theta}} \right| + \left| \log \frac{r e^{i\theta}}{J_n(r e^{i\theta})} \right|.$$

Ignoring the error term again, this becomes

$$\left| \log \frac{r' e^{i\theta} + a_{n+1}}{1 + a_{n+1} r' e^{i\theta}} \right| + \left| \log \frac{r e^{i\theta}}{(\mu_n^2 r^3 e^{i\theta} / (\mu_n^2 r^2 - 1)) - (r / (e^{i\theta} (\mu_n^2 r^2 - 1)))} \right|. \tag{15}$$

Thus, we see that, by making appropriate, independent choices of $a_n \nearrow 1$ and $\mu_n \to +\infty$, we can make (15) go to zero arbitrarily fast as $n \to +\infty$.

It follows that we can arrange conditions (10) and (11) to hold in each annulus $A_n := \{z \in \mathbb{C} : r < |z - z_n| < r'\}$ with constants $\delta_{n,0}$ and $\delta_{n,1}$ that are as small as we need them to be. Thus, by invoking Lemma 4.2, we obtain a sequence of quasiconformal maps ϕ_n that interpolate between J_n and f on the annuli A_n with dilatation as close to 1 as we please, (say) $K_n < 1 + 1/n^2$, by (12) and (13). It follows that the K_n can be made to satisfy

$$K_\infty := \prod_{n=1}^{+\infty} K_n < +\infty.$$

We now define the map

$$g_0(z) := \begin{cases} J_n(z), & z \in \operatorname{int}(C_n), n \geq N, \\ \phi_n(z), & z \in \overline{A_n}, n \geq N, \\ f(z) & \text{elsewhere}, \end{cases}$$

and claim that it satisfies the hypotheses of Lemma 4.3. Indeed, for any $z \in A_n$, the fact that $J_n(C_n)$ surrounds C_{n+1} while C'_{n+1} surrounds $f(C'_n)$ implies that $g_0(z) \in A_{n+1}$, and so the g_0-orbit of every $z \in \mathbb{C}$ meets each A_n at most once, guaranteeing hypothesis (i).

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Hypotheses (ii)–(iv) are also satisfied by the construction of the J_n and our choices of a_n and μ_n, and so we can apply Lemma 4.3 and obtain a K_∞-quasiconformal map ψ such that

$$g(z) := \psi \circ g_0 \circ \psi^{-1}(z)$$

is a transcendental meromorphic function.

We claim that:

(i) the new map g has a wandering domain V;

(ii) V is semi-contracting; and

(iii) V is infinitely connected.

The first claim will follow from the fact that ψ conjugates g_0 to g, and that we left 'enough' of f intact in each U_n. More specifically, recall the round annulus A from §4.1; it satisfies $|B(z)| > c$ for $z \in A$. By Rouche's theorem, we find a topological annulus $A' \subset U_N$ (where, again, N is large enough that the annuli A'_n exist for $n \geq N$) such that, for $z \in A'$, $f^n(z) \in U_{n+N}$ does not intersect the discs $\{z \in \mathbb{C} : |z - 4(n+N)| < c'\} \subset U_{n+N}$, where c' was also defined in §4.1. In particular, for $z \in A'$, the f-orbit of z is not affected by the surgery, and therefore is conjugated by ψ to its g-orbit; it follows that $\psi(A')$ is contained in a Fatou component V of g. Furthermore, since g has a pole at $\psi(4n)$ for $n \geq N$, V is at least doubly connected.

We now show that V is semi-contracting. We begin by showing that V (and each V_n, the Fatou component of g containing $g^n(V)$) is contained in $\psi(U_N)$ (respectively, $\psi(U_{n+N})$). This, in turn, will follow from the fact that we did not modify f outside of a small disc properly contained within each U_n. Indeed, if w is a point on ∂U_N, it belongs to the Julia set of f, and is therefore approached by a sequence $(w_n)_{n\in\mathbb{N}}$ of repelling periodic points of f (see, for instance, [8, Theorem 4]). Since the w_n are periodic points in the Julia set, their orbits do not intersect the discs $\{z \in \mathbb{C} : |z - 4(n+N)| < c'\} \subset U_{n+N}$ for any $n \in \mathbb{N}$, and so g_0 agrees with f on their f-orbits. It follows that the conjugacy ψ takes these f-orbits to corresponding g-periodic orbits $\psi(w_n)$, which we can show to be repelling by the local topological dynamics as follows.

Take one of the w_n, of minimal period $k_n \geq 1$ (say), and apply Koenig's linearization theorem [26, Theorem 8.2] to find a neighbourhood W of w_n and a biholomorphic map $\phi : W \cup f^{k_n}(W) \to \phi(W \cup f^{k_n}(W)) \subset \mathbb{C}$ such that ϕ conjugates $f^{k_n}|_W$ to multiplication by $\alpha_n := (f^{k_n})'(w_n)$, which satisfies $|\alpha_n| > 1$ since w_n is repelling. This allows us to find (if necessary) a smaller neighbourhood $W' \subset W$ such that $f^m(W') \cap \{z \in \mathbb{C} : |z - 4(n+N)| < c'\} = \emptyset$ for all $n \in \mathbb{N}$ and $1 \leq m \leq k_n$, so that $\psi(f^m(z)) = g^m(\psi(z))$ for $z \in W'$ and $1 \leq m \leq k_n$. From this, we may conclude that $\psi(w_n)$ is repelling for g: for all $z \in \psi(W') \setminus \{w_n\}$, there must exist $M \in \mathbb{N}$ such that $g^{Mk_n}(z)$ is not in $\psi(W')$ (see, for instance, [26, p. 84]). It follows that $\psi(w)$, being accumulated by repelling periodic points, belongs to $J(g)$; a similar argument applies to each V_n, $n \in \mathbb{N}$.

Thus, for any points $z, w \in V$, we have by the Schwarz–Pick lemma and the fact that $V_n \subset \psi(U_{n+N})$ that

$$d_{V_n}(g^n(z), g^n(w)) \geq d_{\psi(U_{n+N})}(g^n(z), g^n(w));$$

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if ψ were a conformal map, our work would be done—but it is not, and it does not preserve the hyperbolic metric. Instead, for a domain $D \subset \mathbb{C}$ and points $z, w \in D$, let us define

$$k_D(z, w) := \inf_{\gamma} \int_{\gamma} \frac{1}{d(s, \partial D)} |ds|,$$

where the infimum runs over every rectifiable arc $\gamma \subset D$ joining z and w. This is the *quasihyperbolic metric*; if D is simply connected, standard estimates for the hyperbolic metric [13, p. 13] show that

$$\frac{k_D(z, w)}{2} \leq d_D(z, w) \leq 2k_D(z, w).$$

Additionally, since ψ is $(K_\infty)^{-1}$-Hölder continuous [11, p. 31], we know (see [16, Theorem 3]) that there exists a constant $C > 0$ depending only on K_∞ such that

$$k_{U_{n+N}}(z', w') \leq C \cdot \max\{k_{\psi(U_{n+N})}(z, w), k_{\psi(U_{n+N})}(z, w)^{1/K_\infty}\},$$

where $z' = \psi^{-1}(z)$ and $w' = \psi^{-1}(w)$ are points in U_{n+N}. For points $z' \in A'$, the construction of g implies that

$$\psi(f^n(z')) = g^n(\psi(z)) \quad \text{for all } n,$$

and so taking z' and w' in A' yields

$$\begin{aligned} &k_{U_{n+N}}(f^n(z'), f^n(w')) \\ &\quad \leq C \cdot \max\{k_{\psi(U_{n+N})}(g^n(z), g^n(w)), k_{\psi(U_{n+N})}(g^n(z), g^n(w))^{1/K_\infty}\}. \end{aligned}$$

The left-hand side of this expression is bounded below by $d_{U_{n+N}}(f^n(z'), f^n(w'))/2$, which, since U is semi-contracting, is in turn bounded below by $c(z', w')/2 > 0$. Also, since the exponent on the right-hand side is independent of n, we can get a uniform, positive lower bound $c'(z, w)$ on $k_{\psi(U_{n+N})}(g^n(z), g^n(w))$ regardless of which term is the maximum. Combining everything, we see that

$$\frac{c'(z, w)}{2} \leq d_{\psi(U_{n+N})}(g^n(z), g^n(w)) \leq d_{V_n}(g^n(z), g^n(w)) \quad \text{for } z \text{ and } w \text{ in } \psi(A'),$$

and thus V contains a non-empty open subset that is semi-contracting relative to any z_0 in this subset—we claim that $(d_{V_n}(g^n(z), g^n(w)))_{n\in\mathbb{N}}$ is not a constant sequence, since each $g|_{V_n}$ 'inherits' a critical point from f. Indeed, $f|_{U_n}$ has a single critical point z_n^* for every n, and since z_n^* approaches ∂U_n as $n \to +\infty$ (in the sense outlined in property (D)), we have $z_n^* \in U_n \setminus \{z \in \mathbb{C} : |z - z_n| \leq r'\}$ for all sufficiently large n. Hence, the critical point is not affected by the surgery for sufficiently large n, and $\psi(z_n^*) \in V_n$ is a critical point of g for all large n.

We have only shown that V is semi-contracting on the topological annulus $\psi(A')$. For the remainder of V, we choose some z_0 in $\psi(A')$ and apply Theorem 1.1.

To prove that V is infinitely connected, notice that (since $V_n \subset \psi(U_{n+N})$) all V_n are bounded, and so $\deg g|_{V_n}$ is always finite. By applying [15, Theorem 1.1], we deduce that V must be infinitely connected. This concludes the proof of Theorem 1.2.

https://doi.org/10.1017/etds.2023.27 Published online by Cambridge University Press

Acknowledgements. I would like to thank my supervisors, Phil Rippon and Gwyneth Stallard, for their comments and support. I am also deeply grateful to the referee for their detailed corrections and suggestions, and to Lasse Rempe for pointing out a mistake in the statement of Theorem 1.1.

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