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ON THE DIOPHANTINE EQUATION $x(x+d)(x+2d)+y(y+d)(y+2d) = z(z+d)(z+2d)$

BY

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ABSTRACT—A previous result of the author concerning the parametric representation of infinitely many solutions of the title equation is strongly improved. New classes each containing infinitely many solutions of the equation for specified values of d are stated explicitly. The method of solution hinges heavily on solving the generalized Pell's equation $x^2-Dy^2=c$.

1. **Introduction.** The title equation has been studied only recently for the case $d=1$, being presented in the more simple form (writing $x-1$ for x, $y-1$ for y and $-z-1$ for z)

$$x + y^3+z^3 = x+y+z \tag{1.2}$$

As such it has been investigated by A. Oppenheim [4], S. L. Segal [5], W. Sierpinski [6], and M. Wunderlich [7]. These authors succeeded to show that (1.2) has infinitely many solutions, but could not state an explicit parametric representation of these. This was achieved by the author in a previous paper [1] for the general case (1.1). Since reference will be made to that paper, its main results are stated here in the following two theorems:

THEOREM I. *Infinitely many solutions of the Diophantine equation*

$$x(x+d)+y(y+d) = z(z+d), \qquad d \in Z, \tag{1.3}$$

are given by the parametric presentation

$$\begin{aligned} x &= \tfrac{1}{2}(t(a_{2k+1}-1)+d(b_{2k+1}-1)),\\ y &= \tfrac{1}{2}(t(a_{2k+1}+1)+d(b_{2k+1}-1)),\\ z &= \tfrac{1}{2}(d(a_{2k+1}-1)+2tb_{2k+1}), \end{aligned} \tag{1.4}$$

where (a_{2k+1}, b_{2k+1}) $(k=0, 1, \ldots)$ *are the solution vectors of Pell's equation*

$$a^2-2b^2 = -1. \tag{1.5}$$

THEOREM II. *Let* (s_k, t_k) $(k=1, 2, \ldots)$ *denote infinitely many solution vectors of Pell's equation*

$$s^2-(12a^3-3)t^2 = 1 \quad (a \text{ a natural number} > 1). \tag{1.6}$$

If

$$(a-1)\,|\,(3(2a-1)t_k+s_k, (2a^3+3a-2)t_k+as_k) \tag{1.7}$$

then (x_k, y_k, z_k) *with*

$$\begin{aligned} x_k &= (2a^2+2a-1)t_k+s_k, \\ y_k &= (a-1)^{-1}(3a(2a-1)t_k+as_k), \\ z_k &= (a-1)^{-1}((2a^3+3a-2)t_k+as_k), \end{aligned} \tag{1.8}$$

are infinitely many solutions of the equation

$$(x-1)x(x+1)+(y-1)y(y+1) = (z-1)z(z+1).$$

The reader should note that if the entities $a-1$, $3a(2a-1)t_k+as_k$, $(2a^3+3a-2)t_k+as_k$ have a (greatest) common divisor, then this ought to be cancelled in the fractions of y_k and z_k of (1.8), and the values of x_k, y_k, z_k must be multipled by the corresponding factors, so that these values become integral.

COROLLARY. *Infinitely many solution vectors* (x_k, y_k, z_k) *of the Diophantine equation* $(x-1)x(x+1)+(y-1)y(y+1)=(z-1)z(z+1)$ *are given by the formulas*

$$x_k = 11t_k+s_k; \qquad y_k = 18t_k+2s_k; \qquad z_k = 2_\omega t_k+2s_k, \tag{1.9}$$

where (s_k, t_k) *are solution vectors of Pell's equation* $s_2^k-93t^k=1$.

2. **A much stronger theorem.** Theorem II reveals nothing about the nature of g; essentially, here g depends on a, and it is not known, therefore, whether equation (1.1) has (infinitely many) solutions for any d, or only for a special set of values of d which may even be finite. The following theorem is a much stronger result on the solubility of the title equation; it shows that the latter has infinitely many solutions for every d.

THEOREM 1. *The Diophantine equation*

$$(x-d)x(x+d)+(y-d)y(y+d) = (z-d)z(z+d)$$

has infinitely many solutions for every d given by the formulas

$$\begin{aligned} x_k &= d[(2d^2+6d+3)t_k+s_k], \\ y_k &= (d+1)]3(2d+1)t_k+s_k], \\ z_k &= (2d^3+6d^2+9d+3)t_k+(d+1)s_k \end{aligned} \tag{2.1}$$

where (s_k, t_k) *are solution vectors of Pell's equation*

$$s^2-(12a^3-3)t^2 = 1; \qquad a = d+1. \tag{2.2}$$

Proof. In [1] author has shown that making in

$$(x-d)x(x+d)+(y-d)y(y+d) = (z-d)z(z+d) \tag{2.3}$$

the substitution

$$y = a(z-x) \quad (a \text{ a natural number} > 1) \tag{2.4}$$

we obtain for (2.3) the rational solutions

$$\begin{aligned} x_k &= d[(2a^2+2a-1)t_k+s_k], \\ y_k &= da(a-1)^{-1}[3(2a-1)t_k+s_k], \\ z_k &= d(a-1)^{-1}[(2a^3+3a-2)t_k+as_k], \end{aligned} \tag{2.5}$$

where the (s_k, t_k) are the solution vectors of Pell's equation (1.6). From (2.5) it is obvious that x_k is integral; so it suffices to establish conditions for the integrality of z_k, since then y_k is integral by (2.4). Since (i) $2a^3+3a-2\equiv 3 (\text{mod } a-1)$, (ii) $a\equiv 1 (\text{mod } a-1)$, we obtain

$$(2a^3+3a-2)t_k+as_k \equiv 3t_k+s_k \ (\text{mod } a-1). \tag{2.6}$$

Since $12a^3-3\equiv 9(\text{mod } a-1)$, we obtain from $s_k^2-(12a^3-3)t_k^2=1$,

$$s_k^2-9t_k^2 \equiv 1 \ (\text{mod } a-1). \tag{2.7}$$

Now let

$$(3t_k+s_k, a-1) = g, \tag{2.8}$$

$$3t_k+s_k \equiv 0 \ (\text{mod } g). \tag{2.9}$$

From (2.7), (2.8) we obtain $s_k-9t_k^2\equiv 1(\text{mod } g)$, so that

$$(s_k-3t_k)(s_k+3t_k) \equiv 1 \ (\text{mod } g). \tag{2.10}$$

From (2.9), (2.10) we obtain $0\equiv 1(\text{mod } g)$, hence

$$g = 1. \tag{2.11}$$

Thus

$$(a-1, (2a^3+3a-2)t_k+as_k) = 1, \tag{2.12}$$

and from (2.5) we obtain that z_k is integral if

$$d = a-1, \qquad a = d+1. \tag{2.13}$$

Substituting $a=d+1$ in (2.5), Theorem 1 is easily verified. For large d the solutions of equation (2.3) grow rapidly. We illustrate that by an example. Let be

$$\begin{aligned} d = 47; \qquad a = 48; \qquad 12a^3-3 = 1327101; \\ (1327101)^{1/2} = (1151, [1,766,1,2302]). \end{aligned} \tag{2.14}$$

The smallest solution of $s^2-1327101t^2=1$ is given by

$$s_1 = 884735; \qquad t_1 = 768. \tag{2.15}$$

For easy calculation of these values the reader should compare the papers of the author [2] or [3]. Substituting the values of (2.15) into (2.1) and taking into account (2.14) we obtain that the smallest solution (of this type) of the Diophantine equation $(x-47)\times(x+47)+(y-47)y(y+47)=(z-47)z(z+47)$ is given by: $x=211342033$; $y=52973520$; $z=212445648$.

3. **New classes of solutions of the title equation.** We shall again write the title equation in the form (2.3) and make the substitution

$$y = a(z-x)-d. \qquad (a \text{ a natural number}). \tag{3.1}$$

This substitution is essentially different from the one made by the author in [1] where (2.4) was used. Substituting the value of y from (3.1) into (2.3) we obtain, after easy rearrangements,

$$(a^3-1)z^2-((2a^3+1)x+3\,da^2)z+(a^3-1)x^2 = -d^2(2a+1)-3\,da^2x. \tag{3.2}$$

Multiplying both sides of (3.2) by $4(a^3-1)$, we obtain

$$\begin{aligned}(2(a^3-1)z-(2a^3+1)x-3\,da^2)^2-((2a^3+1)x+3\,da^2)^2 \\ +4(a^3-1)^2x^2+12\,da^2(a^3-1)x = -4d^2(2a+1)(a^3-1).\end{aligned} \tag{3.3}$$

Denoting

$$2(a^3-1)z-(2a^3+1)x-3\,da^2 = v, \tag{3.4}$$

(3.3) takes the form

$$v^2-(12a^3-3)x^2-18\,da^2x = d^2(a^4-4a^3+8a+4). \tag{3.5}$$

Multiplying both sides of (3.5) by $-(12a^3-3)$, we obtain

$$((12a^3-3)x+9\,da^2)^2-(12a^3-3)v^2 = d^2(81a-(a^4-4a^3+8a+4)(12a^3-3)). \tag{3.6}$$

Denoting

$$(12a^3-3)x+9\,da^2 = Ud; \qquad v = Vd, \tag{3.7}$$

(3.6) takes the form

$$U^2-(12a^3-3)V^2 = 81a^4-(12a^3-3)(a^4-4a^3+8a+4). \tag{3.8}$$

To solve Pell's equation (3.8), one needs first a singular solution of it. Since a is a free parameter, this constitutes the whole difficulty of solving (3.8). Surprisingly, a singular solution of (3.8) can be found. We observe first that

$$a^4-4a^3+8a+4 = (a^2-2a-2)^2. \tag{3.9}$$

Thus (3.8) takes the form

$$U^2-(12a^3-3)V^2 = (9a^2)^2-(12a^3-3)(a^2-2a-2)^2, \tag{3.10}$$

and (3.10) has the singular solution

$$U_0 = 9a^2; \qquad V_0 = a^2-2a-2. \tag{3.11}$$

Let (s_k, t_k) be the solution vectors of $s^2-(12a^3-3)t^2=1$. Then infinitely many solution vectors of (3.8) are given by

$$U_k+(12a^3-3)^{1/2}V_k = [9a^2+(12a^3-3)^{1/2}(a^2-2a-2)][s_k+(12a^3-3)^{1/2}t_k],$$

so that

$$\begin{aligned} U_k &= 9a^2s_k+(a^2-2a-2)(12a^3-3)t_k, \\ V_k &= 9a^2t_k+(a^2-2a-2)s_k\cdot(s_k^2-(12a^3-3)t_k^2=1), \end{aligned} \tag{3.12}$$

gives a system of infinitely many solution vectors for (3.8) for $a>1$. The case $a=1$ will be investigated later. Substituting the values of U, V from (3.12) into (3.4)(3.7) we obtain, after easy calculations

$$\begin{aligned} (4a^3-1)x_k &= d(3a^2(s_k-1)+(a^2-2a-2)(4a^3-1)t_k) \\ 2(a^3-1)z_k &= (2a^3+1)x_k+d(3a^2(3t_k+1)+(a^2-2a-2)s_k). \end{aligned} \tag{3.13}$$

Adding the two equations of (3.13) we obtain:

$$2(a^3-1)z_k+2(a\ -1)x_k = 4d(a^2-2a-2)(a^3-1)t_k+d(4a^2-2a-2)(s_k+3t_k).$$

Dividing both sides of this equation by $2(a^3-1)$, we obtain

$$z_k+x_k = 2d(a^2-2a-2)t_k+(a^3-1)^{-1}\, d(2a^2-a-1)(s_k+3t_k). \tag{3.14}$$

We presume again $a>1$. Since $2a^2-a-1=(a-1)(2a+1)$, we obtain from (3.14)

$$z_k+x_k = 2d(a^2-2a-2)t_k+(a^2+a+1)^{-1}(2a+1)(s_k+3t_k)d. \tag{3.15}$$

From (3.13) we obtain $x_k=d(a^2-2a-2)t_k+(4a^3-1)^{-1}3a^2d(s_k-1)$, which finally yields the following expressions for x_k, y_k, z_k:

$$\begin{aligned} z_k &= d(a^2-2a-2)t_k+(a\ +a+1)^{-1}(2a+1)(s_k+3t_k)d \\ &\qquad -3a^2d(4a^3-1)^{-1}(s_k-1), \\ x_k &= d(a^2-2a-2)t_k+\ a^2d(4a^3-1)^{-1}(s_k-1); \\ y_k &= a(z_k-x_k)-d. \end{aligned} \tag{3.16}$$

(3.16) supplies the rational solutions of (2.3). The question of their integrality will be investigated in the next chapter. We now return to the case $a=1$; (4.1) becomes $y=z-x-d$, and substituting this value of y in (2.3), we obtain

$$(z-x-2d)(z-x-d)(z-x) \qquad (z-x)(z^2+xz+x^2-d^2).$$

One solution of this equation is given by $x=z$, $y=0$. Excluding this solution, we can cancel the previous equation by $z-x$, and obtain $(x+d)z=d(x+d)$. The reader will have no difficulties to find the remaining solutions of this equation.

4. **Integrality of solutions.** We shall now investigate the integrality of the vector solutions (x_k, y_k, z_k) given by (3.16). It is easily verified that x_k, y_k, z_k are integral

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if and only if the following two conditions both hold

$$(4.1) \qquad (4a^3-1) \mid 3a^2d(s_k-1),$$

$$(4.2) \qquad (a^2+a+1) \mid (2a+1)(s_k+3t_k)d.$$

We first note that

$$(4.3) \qquad \begin{aligned} &\text{(i)} \quad (4a^3-1, a^2+a+1) = 1, \qquad \text{if } a \not\equiv 1 \pmod 3, \\ &\text{(ii)} \quad (4a^3-1, a^2+a+1) = 3, \qquad \text{if } a \equiv 1 \pmod 3. \end{aligned}$$

We further note that

$$(4.4) \qquad \begin{aligned} &\text{(i)} \quad (4a^3-1, a^2) = 1 \\ &\text{(ii)} \quad (a^2+a+1, 2a+1) = 3, \qquad \text{if } a \equiv 1(3), \qquad \text{and } = 1 \text{ otherwise.} \end{aligned}$$

If we disregard the structure of s_k-1 and s_k+3t_k as to their eventual factorization, we obtain

THEOREM 2. *A sufficient condition that x_k, y_k, z_k from* (3.16) *be rational integers is that the conditions hold*

$$(4.5) \qquad \begin{aligned} d &= (4a^3-1)(a^2+a+1), \qquad \text{if } a \not\equiv 1 \pmod 3, \\ d &= 3^{-2}(4a^3-1)(a^2+a+1), \qquad \text{if } a \equiv 1 \pmod 3. \end{aligned}$$

Of course, in this case nothing can be said as to whether or not (x_k, y_k, z_k) is a primitive solution of (2.3). We proceed to sharpen the conditions for d of Theorem 2. We first investigate the integrality of x_k and ask the question: what are the conditions for

$$(4.7) \qquad (4a^3-1) \mid 3a^2(s_k-1).$$

This would amount to

$$(4.8) \quad 3^{-1}(4a^3-1) \mid (s_k-1) \quad \text{for } a \equiv 1(3); \qquad (4a^3-1) \mid (s_k-1) \text{ otherwise.}$$

From $s_k^2-3(4a^3-1)t_k^2=1$, we obtain

$$(4.9) \qquad (s_k-1)(s_k+1) = 3(4a^3-1)t_k^2.$$

Let be

$$(4.10) \quad 4a^3-1 = p, \quad a \not\equiv 1(3); \quad 4a^3-1 = 3p, \quad a \equiv 1(3), \quad p \text{ a prime.}$$

In both cases we obtain from (4.9)

$$(4.11) \qquad p \mid (s_k-1) \quad \text{or} \quad p \mid (s_k+1).$$

The reader will note that not both of the relations (4.11) can hold, since in view of (4.10), $p>2$. Now if $p \mid (s_k+1)$, then, taking for s_k the solution $-s_k$, we obtain $p \mid -s_k+1$, so that in any case we can presume

$$(4.12) \qquad p \mid (s_k-1). \qquad (s_k < 0 \text{ or } s_k > 0).$$

We now prove

THEOREM 3. *Let* (s_k, t_k) *be a solution vector of Pell's equation* $s^2-(12a^3-3)t^2=1$; *then*

$$((a^2+a+1), s_k+3t_k) = 1. \tag{4.13}$$

Proof. We have $1=s_k^2-3(4a^3-1)t_k^2=s_k^2-3(4(a^3-1)+3)t_k^2\equiv s_k^2-9t_k^2 (\text{mod } a^3-1)$, so that

$$s_k^2-9t_k^2 \equiv 1 \ (\text{mod } a^2+a+1). \tag{4.14}$$

From (4.14) we obtain $(s_k-3t_k)(s_k+3t_k)\equiv 1 (\text{mod } a^2+a+1)$ which proves Theorem 3. This and the results of (4.10) yields

THEOREM 4. *Let* $4a^2-1=p$ *for* $a\not\equiv 1$ *(mod 3), or* $4a^3-1=3p$ *for* $a\equiv 1$ *(mod 3), p an odd prime in both cases. Then a sufficient condition that* x_k, y_k, z_k *from* (3.16) *be integers is that*

$$d = a^2+a+1, \quad \text{for } a \equiv 1 \ (\text{mod } 3); \qquad d = 3^{-1}(a^2+a+1) \text{ otherwise.} \tag{4.15}$$

The question when $4a^3-1$ or $3^{-1}(4a^3-1)$ are primes, remains, of course, open, and will probably continue to do so for a long time. We shall now construct integers a such that $(4a^3-1) \mid (s_k-1)$, where again (s_k, t_k) are solution vectors of

$$s^2-3(4a^3-1)t^2 = 1.$$

Thus we shall be able to avoid the question whether or not $4a^3-1$ or $3^{-1}(4a^3-1)$ are primes. Let be

$$a = 3k^2. \quad (k \text{ a natural number}) \tag{4.16}$$

Then $12a^3-3=(18k^3)^2-3$, so that

$$12a^3-3 = D^2-d; \qquad D = 18k^3, \qquad d = 3, \qquad d \mid D. \tag{4.17}$$

In [2] the author has proved the formula

$$D^2-d = (D-1, [1, 2(D-d)/d, 1, 2(D-1)] \quad \text{for} \quad d \mid D. \tag{4.18}$$

In our case we obtain

$$12a^3-3 = (18k^3-1, [1, 2(6k^3-1), 1, 2(18k^3-1)]). \tag{4.19}$$

Using formulas (2.2) from [1] we obtain, since in our case, by (4.18) the length of the period is $n=4$,

$$s_1 = A^{(4)}+(18k^3-1)A^{(5)}, \qquad t_1 = A^{(5)}. \tag{4.20}$$

By means of formula $(2.2)^{(1)}$ we now calculate

$$A^{(2)} = 1; \qquad A^{(3)} = 1; \qquad A^{(4)} = 1+2(6k^3-1) = 12k^3-1;$$
$$A^{(5)} = 1+(12k^3-1) = 12k^3,$$

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and obtain from (4.20)

$$s_1 = 216k^6 - 1; \qquad t_1 = 12k^3. \tag{4.21}$$

From (4.21) we obtain $s_1 - 1 = 216k^6 - 2$, so that

$$s_1 - 1 = 2(108k^6 - 1); \qquad s_1 + 1 = 216k^6. \tag{4.22}$$

But $4a^3 - 1 = 108k^6 - 1$, so that indeed

$$(4a^3 - 1) \mid (s_1 - 1). \tag{4.23}$$

We further obtain, from the known formula $s_k + (12a^3-3)^{1/2}t_k = (s_1 + (12a^3-3)^{1/2}t_1)^k$

$$s_k = s_1^k + \binom{k}{2} s_1^{k-2}(12a^3-3) + \binom{k}{4} s_1^{k-4}(12a^3-3)^2 + \dots \tag{4.24}$$

From (4.24) we obtain $s_k - 1 = s_1^k - 1 + m(12a^3-3)$ where m is some integer. Since $(s_1-1) \mid (s_1^k - 1)$ and $(4a^3-1) \mid (s_1-1)$,

$$(4a^3 - 1) \mid (s_k - 1). \tag{4.25}$$

On basis of (4.24) we can now state

THEOREM 5. *Let $a = 3k^2$ be a natural number; then the solution vectors of (s_k, t_k) of $s^2 - (12a^3-3)t^2$ are all such that $(4a^3-1) \mid (s_k - 1)$; a sufficient condition that the solution vectors (x_k, y_k, z_k) from* (3.16) *of $x(x^2-d^2) + y(y^2-d^2) = z(z^2-d^2)$ have rational integral components is given by*

$$d = 9k^4 + 3k^2 + 1. \tag{4.27}$$

We shall illustrate theorem 5, by a numerical example. Let $k=1$; $a=3$; $12a^3-1 = 321 = 3 \cdot 107$; $s_1 = 216k^6 - 1 = 215$; $t_1 = 12k^3 = 12$. From (3.16) we now calculate, since $d = 13$ from (4.27),

$$z_1 = 1211; \qquad x_1 = 858; \qquad y_1 = 1046.$$

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