Fourth Meeting, 10*th February* 1899.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

On the Eight Queens Problem.

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This is the problem discussed in my paper bearing the not very happy title "On the different non-linear arrangements of eight men on a chess-board", which was read to the Edinburgh Mathematical Society on 14th March 1890, and is printed in its *Transactions*, Vol. VIII, p. 30. At that time I was not aware that the problem had been discussed by any previous writer, and I treated it as an entirely new one. I have since learnt that a good deal has been written about it, and I propose on the present occasion to give briefly the history of the problem, and the results which have been arrived at; also to communicate some new results which I have obtained.

For those who are familiar with the game of chess, the best statement of the problem is as follows:—To find the different ways in which eight queens may be arranged on a chess-board, so that no two of them shall be in check of each other. For those who know nothing of that game, the problem may be stated thus:— To find the different ways in which eight pieces may be placed on a chess-board, so that no line joining two of them, shall be parallel either to a side or to a diagonal of the board.

My knowledge of the history of the problem is mostly derived from the following sources:—

1. Mr J. W. L. Glaisher's paper "On the Problem of the Eight Queens" in the *Philosophical Magazine* for December 1874. This is the work of an accomplisht mathematician, but the treatment of the problem is not entirely satisfactory, the explanation of the processes being often too brief: at all events, I found that much careful study was necessary before I could entirely understand it.

2. "Mathematical Recreations and Problems", by W. W. Rouse Ball (Macmillan & Co., 2nd Edition, 1892). This contains, on pages 85–88, a brief, and not entirely accurate, summary of the history of the problem, and of the results obtained.

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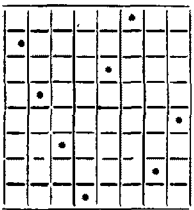
3. Dr Pein's work entitled "Aufstellung von n Königinnen auf einem Schachbrett von n^2 Feldern, derart dass keine von einer andern geschlagen werden kann. (Von $n=4$ bis $n=10$). Von Dr August Pein, Oberlehrer." This bears the name of the printer, B. G. Teubner, Leipzig; but seems not to have been publisht in the usual way, as it is headed "Städtische Realschule zu Bochum. Beilage zu dem Jahresbericht über das Schuljahr, 1888-89." It is a very lengthy paper, is illustrated by numerous woodcuts, and deals with the problem in a very complete manner; but it seems to me to contain a good deal of unnecessary detail.

The problem seems to have been first proposed by Dr Nauck in the Leipzig *Illustrated Times* of 1st June 1850, and the complete solution was given by him in the number for the following 22nd of Sept. Meantime, the question had attracted the attention of Gauss, and it is discussed in his publisht correspondence with Schumacher.

Mr Glaisher states that the problem was proposed by Nauck to Gauss, and that the latter, after finding the number of solutions to be 76 and then 72, ultimately arrived at the correct number, 92; and Mr Ball follows him in saying that the earliest solution was given by Gauss; but these statements appear to be erroneous. Dr Pein has given in his paper several extracts, both from Gauss's correspondence with Schumacher, and from Nauck's original papers; and it appears from these that Nauck first stated (1st June 1850) that there were 60 solutions, but on 22nd September he gave the correct number, 92; also that Gauss said, in his letter of 1st September, that the number was not 60 but 76; and on 12th September that this must be reduced to 72. In his letter of 27th September, he mentions that Nauck had stated that the correct number was 92, but expresses a doubt whether that might not, like the 60 previously given, have to be corrected. It is clear, therefore, that Gauss had not himself at that time verified this number. It is true that he gives the twelve fundamental solutions, from which the whole 92 can be deduced; but Nauck had previously explained fully how from any one solution that is not symmetrical, seven others can be deduced. Glaisher's authority for his statements appears to be a paper by Dr Siegmund Günther, entitled "Zur Mathematischen Theorie des Schachbretts", which was contained in Grunert's "Archiv der Mathematik und Physik", and which, Glaisher says, gives an interesting account of the history of the problem. I have not seen this paper, and therefore cannot say whether Günther was responsible for the mistake.

As already mentioned, there are 92 ways in which the eight queens may be arranged so as to satisfy the required conditions; or 92 solutions of the problem. But these are not all independent of each other; for, if we have got any one solution, we can generally get three others by turning the chess-board through 1, 2, and 3 right angles. Four other solutions can then be got, by taking the reflection of the first solution in a mirror, and turning this through 1, 2, and 3 right angles. In this way eight solutions are got, which are so connected, that all eight can be obtained from any one of them by rotation and reflection.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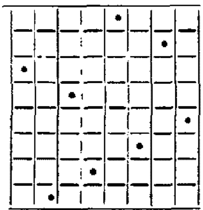
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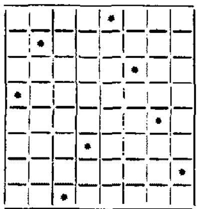
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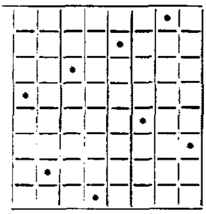
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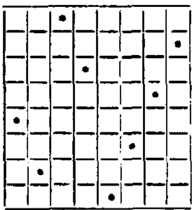
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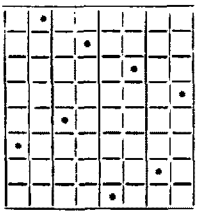
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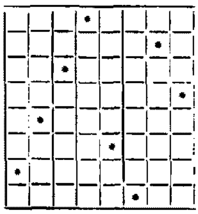
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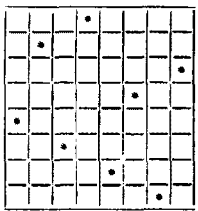
H

The connection between the eight solutions, which may be called the different aspects of a single solution, will be best understood by considering the diagrams here given. Starting with the aspect A, we get B, C, D, by turning the board round in the opposite direction to the hands of a clock, through 1, 2, and 3 right angles respectively. Then E is got from A by reflection in a mirror passing through a vertical side of the square; and F, G, H are got by turning E similarly through 1, 2, and 3 right angles. We see also that, by the same kind of reflection, H may be got from B; G from C; and and F from D; so that the eight aspects may be arranged in the following pairs:—

A, E; B, H; C, G; D, F.

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If the mirror passes through the top or bottom side of A, this gives us the aspect G; and by the same kind of reflection the aspects may be grouped in the following pairs:—

A, G; B, F; C, E; D, H.

If the mirror is parallel to the diagonal which runs from the left hand top corner (which we will call the first diagonal), reflection of A gives us F; and by similar reflections we can group the aspects in the following pairs:—

A, F; B, E; C, H; D, G,

Lastly, if the mirror is parallel to the other diagonal, reflection gives us the following pairs:—

A, H; B, G; C, F; D, E.

By reflecting the aspects alternately in two mirrors parallel respectively to a vertical side, and to the first diagonal, we get all the aspects from A, in the order

A, E, B, H, C, G, D, F.

By similar alternate reflections in mirrors parallel to a vertical side and the second diagonal, we get the eight aspects in the order

A, E, D, F, C, G, B, H;

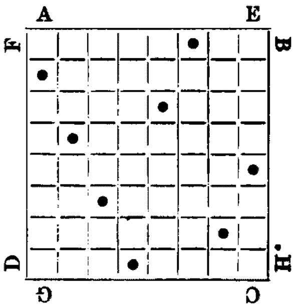
and by reflections in mirrors parallel to a horizontal side and to the first and second diagonals respectively, we get the eight aspects in the two orders

A, G, D, H, C, E, B, F,

and A, G, B, F, C, E, D, H.

The best way of representing any solution, is by writing down the figures that indicate the places of the queens in the various columns: thus, counting the squares from the top, solution A is represented by 24683175. The representations of the seven other aspects of this solution are most easily got by reading them off from the diagram.

A, 24683175	E, 57138642 = iA
B, 38471625 = rpA	F, 61528374 = pA
C, 42861357 = irA	G, 75316824 = rA
D, 47382516 = ipA	H, 52617483 = $irpA$



It is not difficult to obtain from any one aspect of the solution the seven others, without a diagram or a chessboard: but in order to do this, it is convenient to arrange the aspects in a different order. From A we get, by inverting the order of the figures, E, which therefore I denote by iA. Next, substituting for each figure in A its difference from 9, we get G. This process, which corresponds to reflection in a mirror parallel to the horizontal sides, I call reversion; and I denote G by rA. Inversion of the figures gives us C, which I accordingly denote by irA. We have thus got the four aspects which are obtained by reflection in mirrors parallel to the sides of the board.

In order to find the other aspects, we must transform A in a more fundamental manner: we must, in fact, interchange the rows and columns. Thus in A

1	stands	in	the	6th	place;	therefore	put	6	in	the	1st	place
2	,,	,,	,,	1st	,, ;	,,	,,	1	,,	,,	2nd	,,
3	,,	,,	,,	5th	,, ;	,,	,,	5	,,	,,	3rd	,,

and so on.

This process, which is the same thing as reflection in a mirror parallel to the first diagonal, I call perversion. When applied to A, it gives F, which I denote by pA. Inversion then gives us D, which is consequently ipA. Again, reversion of F gives B, which is therefore rpA; and lastly, inversion of B gives H, which is therefore irpA. The most suitable arrangement now is

A,	24683175 = A	iA,	57138624 = E
rA,	75316824 = G	irA,	42861357 = C
pA,	61528374 = F	ipA,	47382516 = D
rpA,	38471625 = B	irpA,	52617483 = H

(It may be useful to give here a brief statement of the results we get by combining any two of these symbols of operation, i, r, and p. We have $i^2=1$, $r^2=1$, $p^2=1$; $ir=ri$; $ip=pr$; $rp=pi$.)

If it should happen that rotation through two right angles reproduces the original arrangement, then the solution will be symmetrical, and have only four aspects.

The 92 arrangements which satisfy the conditions of the problem, may thus be grouped under 12 distinct solutions, 11 of which have eight aspects each, while the twelfth is symmetrical and has only four. The 12 solutions (which were given by Gauss, but apparently not by Nauck) are represented by

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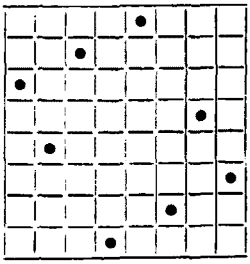
15863724	25713864	26831475	*35281746
16837425	25741863	27368514	35841726
24683175	26178435	27581463	36258174



The symmetrical solution is 35281746, and is represented in the annext figure. The sum of each pair of figures equidistant from the beginning and end, 3, 6; 5, 4; 2, 7; 8, 1; is 9; and it is easily seen that this is the condition that an arrangement of eight men, representing a permutation of the numbers 1, 2,...8, may be symmetrical, or remain unaltered when turned round through two right angles. Such a solution I call "centric".

Gauss has given the following rule for testing whether the arrangement corresponding to any permutation, is a solution of the problem:—

To the successive figures in the permutation add the numbers 1, 2, 3,...8, respectively; then all the totals must be different. Again, add to the same figures the numbers 8,...3, 2, 1, respectively; then all these totals must also be different. Taking, for instance, the first solution, we find

1	5	8	6	3	7	2	4
1	2	3	4	5	6	7	8
2,	7,	11,	10,	8,	13,	9,	12

1	5	8	6	3	7	2	4
8	7	6	5	4	3	2	1
9,	12,	14,	11,	7,	10,	4,	5

He remarks that the problem may be stated without any reference to a chessboard, as follows:—Required to arrange the numbers 1, 2, 3,...8 in such an order that, if the successive figures are increast respectively by 1, 2, 3,...8, all the totals shall be different; and that, if they are increast respectively by 8,...3, 2, 1, these totals also shall all be different.

Gauss describes very clearly the process of systematic trial by which the problem can be solved. He begins with 1 in the first column, and 3 in the second, and tries to find a solution beginning with 13. By the conditions of the problem, neither 2 nor 4 is admissible in the third column, but only 5, 6, 7, or 8. We must therefore make trial of the beginnings

135...,   136...,   137...,   138....

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Taking then 135, the conditions exclude 4 and 6 from the fourth place. Only 2, 7, 8, therefore, remain, and we have to make trial of the beginnings 1352..., 1357..., 1358....

Taking 1352, the conditions exclude 6 and 7 from the next place, so that the only beginnings that have to be tried are

13524... and 13528....

Having regard to the conditions, neither 6, nor 7, nor 8, can stand in the sixth place, after 13524, and this beginning therefore is to be rejected. Similarly the beginning 13528 is to be rejected, because neither 4, nor 6, nor 7, can come in the sixth place. The beginning 1352 is therefore inadmissible. Proceeding in the same way with 1357 and 1358, we find that both of these are also inadmissible. The beginning 135 is therefore inadmissible, and we have to make trial in a similar way of 136, 137, and 138.

It will be seen that this is exactly the same process as I described in my paper. The actual manipulation, however, adopted by Gauss differed from mine. He used cross-ruled paper, or a slate with lines cut pretty deeply in it; and he marked with pencil a cross (×) on each square as soon as it is supposed to be occupied by one of the Queens, and a nought (0) on each of the squares from which a Queen is thereby excluded; and he subsequently rubbed out, as the process proceeded, the marks that were not required. My plan is to place a pawn on each square supposed to be occupied by a Queen; and the removal of the pawns from the board is evidently a much easier process than rubbing out the marks.*

It is obvious that the problem is not restricted to the ordinary chess-board, containing eight squares in a side and sixty-four altogether, but applies to a board with n squares in a side, which therefore contains n^2 squares altogether; in which case, of course, n queens are to be arranged on it, so that no two of them shall be in check of each other.

In the year 1874 it was suggested by Dr Siegmund Günther that our problem might be solved by means of determinants (Grunert's *Archiv der Mathematik und Physik*, 1874, vol. 56, pp. 281–292; see also Günther's *Lehrbuch der Determinanten-Theorie für Studierende. Erlangen*, 1875. Ch. 2, §11, p. 46).

* I found it was of great assistance in working with a board of 11 squares in a side, to have series of lines ruled on the board parallel to the two diagonals, and drawn with black and red ink alternately.

This suggestion was taken up and improved upon by Glaisher, in the paper mentioned above. He says:—

"Dr Günther remarks that if the determinant

$$\begin{vmatrix} a_1 & c_2 & e_3 & g_4 & k_5 & . & . & . & . \\ b_2 & a_3 & c_4 & e_5 & g_6 & . & . & . & . \\ d_3 & b_4 & a_5 & c_6 & e_7 & . & . & . & . \\ f_4 & d_5 & b_6 & a_7 & c_8 & . & . & . & . \\ h_5 & f_6 & d_7 & b_8 & a_9 & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & b_{2n-2} & a_{2n-1} \end{vmatrix}$$

"be expanded, and all the terms be rejected in which either the "same letter or the same suffix appears more than once, then the "terms that remain will give all the solutions of the problem. The "reason for the rule is evident: from the nature of a determinant "each term involves one constituent from each row and one from "each column, and the terms thus represent all the positions in "which the queens cannot take one another castle-fashion; the "omission of the terms in which the same letter or suffix appears "more than once, excludes the cases in which two or more queens "lie on the same diagonal (*i.e.*, can take one another bishop-fashion); "so that the terms that remain are the solutions. Dr Günther "develops the determinants for boards of 9, 16, and 25 squares, but, "owing to the number of terms involved, does not proceed further; "he remarks that for the chess-board of 64 squares it would be "necessary to calculate 20,160 terms."

Glaisher remarks that it would be quite out of the question to actually write down 20,000 terms; and he points out various ways of shortening the work, which render it unnecessary to write down all the terms of the determinant, as Günther apparently did. Taking, for instance, the board of 25 squares, for which the determinant becomes

$$\begin{vmatrix} a_1 & c_2 & e_3 & g_4 & k_5 \\ b_2 & a_3 & c_4 & e_5 & g_6 \\ d_3 & b_4 & a_5 & c_6 & e_7 \\ f_4 & d_5 & b_6 & a_7 & c_8 \\ h_5 & f_6 & d_7 & b_8 & a_9 \end{vmatrix}$$

Glaisher points out that, if a solution contains a_9, and we remove the row and the column containing a_9, we get a board of 16 squares, on which the four queens can be arranged as required by the problem. If, then, we already know the solutions for $n=4$, we shall get from each of them which does not contain the letter a, a solution for $n=5$, by simply adding a_9. But it is easily seen that the only solutions when $n=4$ are

$$c_2e_5d_3b_8 \quad \text{and} \quad e_3b_2c_8d_5.$$

Hence we get two solutions for the 5-board,

$$c_2e_5d_3b_8a_9 \quad \text{and} \quad e_3b_2c_8d_5a_9,$$

which in the usual notation will be represented by 24135 and 31425. (It is to be borne in mind that Glaisher deals with the places of the queens in the several rows, not in the columns, as we have done above.)

These, however, are only aspects of the same solution, as each of them can be got from the other by reflection in the first diagonal. Glaisher then shows how to get the six solutions containing a_1, h_5, and k_5, respectively; but as, in consequence of the double symmetry of the solutions for the 4-board, all these are only other aspects of the solution we have already got, it is unnecessary to say anything more about them.

Whatever the value of n, a little consideration will show that, if we have all the solutions for the $(n-1)$-board, and in one of these solutions a diagonal of the board is open; that is to say, a queen does not stand on any square of the diagonal; then we shall get a solution for the n-board, by adding a new column and a new row, intersecting in that diagonal, and placing a queen on the new square thus added to it. We can do this for each end of the diagonal; so that, in general, we get two solutions for the n-board when a diagonal of the $(n-1)$-board is open; and if both diagonals are open, we get four solutions. These solutions Glaisher calls "ultimate" solutions; but I prefer to call them "corner" solutions.

Having thus got all the solutions that contain a corner square, Glaisher says that it would be useless to write down the terms of the determinant that contain one of the corner constituents, and we may therefore replace these by zeros, and take the simpler determinant:—

$$\begin{vmatrix} \cdot & c_2 & e_3 & g_4 & \cdot \\ b_2 & a_3 & c_4 & e_5 & g_6 \\ d_3 & b_4 & a_5 & c_6 & e_7 \\ f_4 & d_5 & b_6 & a_7 & c_8 \\ \cdot & f_6 & d_7 & b_8 & \end{vmatrix}$$

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In expanding the determinant the signs of the terms are immaterial, and we proceed as if all the terms were positive. We thus get

$$c_2 \begin{vmatrix} b_2 & c_4 & e_5 & g_6 \\ d_3 & a_5 & c_6 & e_7 \\ f_4 & b_6 & a_7 & c_8 \\ \cdot & d_7 & b_8 & \cdot \end{vmatrix} + e_3 \begin{vmatrix} b_2 & a_3 & e_5 & g_6 \\ d_3 & b_4 & c_6 & e_7 \\ f_4 & d_5 & a_7 & c_8 \\ \cdot & f_6 & b_8 & \cdot \end{vmatrix} + g_4 \begin{vmatrix} b_2 & a_3 & c_4 & g_6 \\ d_3 & b_4 & a_5 & e_7 \\ f_4 & d_5 & b_6 & c_8 \\ \cdot & f_6 & d_7 & \cdot \end{vmatrix}$$

Now if we have found all the solutions containing c_2, we can get those containing g_6, b_8, f_4, by turning the board through 1, 2, and 3 right angles, respectively; and those containing g_4, c_8, f_6, b_2, by reflection. Hence we may put each of these seven constituents equal to zero in the second and third terms. When this is done, all the constituents in the fourth row of the second term are replaced by zeros, and the determinant therefore vanishes. The third term, being multiplied by g_4 which is to be replaced by zero, also vanishes. Glaisher, however, does not notice that the second term vanishes, but proceeds:—In the first term, which is multiplied by c_2, we have to reject from the determinant every constituent which contains the letter c, or the suffix 2: or, in other words, we may put a zero in the place of every constituent that contains either c or 2. Similarly, in the second term, we reject every constituent which contains e or 3. We thus get the simplified determinants

$$c_2 \begin{vmatrix} \cdot & \cdot & e_5 & g_6 \\ d_3 & a_5 & \cdot & e_7 \\ f_4 & b_6 & a_7 & \cdot \\ \cdot & d_7 & b_8 & \cdot \end{vmatrix} + e_3 \begin{vmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & b_4 & c_6 & \cdot \\ \cdot & d_5 & a_7 & \cdot \\ \cdot & f_6 & b_8 & \cdot \end{vmatrix}$$

(Here, it will be seen, Glaisher has omitted to put zeros for f_6 and b_8 in the second term; and this error is copied by Pein.)

The determinant which is multiplied by e_3, vanishes; and the other term becomes

$$c_2e_5 \begin{vmatrix} d_3 & \cdot & \cdot \\ f_4 & b_6 & \cdot \\ \cdot & d_7 & \cdot \end{vmatrix} + c_2g_6 \begin{vmatrix} d_3 & a_5 & \cdot \\ f_4 & \cdot & a_7 \\ \cdot & d_7 & b_8 \end{vmatrix}$$

putting zeros for a_5 and e_7 in the first of these minor determinants, and for b_6 in the second.

The first term vanishes, and the second gives us $c_2g_6a_5f_4b_8$, since we reject the combination $d_3a_7d_7$.

This solution, in the usual notation, is 25314, and is easily seen to be doubly symmetrical.

(The above explanation of the process is a little fuller than that given by Glaisher, which is so condenst as to be difficult to follow.)

If we employ in this case the method of systematic trial described by Guass, we shall arrive at the solutions more expeditiously. The corner solutions, if any, having been got from the solution for the 4-board, as explained above, we have to begin with 2; and we see at once that, by the conditions of the problem, this can only be followed by 4, or by 5. We now see that 24 can only be followed by 1, and this leads at once to 24135. This, however, is a corner solution; and is found to be only another aspect of a corner solution already found.

Next, beginning with 25, we have 251, or 253; and the former of these can only be followed by 4; but 2514 must be rejected, because 3 cannot come after it, according to the conditions. Passing on then to 253, we see at once that it can only be followed by 1, and we get the solution 25314.

It is useless to begin with 3, because any solution we should thus get, must have been already got, as it is obviously impossible that a solution should contain the middle points of all the 4 sides. It is also useless to begin with 4; since any solution beginning with 4 is another aspect of a solution, already got, beginning with 2.

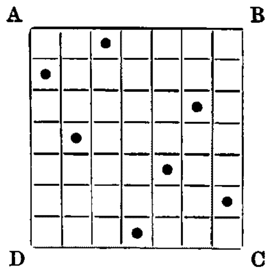
Passing on now to the 8-board, or the ordinary chess-board, Glaisher gives no particulars as to his method of developing the determinant; but Pein, who has used Glaisher's method, tells us that, in order to get the 9 distinct solutions that begin with b_2, and d_3, he has had to investigate 26 minor determinants of the fifth degree. It will be noticed that he deals with the columns of the board, whereas Glaisher deals with the rows. Pein has solved the problem for the 9-board, and the 10-board; and he says that, in order to get the solutions that begin with b_2, d_3, and f_4, he has had to investigate, in the former case, 134 minor determinants of the fifth order; and in the latter, 296 minor determinants of the sixth order.

The labour of writing down and expanding so many determinants, will evidently be very great; and the use of determinants should possess some very decided countervailing advantages, to make it preferable to the method of systematic trial, as explained by Gauss. After careful consideration, however, and trial of both methods, I think that the latter is greatly to be preferred. In order to compare the two methods, it will be convenient to show in some detail how the method of determinants is to be employed in the case of the ordinary chess-board.

I find it desirable to employ a more symmetrical form of determinant, than the one proposed by Günther and adopted by Glaisher; and I write the determinant for the ordinary chess-board as follows:—

$$\begin{vmatrix} \Delta_1 & A_2 & B_3 & C_4 & D_5 & E_6 & F_7 & G_8 \\ a_2 & \Delta_3 & A_4 & B_5 & C_6 & D_7 & E_8 & F_9 \\ b_3 & a_4 & \Delta_5 & A_6 & B_7 & C_8 & D_9 & E_{10} \\ c_4 & b_5 & a_6 & \Delta_7 & A_8 & B_9 & C_{10} & D_{11} \\ d_5 & c_6 & b_7 & a_8 & \Delta_9 & A_{10} & B_{11} & C_{12} \\ e_6 & d_7 & c_8 & b_9 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\ f_7 & e_8 & d_9 & c_{10} & b_{11} & a_{12} & \Delta_{13} & A_{14} \\ g_8 & f_9 & e_{10} & d_{11} & c_{12} & b_{13} & a_{14} & \Delta_{15} \end{vmatrix}$$

We first find the corner solutions from the solutions for the 7-board, in which the diagonals are open. The only solution of this kind for the 7-board is, 2417536; and in this the second diagonal, B D, is open. Adding a fresh column to the board, along A D, and a fresh row along D C, we get a square at D on the 8-board, on which a queen may be placed; and we thus get the solution, 82417536. Also, adding a column along B C, and a row along A B, we get a square at B, on which a queen may be placed; and we thus get the solution 35286471. Each of these solutions has 8 aspects, which are to be found in the way explained above.



It may readily be ascertained without a diagram, whether any given solution has a diagonal open or not. Thus, taking the above solution for the 7-board, 2417536, we count from the left to the right, and find that 5 stands in the 5th place; therefore the first diagonal is closed. But, counting from right to left, we find that no number, r, in the solution, stands in the rth place; and therefore,

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the second diagonal is open. Taking another solution for the 7-board, for instance, 2637415, we see that, counting from left to right, 3 stands in the 3rd place; and counting from right to left, 6 stands in the 6th place; both diagonals are therefore closed.

Having thus got all the corner solutions, we now, as explained by Glaisher, put zeros in place of the corner constituents Δ_1, Δ_{15}, G_8, g_8. This is equivalent to saying that in our systematic trials we need not begin, as Gauss did, with 1 in the 1st place; because this can only give us corner solutions that we have already found by a simpler process.

Proceeding now according to Glaisher's method, we have to develop the following:—

$$A_2 \begin{vmatrix} a_2 & A_4 & B_5 & C_6 & D_7 & E_8 & F_9 \\ b_3 & \Delta_5 & A_6 & B_7 & C_8 & D_9 & E_{10} \\ c_4 & a_6 & \Delta_7 & A_8 & B_9 & C_{10} & D_{11} \\ d_5 & b_7 & a_8 & \Delta_9 & A_{10} & B_{11} & C_{12} \\ e_6 & c_8 & b_9 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\ f_7 & d_9 & c_{10} & b_{11} & a_{12} & \Delta_{13} & A_{14} \\ \cdot & e_{10} & d_{11} & c_{12} & b_{13} & a_{14} & \cdot \end{vmatrix} \quad \cdot \quad \cdot \quad (1)$$

$$+ B_3 \begin{vmatrix} a_2 & \Delta_3 & B_5 & C_6 & D_7 & E_8 & F_9 \\ b_3 & a_4 & A_6 & B_7 & C_8 & D_9 & E_{10} \\ c_4 & b_5 & \Delta_7 & A_8 & B_9 & C_{10} & D_{11} \\ d_5 & c_6 & a_8 & \Delta_9 & A_{10} & B_{11} & C_{12} \\ e_6 & d_7 & b_9 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\ f_7 & e_8 & c_{10} & b_{11} & a_{12} & \Delta_{13} & A_{14} \\ \cdot & f_9 & d_{11} & c_{12} & b_{13} & a_{14} & \cdot \end{vmatrix} \quad \cdot \quad \cdot \quad (2)$$

$$+ C_4 \begin{vmatrix} a_2 & \Delta_3 & A_4 & C_6 & D_7 & E_8 & F_9 \\ b_3 & a_4 & \Delta_5 & B_7 & C_8 & D_9 & E_{10} \\ c_4 & b_5 & a_6 & A_8 & B_9 & C_{10} & D_{11} \\ d_5 & c_6 & b_7 & \Delta_9 & A_{10} & B_{11} & C_{12} \\ e_6 & d_7 & c_8 & a_{10} & \Delta_{11} & a_{12} & B_{13} \\ f_7 & e_8 & d_9 & b_{11} & a_{12} & \Delta_{13} & A_{14} \\ \cdot & f_9 & e_{10} & c_{12} & b_{13} & a_{14} & \cdot \end{vmatrix} \quad \cdot \quad (3)$$

+ determinants multiplied by D_5, E_6, and F_7.

Now, if we have found all the solutions containing A_2, we can as Glaisher says, get those containing f_7, a_{14}, F_9, by turning the board through 1, 2, 3, right angles respectively; and those containing a_2, F_7, A_{14}, f_9, by reflection. We may, therefore, replace these 7 constituents by zeros in (2) and (3), and the subsequent determinants.

We have also in the first determinant, to replace by zeros all the constituents that contain A or 2; in the second determinant, all that contain B or 3; and so on; and we thus get the simplified expressions:—

$$A_2 \begin{vmatrix} \cdot & \cdot & B_5 & C_6 & D_7 & E_8 & F_9 \\ b_3 & \Delta_5 & \cdot & B_7 & C_8 & D_9 & E_{10} \\ c_4 & a_6 & \Delta_7 & \cdot & B_9 & C_{10} & D_{11} \\ d_5 & b_7 & a_8 & \Delta_9 & \cdot & B_{11} & C_{12} \\ e_6 & c_8 & b_9 & a_{10} & \Delta_{11} & \cdot & B_{13} \\ f_7 & d_9 & c_{10} & b_{11} & a_{12} & \Delta_{13} & \cdot \\ \cdot & e_{10} & d_{11} & c_{12} & b_{13} & a_{14} & \cdot \end{vmatrix} \qquad - \quad - \quad (4)$$

$$+B_3 \begin{vmatrix} \cdot & \cdot & \cdot & C_6 & D_7 & E_8 & \cdot \\ \cdot & a_4 & A_6 & \cdot & C_8 & D_9 & E_{10} \\ c_4 & b_5 & \Delta_7 & A_8 & \cdot & C_{10} & D_{11} \\ d_5 & c_6 & a_8 & \Delta_9 & A_{10} & \cdot & C_{12} \\ e_6 & d_7 & b_9 & a_{10} & \Delta_{11} & A_{12} & \cdot \\ \cdot & e_8 & c_{10} & b_{11} & a_{12} & \Delta_{13} & \cdot \\ \cdot & \cdot & d_{11} & c_{12} & b_{13} & \cdot & \end{vmatrix} \qquad - \quad - \quad (5)$$

$$+C_4 \begin{vmatrix} \cdot & \Delta_3 & \cdot & & D_7 & E_8 & \cdot \\ b_3 & \cdot & \Delta_5 & B_7 & \cdot & D_9 & E_{10} \\ & b_5 & a_6 & A_8 & B_9 & \cdot & D_{11} \\ d_5 & c_6 & b_7 & \Delta_9 & A_{10} & B_{11} & \cdot \\ e_6 & d_7 & c_8 & a_{10} & \Delta_{11} & A_{12} & B_{13} \\ \cdot & e_8 & d_9 & b_{11} & a_{12} & \Delta_{13} & \cdot \\ \cdot & \cdot & e_{10} & c_{12} & b_{13} & \cdot & \cdot \end{vmatrix} \qquad - \quad - \quad (6)$$

+ determinants multiplied by D_5 and E_6.

Thus far we have followed the instructions given by Glaisher; but further simplifications are possible, which he does not mention, although he was probably aware of them. Glaisher calls a solution which begins with A_2, a penultimate solution; one that begins with B_3, an antepenultimate; and one that begins with C_4, a preantepenultimate; but I prefer to call them second-place, third-place, and fourth-place solutions.

It is to be noticed that, when we have got the solutions that begin with A_2, B_3, or C_4, we can get those that begin with D_5, E_6, F_7, by the process I have termed reversion; and we may therefore reject the determinants multiplied by these quantities. Again, when we have got the solutions that begin with B_3, these will give us all the 3rd place solutions, so that we may in the determinant multiplied by C_4, put zeros for b_3, e_6, e_{10}, b_{13}, B_{13}, E_{10}. We thus get instead of (6) the following:—

$$C_4 \begin{vmatrix} . & \Delta_3 & . & . & D_7 & E_8 & . \\ . & . & \Delta_5 & B_7 & . & D_9 & . \\ . & b_5 & a_6 & A_8 & B_9 & . & D_{11} \\ d_5 & c_6 & b_7 & \Delta_9 & A_{10} & B_{11} & . \\ . & d_7 & c_8 & a_{10} & \Delta_{11} & A_{12} & . \\ . & e_8 & d_9 & b_{11} & a_{12} & \Delta_{13} & . \\ . & . & . & c_{12} & . & . & . \end{vmatrix} \quad - \quad - \quad (7)$$

Proceeding now to develop (4) we have first:

$$A_2B_5 \begin{vmatrix} b_3 & . & . & C_8 & D_9 & E_{10} \\ c_4 & a_6 & . & . & C_{10} & D_{11} \\ & b_7 & \Delta_9 & . & . & C_{12} \\ e_6 & c_8 & a_{10} & \Delta_{11} & . & . \\ f_7 & d_9 & b_{11} & a_{12} & \Delta_{13} & . \\ . & e_{10} & c_{12} & b_{13} & a_{14} & . \end{vmatrix} \quad - \quad - \quad (8)$$

$$= A_2B_5b_3 \begin{vmatrix} a_6 & . & . & C_{10} & D_{11} \\ . & \Delta_9 & . & . & C_{12} \\ c_8 & a_{10} & \Delta_{11} & . & . \\ d_9 & . & a_{12} & \Delta_{13} & . \\ e_{10} & c_{12} & . & a_{14} & . \end{vmatrix} + A_2B_5C_8 \begin{vmatrix} c_4 & a_6 & . & . & D_{11} \\ . & b_7 & \Delta_9 & . & . \\ e_6 & . & a_{10} & . & . \\ f_7 & d_9 & b_{11} & \Delta_{13} & . \\ . & e_{10} & c_{12} & a_{14} & . \end{vmatrix}$$

$$+ A_2B_5D_9 \begin{vmatrix} c_4 & a_6 & . & . & . \\ . & b_7 & . & . & C_{12} \\ e_6 & c_8 & a_{10} & \Delta_{11} & . \\ f_7 & . & b_{11} & a_{12} & . \\ . & e_{10} & c_{12} & b_{13} & . \end{vmatrix} + A_2B_5E_{10} \begin{vmatrix} c_4 & a_6 & . & . & . \\ . & b_7 & \Delta_9 & . & . \\ e_6 & c_8 & . & \Delta_{11} & . \\ f_7 & d_9 & b_{11} & a_{12} & \Delta_{13} \\ . & . & c_{12} & b_{13} & a_{14} \end{vmatrix}$$

Here it may be useful to mention a practical hint that Glaisher gives, namely, that in forming the minor determinants it is convenient to write down the constituents not already obliterated, and then to scratch through with the pen, those that have to be replaced by zeros, in consequence of the new factor we have taken into the multiplier.

Proceeding now to develop the determinant multiplied by $A_2B_5b_3$, we have as the first minor determinant

$$A_2B_5b_3a_6 \begin{vmatrix} \Delta_9 & . & . & C_{12} \\ . & \Delta_{11} & . & . \\ . & . & \Delta_{13} & . \\ c_{12} & . & . & . \end{vmatrix}$$

and it is clear that this does not give a solution.
The second minor determinant is—

$$A_2B_5b_3C_{10} \begin{vmatrix} . & \Delta_9 & . & . \\ c_8 & . & \Delta_{11} & . \\ d_9 & . & a_{12} & . \\ . & c_{12} & . & . \end{vmatrix}$$

This vanishes, and of course gives no solution.
The third minor is—

$$A_2B_5b_3D_{11} \begin{vmatrix} . & \Delta_9 & . & . \\ c_8 & a_{10} & . & . \\ d_9 & . & a_{12} & \Delta_{13} \\ e_{10} & c_{12} & . & a_{14} \end{vmatrix}$$

As Δ_9 is the only constituent in the first row, and a_{12} the only one in the third column, the expanded determinant must contain both these; and we must then replace Δ_{13} and a_{14} by zeros, so that the fourth column contains only zeros, and the determinant vanishes.

We thus see that we cannot have a solution beginning with $A_2B_5b_3$, that is, with 241.

Next, taking the determinant multiplied by $A_2B_5C_8$, we see that, as D_{11} is the only constituent in the last column, it must be a factor in the expanded determinant, and we thus have to deal with the simpler determinant—

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$$A_2B_5C_8D_{11}\begin{vmatrix} . & b_7 & \Delta_9 & . \\ e_6 & . & a_{10} & . \\ f_7 & d_9 & . & \Delta_{13} \\ . & e_{10} & c_{12} & a_{14} \end{vmatrix}$$

$$= A_2B_5C_8D_{11}b_7\begin{vmatrix} e_6 & a_{10} & . \\ . & . & \Delta_{13} \\ . & c_{12} & a_{14} \end{vmatrix} + A_2B_5C_8D_{11}\Delta_9\begin{vmatrix} e_6 & . & . \\ f_7 & . & . \\ . & e_{10} & a_{14} \end{vmatrix}$$

The first of these minor determinants gives the solution—

$$A_2B_5C_8D_{11}b_7e_6\Delta_{13}c_{12} \quad \text{or} \quad 24683175.$$

The second evidently can give no solution, and we thus find that this is the only solution beginning with 246.

Taking now the determinant multiplied by $A_2B_5D_9$, we see C_{12} is a factor of it, and we have to expand the simpler determinant

$$\begin{vmatrix} c_4 & a_6 & . & . \\ e_6 & c_8 & a_{10} & \Delta_{11} \\ f_7 & . & b_{11} & . \\ . & e_{10} & . & b_{13} \end{vmatrix} = c_4\begin{vmatrix} . & a_{10} & \Delta_{11} \\ . & b_{11} & . \\ e_{10} & . & b_{13} \end{vmatrix} + a_6\begin{vmatrix} . & . & \Delta_{11} \\ f_7 & b_{11} & . \\ . & . & b_{13} \end{vmatrix}$$

and it is obvious on inspection that neither of these gives a solution.

Lastly, taking the determinant multiplied by $A_2B_5E_{10}$, the first minor is—

$$A_2B_5E_{10}c_4\begin{vmatrix} b_7 & \Delta_9 & . & . \\ . & . & \Delta_{11} & . \\ d_9 & b_{11} & a_{12} & \Delta_{13} \\ . & . & b_{13} & a_{14} \end{vmatrix}$$

Here we see that Δ_{11} is a factor, since it is the only constituent in the second row; and as this involves replacing both Δ_9 and b_{11} by zeros, the determinant vanishes, and we get no solution.

The second minor is—

$$A_2B_5E_{10}a_6\begin{vmatrix} . & \Delta_9 & . & . \\ . & . & \Delta_{11} & . \\ f_7 & b_{11} & . & \Delta_{13} \\ . & c_{12} & b_{13} & . \end{vmatrix}$$

Here Δ_9 must be a factor, and this involves replacing Δ_{13} by zero, so that the determinant vanishes.

We thus finally conclude that there is only one solution which begins with A_2B_5 or with 24.

As a final example, I will take the beginning 257 or $A_2C_6D_9$. The determinant to be expanded in this case is—

$$\begin{vmatrix} c_4 & . & \Delta_7 & . & . \\ d_5 & b_7 & a_8 & . & . \\ . & c_8 & . & \Delta_{11} & B_{13} \\ f_7 & . & c_{10} & a_{12} & . \\ . & e_{10} & d_{11} & b_{13} & . \end{vmatrix} = B_{13}\begin{vmatrix} c_4 & . & \Delta_7 & . \\ d_5 & b_7 & a_8 & . \\ f_7 & . & c_{10} & a_{12} \\ . & e_{10} & d_{11} & . \end{vmatrix}$$

$$= B_{13}a_{12}\begin{vmatrix} c_4 & . & \Delta_7 \\ d_5 & b_7 & . \\ . & e_{10} & d_{11} \end{vmatrix}$$

This gives us the two products $c_4b_7d_{11}$, and $\Delta_7d_5e_{10}$; and arranging the constituents in the proper order, we get the solutions

$$A_2C_6D_9\,c_4\,b_7B_{13}a_{12}d_{11} \quad \text{or} \quad 25713864,$$

and $$A_2C_6D_9\Delta_7d_5B_{13}a_{12}e_{10} \quad \text{or} \quad 25741863.$$

On comparing the two processes—systematic trial and the use of determinants—we see that each step in the one process usually corresponds exactly to a step in the other.

When we replace Δ_1 by zero, this, as already remarkt, is equivalent to saying that we need not make trial of any combination that begins with 1. When we replace G_8 by zero, this is equivalent to saying that we are not to make trial of any combination that begins with 8; and when we similarly replace g_8 and Δ_{15}, this implies that we are not to try any combination that has 1 or 8 in the last place.

When in the determinant (2) (that is multiplied by B_3) we replace F_7 by zero, this implies that we are to try no combination that begins with 7. Replacing a_2 by zero implies that we are to try no combination that contains 1 in the 2nd place. Similarly replacing f_7 and A_{14} implies that we are not to try any combination in which either 1 or 8 stands in the 7th place; and, lastly, replacing f_9 and a_{14} by zero implies that we are not to try any combination in which 2 or 7 occupies the last place.

Proceeding next to the determinant (3) (that is multiplied by C_4) the directions as to replacing certain constituents by zeros imply that, in trying the combinations that begin with 4, we may neglect those in which either 1 or 8 stands in the 2nd or 3rd or in the 6th or 7th place; or in which 2, or 3, or 6, or 7, stands in the 8th place.

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It is obvious, however, that very little labor is saved by excluding the combinations in which e_8, f_7, f_9, e_{10}, b_{13}, a_{14}, A_{14}, and B_{13} appear; and in practice it will be found convenient to try those combinations, so as to secure that we shall get at least two aspects of each solution.

The principal difference between the two processes is that, when determinants are used, each step in the process is recorded, so that any error can be easily traced and corrected.

Much time is occupied in writing down the determinants; but the process has the advantage of enabling us to take a comprehensive view of the combinations which we are to try. It sometimes also has the advantage of enabling us, when we are dealing with determinants containing 4 or 5 rows, to see more quickly than by the other method what combinations will give a solution. On the other hand, the method of systematic trial has the great advantage of furnishing a complete check on our work, because this can easily be so arranged that we shall obtain two (or more) aspects of each solution. Although, as already stated, I prefer the method of systematic trial, others may be of a different opinion; and each operator will probably prefer the method to which he is more accustomed.

We have seen that Glaisher gets the corner solutions for the n-board, from the solutions for the $(n-1)$-board in which the diagonals are open; and this has led Mr Rouse Ball to say that, "his method consists in deducing the solutions for a board of n^2 "cells, from one of $(n-1)^2$ cells". This statement, however, is very inaccurate; as there is no method, or at all events, none known at present, by which all the solutions for the n-board can be got from those of the $(n-1)$-board. The same erroneous idea appears again in the following passage: "The solutions for a board of 9^2 cells, "were given first by Prof. P. H. Schoute, of Holland, in the *Eigen* "*Haard*, and from them M. Delannoy constructed the solutions "for a board of 10^2 cells. The solutions are quoted by Lucas "(*Récréations*), Vol. II, pp. 238–240". I am not in a position to give any further information as to the methods followed by these gentlemen, the dates of their papers, or the results they have obtained.

As already mentioned, Pein has obtained the solutions for the 9-board, and for the 10-board; and the former agree exactly with those which I gave in my paper above referred to. I have since

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obtained the solutions for the 11-board, and a list of them is appended to this paper. The corner solutions were got by means of Pein's solutions for the 10-board. I find there are:—

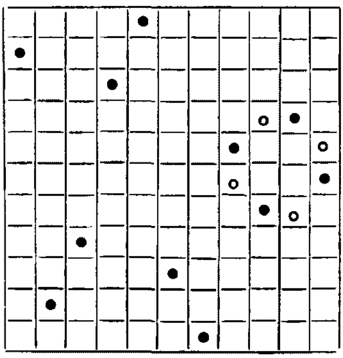
48	Corner solutions,	of	which	0	are	centric.
179	2nd place ,,	,,	,,	4	,,	,,
82	3rd place ,,	,,	,,	4	,,	,,
30	4th place ,,	,,	,,	4	,,	,,
2	5th place ,,	,,	,,	0	,,	,,
341	solutions in all	,,	,,	12	,,	,,

This was a very heavy piece of work, and occupied most of my leisure time for several months. The results thus obtained have settled a question I raised in my former paper. I there put forward the conjecture that it might be impossible that there should ever be four queens arranged in a solution, in the same position as they occupy in the solution on the 4-board; but this turns out not to be the case; for in the following solutions for the 11-board,

2.10.8.3.1.9.11.5.7.4.6,

and 2.10.8.3.1.9.11.6.4.7.5,

being Nos. 173 and 174 of the second-place solutions, the last 4 men in each case are so arranged. The two arrangements are shown in the annext figure.



I have also been able to settle a question raised by Pein. The solutions, when the number of sides is 4 and 5, are doubly symmetrical; each remaining unaltered when the board is rotated thro' a right angle. Pein says that such doubly symmetrical solutions seem to occur only when the number of squares in the side of the board is 4 or 5; but it will be found that the following solution for the 12-board is doubly symmetrical:—

5.3.11.6.12.9.4.1.7.2.10.8.

It is easy to see that a doubly symmetrical solution can only occur when the number of squares in a side is of the form 4N or 4N + 1.

The following are the numbers of solutions for the different boards:—

Number of Squares in a side.	Distinct Solutions.	Of which are centric.	Total Solutions.
4	1	1	2
5	2	1	10
6	1	1	4
7	6	2	40
8	12	1	92
9	46	4	352
10	92	3	724
11	341	12	2680

It will, I imagine, be scarcely practicable to obtain results for larger boards, unless a number of persons co-operate in the work. This would be very easy to arrange, if there were a sufficient number of volunteers; as the work is of such a nature that it could be readily divided among a number of persons.

Solutions for the Board containing 11 Squares in a side.

Note.—The solutions marked with a star (*) are centric.

(Here *t* is put for 10, and *e* for 11.)

Corner Solutions

No. 1.	13579 *e* 2468*t*	No. 16.	——*e*8 3 5926*t*
2.	13692 8 *e*475*t*	7.	14938 2 *e*6*t*75
3.	13794 2 *t*6*e*58	8.	——73 *t* 625*e*8
4.	——*t*2 9 64*e*85	9.	——*t* 2 *e*6835
5.	——8 *e* 24695	20.	14*t*7*e* 3 82596
6.	——*e*4 8 *t*5269	1.	1579*e* 3 8*t*246
7.	13869 2 5*e*47*t*	2.	——*t*4 *e* 93682
8.	——*t*4 9 *e*5726	3.	158*t*7 3 *e*2469
9.	——*e*7 2 *t*6495	4.	1596*t* 2 *e*3748
10.	13*t*79 *e* 24685	5.	164*t*8 3 52*e*97
1.	——84 *e* 96257	6.	——*e*8 5 2*t*379
2.	14692 *t* 538*e*7	7.	1683*e* 9 25*t*47
3.	——*t*3 *e* 85297	8.	——*t*4 9 52*e*37
4.	1479*e* 2 *t*6358	9.	——*e*7 4 2953*t*
5.	——*t*2 5 8*e*369	30.	16938 *e* 42*t*57

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No. 31.	——7t 4 258e3
2.	16e5t 4 93827
3.	1748e 5 t2639
4.	1753e 9 42t86
5.	——8e 4 t3692
6.	——— 9 42t36
7.	17938 t 425e6
8.	——63 t 4e852
9.	——e3 5 t2468
No. 40.	17t3e 9 42586
1.	——83 – ——e6
2.	18497 3 e625t
3.	——e– – t—95
4.	185e6 3 t4792
5.	18639 e 4t752
6.	——e3 5 94t72
7.	1946e 3 t7582
8.	1963e 4 8t572

Second-Place Solutions.

No. 1.	2417t 3 e8596
2.	24693 1 e75t8
3.	——– – –8–7t
4.	24718 e 5396t
5.	——9e 5 t1683
6.	*——– 6 1358t
7.	——t3 e 61958
8.	——e– 5 t—6–
9.	24819 6 3e75t
10.	——e 6 t7539
1.	——t3 1 6e975
2.	——5 – —3–9
3.	——– – e7369
4.	– –e3 6 t5197
5.	——– t 7——6
6.	——5 1 t6397
7.	——— t 61379
8.	24973 e 615t8
9.	——t 1 e5863
20.	——– – –6–35
1.	——e5 t 16837
2.	——8 5 317t6
3.	24t39 e 58617
4.	——79 e 31685
5.	——e 6 31859
6.	——83 e 71695
7.	——5 e 13697
8.	24e73 t 61958
No. 29.	——t 3 58196
30.	——95 1 t7368
1.	25184 e 936t7
2.	——9e 8 473t6
3.	——e7 t 63948
4.	2531t 8 e4796
5.	2579e 3 168t4
6.	——— 8 413t6
7.	——t3 e 64918
8.	——4 e 13968
9.	——— – –8639
40.	——e1 6 4t839
1.	——3 8 4196t
2.	25813 t 74e96
3.	——e 4 7t396
4.	——e1 – ——–
5.	*——3 6 9147t
6.	——– t 74196
7.	——7 t 31946
8.	——9 3 147t6
9.	——– 4 613t7
50.	*——– 6 3147t
1.	25964 e 137t8
2.	25t6e 3 74819
3.	——83 e 91647
4.	——4 e 73169
5.	——e 4 13697
6.	25e6t 3 18497

No. 57.	——91 t 47386
8.	———4 6 3t718
9.	2639e 4 t7518
60.	2683e 9 514t7
1.	——t4 1 3e975
2.	———5 1 9e374
3.	——e1 3 7t495
4.	———4 1 953t7
5.	———5 1 t4793
6.	———7 1 3594t
7.	2691t 4 7e835
8.	——e1 – –t—–
9.	26t14 9 73e85
70.	———e 9 53847
1.	——31 8 5e974
2.	———7 e 48159
3.	——71 3 58e94
4.	26e37 4 t1958
5.	2716e 5 t4938
6.	——8e 9 4t536
7.	——e9 6 4t835
8.	2738e 4 1t596
9.	——t8 1 e4695
80.	——e8 4 1t596
1.	———9 5 1t468
2.	27531 9 e64t8
3.	——8e 1 4693t
4.	——— 9 3641t
5.	——— – 4–13–
6.	275t1 4 e8396
7.	——— 6 4e839
8.	———8 6 e3194
9.	——e1 t 63948
90.	27915 t 63e84
1.	——41 5 t6e38
2.	——61 t 4e853
3.	———3 1 ——t5
4.	——— – e5—4
5.	——e5 1 4t863

No. 96.	27t31 9 58e46
7.	———5 8 e4619
8.	——— 9 14e86
9.	———9 6 4e185
100.	——41 5 9e386
1.	——— – —6–3
2.	——— 8 e3695
3.	———9 1 3e685
4.	——63 1 e8594
5.	——— 9 e4815
6.	——83 5 9e146
7.	27e35 9 1t468
8.	———8 6 4t159
9.	——48 1 3t695
110.	——81 4 t3—–
1.	———3 1 t5964
2.	28394 t 16e57
3.	——— – 5e61–
4.	——e7 5 t1469
5.	——— t 41596
6.	28519 e 637t4
7.	——3t 6 e1479
8.	——7– – ——93
9.	*——91 6 e374t
120.	———4 t 7e316
1.	28613 e 7t495
2.	——39 e 157t4
3.	——4t 1 e5793
4.	——91 t 53e74
5.	28t14 e 96357
6.	——39 6 41e57
7.	——49 3 57e16
8.	——73 1 e4695
9.	29184 e 7t635
130.	29317 t 6e584
1.	———8 t 57e46
2.	——6t 1 5e847
3.	——8t 4 61e75
4.	———e 1 6t574

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No. 135.	——t7	1	e5864
6.	29514	t	7e386
7.	——8	4	e73t6
8.	——e	6	t7483
9.	——3t	6	e1748
140.	——81	3	6e7t4
1.	——	4	7e63t
2.	——t	3	6e147
3.	29613	8	e74t5
4.	——7	4	e853t
5.	——31	8	e57t4
6.	——	t	7–e8–
7.	——t	8	e4175
8.	——41	t	57e38
9.	——t5	1	e4738
150.	——	e	1——
1.	29731	6	t5e84
2.	——41	e	6t358
3.	——t	1	5e863
4.	29e61	3	t7485
5.	2t374	8	e5169
6.	——	e	96158
7.	——9	4	61e58

No. 158.	2t571	4	e8693
9.	——4	1	——39
160.	2t615	8	e3794
1.	——	e	97483
2.	——9	e	3——5
3.	——39	4	18e57
4.	——	e	4–1—
5.	——49	1	37e85
6.	——93	1	e8574
7.	——	5	—1—
8.	——5	1	47e83
9.	——e3	1	95847
170.	2t73e	9	41586
1.	——41	5	9e683
2.	——	9	63e85
3.	2t831	9	e5746
4.	——	–	–6475
5.	——5	–	–1746
6.	——49	3	6e157
7.	——63	9	e1475
8.	——e4	1	36975
9.	——5	3	16497

Third-Place Solutions.

No. 1.	3528e	4	196t7
2.	3579e	2	41t86
3.	——t4	e	18629
4.	——6	–	—4—
5.	3581e	6	27t49
6.	*——t1	6	e2479
7.	——	–	—794
8.	——e9	4	61t27
9.	35914	e	862t7
10.	——t	7	e8246
1.	——26	e	174t8
2.	——t	7	e8146
3.	——e4	7	162t8
4.	35t49	e	27168

No. 15.	——86	–	——49
6.	362t1	4	8e975
7.	3641e	9	528t7
8.	3681e	7	t2594
9.	——	9	524t7
20.	——t1	4	2e975
1.	——e2	7	t4195
2.	——4	1	75t29
3.	36924	e	851t7
4.	——8	e	41t57
5.	——	–	–71t5
6.	——71	e	25t84
7.	——2	e	814t5
8.	——e2	5	t8147

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No. 29.	36t24 1 8e975
30.	——72 e 58194
1.	——— – 84—5
2.	36e14 7 t8259
3.	——8 5 297t4
4.	——51 4 t7928
5.	——91 5 28t74
6.	3748e 9 152t6
7.	——t1 5 e8629
8.	*——— 6 -285-
9.	——e1 5 t6928
40.	37915 t 8e246
1.	——8 5 e2t64
2.	——26 1 e58t4
3.	—t— – -859-
4.	——41 9 e6825
5.	——82 5 1e964
6.	37e81 4 692t5
7.	38241 9 e6t75
8.	*——71 6 e5t49
9.	——9 e 15t64
50.	——96 t 1e574
1.	38429 5 1e6t7
2.	——e1 t 75296
3.	3861t 7 e4295
4.	——e2 5 194t7
5.	38t41 9 e6275
No 56.	——6 1 e7529
7.	*——51 6 e7249
8.	——74 1 e9625
9.	38e52 9 147t6
60.	——92 5 ———
1.	39248 1 e6t75
2.	——e 7 t6158
3.	——t1 6 8e475
4.	394t1 6 e2758
5.	——— e 6——
6.	——5 – -1-2-
7.	3961e 7 528t4
8.	——t5 e 17428
9.	39742 e 61t85
70.	3t296 1 e8574
1.	3t419 e 68257
2.	——28 e 71695
3.	——71 e 62958
4.	——e 2 61—-
5.	3t64e 1 85297
6.	——95 1 e7248
7.	3t716 e 28594
8.	——2- – 1——
9.	——e 5 81946
80.	——41 e 86295
1.	3t842 7 e1596
2.	——5e 1 62974

Fourth-Place Solutions.

No. 1.	427t6 e 91358
2.	428t3 7 e1695
3.	——7 1 -9-3-
4.	42936 e 1t758
5.	——e1 8 63t75
6.	——3 – -1——
7.	42t5e 8 13697
8.	46931 e 7t258
9.	——e1 5 2t738
10.	——2 7 318t5
No. 11.	46t25 e 81397
2.	——7 e 31958
3.	——5e 1 37928
4.	47931 8 e52t6
5.	——e2 5 18t36
6.	*47t31 6 e9258
7.	——— 8 —-625
8.	*——e 6 19258
9.	——63 1 e9528
20.	483e2 7 1t596

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No. 21.	——9 1 6t275	No 26.	49382 e 16t57
2.	—— 5 1t726	7.	*495t1 6 e2738
3.	48591 3 e72t6	8.	*4972e 6 1t538
4.	—— - -—t2–	9.	—3e 8 15t26
5.	48t59 e 13627	30.	4t36e 1 85297

FIFTH-PLACE SOLUTIONS.

No. 1.	524t8 e 13697	No. 2.	536t2 e 18497

On a Problem of Lewis Carroll's.

By Professor STEGGALL.

Fifth Meeting, 10th March, 1899.

ALEXANDER MORGAN, Esq., M.A., D.Sc., President, in the Chair.

Centrobaric Spherical Surface Distribution.

By Professor TAIT.

The following is a simple geometrical demonstration of the well-known theorem that, if matter be distributed over a sphere with a surface-density (*i.e.*, mass per unit of surface) inversely as the cube of the distance from either of two points which are the inversions of each other with respect to the sphere, it will act upon all external masses as if it were collected at the interior point:—and upon all internal masses as if a definite multiple of its mass were concentrated at the exterior point.

Suppose a cone of very small angle, whose vertex is S, to cut out small areas, P and Q, from a spherical surface. (Fig 5.) Then we have, obviously,

$$\frac{P}{SP^2} = \frac{Q}{SQ^2}$$

And, of course, the rectangle SP.SQ is constant, say c^2.
Let R be any point, outside the sphere if S be inside, and *vice versâ*; and take T (always *inside* the sphere) on RS so that SR.ST$=c^2$.

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