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L^p SPACES GENERATED BY CERTAIN OPERATOR VALUED MEASURES

BY

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1. **Introduction.** In this paper we investigate the structure of certain spaces of operator valued measures and the L^p spaces they generate. The work is motivated by our earlier paper [1] in which we studied the L^p spaces generated by matrix valued measures. The present results can thus be regarded as a generalization of this "finite dimensional" situation.

Let H be a separable Hilbert space and $\boldsymbol{\mu}$ a measure defined on $\mathcal{B}$—the bounded Borel subsets of the real line R—so that for each $S \in \mathcal{B}$, $\boldsymbol{\mu}(S): H \to H$ is a compact Hermitian operator. The *variation* μ of $\boldsymbol{\mu}$ is defined by

$$\mu(S) = \sup\left\{\sum_{i=1}^{j} \|\boldsymbol{\mu}(S_i)\| \mid S_1, \ldots, S_j \in \mathcal{B}, \text{ pairwise disjoint}, S_i \subset S\right\}$$

for $S \in \mathcal{B}$. We shall assume $\mu(S) < \infty \forall S \in \mathcal{B}$ and that μ is absolutely continuous with respect to the non-negative regular σ-finite Borel measure ν defined on the real line. In this case we say that $\boldsymbol{\mu}$ *is absolutely continuous with respect to* ν and write $\boldsymbol{\mu} \ll \nu$.

When $H = R$, the classical Radon-Nikodým theorem states that

$$\boldsymbol{\mu}(S) = \int_S \mathbf{m}(s)\, d\nu(s) \qquad \forall S \in \mathcal{B} \tag{1}$$

for some $\mathbf{m}(s)$, real valued and locally ν-integrable. The space $L^p(\boldsymbol{\mu})$ $(p \geq 1)$, consisting of all functions $f: R \to \mathbb{C}$ with

$$\|f\| = \left[\int_R |f(s)|^p\, \mathbf{m}(s)\, d\nu(s)\right]^{1/p} < \infty$$

(modulo functions of zero norm) is by now a standard Banach space having desirable properties such as reflexivity and uniform convexity (for $p > 1$) etc.

Let $\mathbb{C}$ denote the complex numbers. When $H = \mathbb{C}^n$, a straightforward extension of the Radon–Nikodým theorem again yields the representation (1) with $\mathbf{m} \in L^1(\nu): R \to C(H)$. Here $C(H)$ denotes the compact Hermitian operators on H—isomorphic with $n \times n$ complex Hermitian matrices in this case. We may

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define L^p spaces by considering those functions $f: R \to \mathbb{C}^n$ for which

$$\|f\| = \left[\int_R |\langle f(s), \mathbf{m}(s) f(s)\rangle|^{p/2} \, d\nu(s)\right]^{1/p} < \infty. \tag{2}$$

We have shown in [11] that again a class of Banach spaces results with the above desirable properties—at least in the case when $\boldsymbol{\mu}$ takes positive semi-definite values. Further $L^p(\boldsymbol{\mu})$ decomposes into the direct sum of $n+1$ subspaces corresponding to the partition $E_0, E_1, \ldots, E_n$ of R where

$$E_r = \{s \in R \mid \mathbf{m}(s) \quad \text{has rank } r\}, \qquad r = 0, 1, \ldots, n.$$

Our present purpose, then, is to consider a separable Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. $\mathcal{P}(H)$ will denote the positive cone of compact Hermitian positive semi-definite operators on H and $\boldsymbol{\mu}: \mathcal{B} \to \mathcal{P}(H)$ will denote a measure on $\mathcal{B}$, absolutely continuous with respect to the regular non-negative σ-finite Borel measure ν defined on R.

To establish the representation (1) is no longer a trivial exercise in the infinite dimensional case (and is, in general, impossible) but we show that if $\boldsymbol{\mu}$ is $\mathcal{P}(H)$ valued then so is $\mathbf{m}$. For this we shall use the following assumption, typical of infinite dimensional Radon-Nikodým theorems, see for example [8], [9]:

(3) $\{\boldsymbol{\mu}(S)/\nu(S) \mid S \in \mathcal{B}\}$ *is relatively weakly compact in* $C(H)$.

As an example of such a situation, consider the measure $\boldsymbol{\mu}$ defined on $\mathcal{B}$ by

$$[\boldsymbol{\mu}(S)x]_i = x_i \int_S f(i, s) \, d\nu(s), \qquad x \in H,$$

where x_i denotes the ith component of x relative to a fixed orthonormal basis of H. Here we assume $f(i, s)$ is a real valued measurable function of s for each $i = 1, 2, \ldots$ and satisfies $|f(i, s)| \le g(i)$ where $g(i) \to 0$ as $i \to \infty$. Clearly $|[\boldsymbol{\mu}(S)]_{ij}| \le g(i)\nu(S) \to 0$ as $i \to \infty$ so that the eigenvalues of $\boldsymbol{\mu}(S)$ have finite multiplicity and 0 is their only limit point. Hence $\boldsymbol{\mu}(S)$ is a compact operator on H. Further "diagonal" operators such as $\boldsymbol{\mu}(S)$ may be identified in an obvious way with real sequences convergent to zero—i.e. a subspace of c_0. A bounded set $K \subseteq c_0$ is (strongly) relatively compact if the sequence convergence to zero is uniform over K. Since

$$|[\boldsymbol{\mu}(S)/\nu(S)]_{ij}| \le g(i) \to 0 \quad \text{as} \quad i \to \infty$$

this convergence is uniform over $S \in \mathcal{B}$ and so (3) holds even in the strong topology on $C(H)$.

We define L^p spaces by analogy with (2) and investigate their properties. These results not only extend [1] to the infinite dimensional case, but we have been able to simplify our earlier analysis. We show that $L^p(\boldsymbol{\mu})$ is separable and uniformly convex and smooth for $1 < p < \infty$. Again rather more is achieved than in [1].

2. **The spaces** $L^p(\boldsymbol{\mu})$. Our task in this section is to show that the assumptions (3) and $\boldsymbol{\mu}:\mathcal{B}\to\mathcal{P}(H)$ yield a representation of the form (1) which can be used to define function spaces $L^p(\boldsymbol{\mu})$.

THEOREM 1. *Suppose* $\boldsymbol{\mu}:\mathcal{B}\to\mathcal{P}(H)$ *and that* (3) *holds where* $\boldsymbol{\mu}\ll\nu$. *Then there exists a* ν-*essentially unique function* $\mathbf{m}:R\to\mathcal{P}(H)$, *locally* ν-*integrable and such that*

$$\boldsymbol{\mu}(S)=\int_S \mathbf{m}(s)\,d\nu(s),\qquad \forall S\in\mathcal{B}. \tag{1}$$

Proof. A result of Phillips [9; Theorem 5.5]—see also [8; Theorem 2] and [10; p. 48]—gives a locally ν-integrable function $\mathbf{m}: R\to C(H)$ and satisfying (1). ν-essential uniqueness of $\mathbf{m}$ follows readily from [10, Corollary 2.6, p. 33].

It remains to show that $\mathbf{m}$ takes values in $\mathcal{P}(H)$ given that $\boldsymbol{\mu}$ takes values there. The result follows readily by noticing that $\mathcal{P}(H)$ is a closed convex cone in $C(H)$ and appealing to [10, Lemma 2.4(d), p. 33]. This result gives $\mathbf{m}(s)\in\mathcal{P}(H)$ ν-almost everywhere; but by modification of $\mathbf{m}$ on a ν-null set we may choose $\mathbf{m}(s)\in\mathcal{P}(H)$, $\forall s\in R$. This completes the proof of the theorem.

Because of the special nature of $C(H)$, (see, for example, [4]), we were hopeful that less than assumption (3) would be needed. However, the following example shows that (3) cannot be dispensed with entirely.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. For each $S\in\mathcal{B}$ define

$$\mu_{ij}(S)=\delta_{ij}\int_S \sin(is)\,ds,\qquad i,j=1,2,\ldots$$

where δ_{ij} is Kronecker's symbol. Then $[\mu_{ij}(S)]$ can be regarded as the matrix representation relative to $\{e_n\}$ of an operator $\boldsymbol{\mu}(S):H\to H$. If λ denotes Lebesgue measure we have $\|\boldsymbol{\mu}(S)\|\le\lambda(S)$. Further, $\boldsymbol{\mu}(S)$ is compact since by the Riemann-Lebesgue lemma [11, Theorem 1, p. 11], $\mu_{ii}(S)\to 0$ as $i\to\infty$. However, the density $\mathbf{m}(s)$ of $\boldsymbol{\mu}$ with respect to λ is given by

$$\mathbf{m}(s)=[m_{ij}(s)]=[\delta_{ij}\sin(is)]$$

and, except when s is an integral multiple of π, this is not the matrix representation of a compact operator since $\sin(is)\not\to 0$ as $i\to\infty$. Thus we have a compact operator valued measure $\boldsymbol{\mu}$, absolutely continuous with respect to Lebesgue measure and whose density is almost everywhere non-compact operator valued.

At this stage the following assumption will be made concerning the nature of $\mathbf{m}(s)$:

(4) *For* ν-*almost all* s, $\mathbf{m}(s)$ *is an operator with finite dimensional range.*

The dimension of the range of $\mathbf{m}(s)$ will of course vary with s and indeed these dimensions are not assumed uniformly bounded with respect to s. Remarks on the *raison d'être* for this assumption will be made in the next paragraph. We shall also discuss the situation in which (4) is not satisfied.

We come now to the definitions of the spaces $L^p(\boldsymbol{\mu})$. For $1 \le p < \infty$, the space $L_0^p(\boldsymbol{\mu})$ will consist of all Borel measurable functions $f: R \to H$ for which

$$\|f\|_p = \left[\int_R \langle f(s), \mathbf{m}(s)f(s)\rangle^{p/2}\, d\nu(s)\right]^{1/p} < \infty.$$

The space $L_0^\infty(\boldsymbol{\mu})$ will consist of all Borel measurable functions $f: R \to H$ for which

$$\|f\|_\infty = \nu - \operatorname*{ess\,sup}_{s \in R} \langle f(s), \mathbf{m}(s)f(s)\rangle^{1/2} < \infty.$$

Notice that $\langle f(s), \mathbf{m}(s)f(s)\rangle \ge 0 \forall s \in R$ so that the quantities $\|f\|_p$, $\|f\|_\infty$ are non-negative real numbers. Easy calculations show that if $\alpha \in \mathbb{C}$ and $f, g \in L_0^p(\boldsymbol{\mu})$ then $\|\alpha f\|_p = |\alpha|\, \|f\|_p$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for any $1 \le p \le \infty$. We define N_p to be the subspace of $L_0^p(\boldsymbol{\mu})$ for which $\|f\|_p = 0$ and finally set $L^p(\boldsymbol{\mu}) = L_0^p(\boldsymbol{\mu})/N_p$. Thus we have so far that $L^p(\boldsymbol{\mu})$ is a normed vector space.

THEOREM 2. *The spaces $L^p(\boldsymbol{\mu})$ are independent of the measure ν used to define them.*

Proof. The proof follows, *mutatis mutandis*, the proof of the corresponding theorem for the finite dimensional case [1, Theorem 1]. The only point needing additional explanation is the following.

If ν and $\hat{\nu}$ are two σ-finite regular Borel measures defined on R such that $\boldsymbol{\mu} \ll \nu$ and $\boldsymbol{\mu} \ll \hat{\nu}$ and $\boldsymbol{\mu}$ satisfies (3) with respect to both ν and $\hat{\nu}$, then $\boldsymbol{\mu}$ has a density, $\mathbf{n}$, with respect to $\nu + \hat{\nu}$ where $\mathbf{n}: R \to \mathcal{P}(H)$. To see this we note that we can write

$$\frac{\boldsymbol{\mu}(S)}{\nu(S)+\hat{\nu}(S)} = \frac{\boldsymbol{\mu}(S)}{\nu(S)} \cdot \frac{\nu(S)}{\nu(S)+\hat{\nu}(S)} \quad \text{if} \quad \nu(S) > 0.$$

$$= \frac{\boldsymbol{\mu}(S)}{\hat{\nu}(S)} \cdot \frac{\hat{\nu}(S)}{\nu(S)+\hat{\nu}(S)} \quad \text{if} \quad \nu(S) = 0.$$

For a subset $X \subset \mathcal{P}(H)$ we denote by $\overline{co}X$ the set

$$\left\{\sum_{i=1}^n \alpha_i x_i \mid x_i \in X,\ \alpha_i \in \mathbb{C},\ 1 \le i \le n,\ \sum_{i=1}^n |\alpha_i| \le 1\right\}.$$

Then our remarks above show that

$$\left\{\frac{\boldsymbol{\mu}(S)}{\nu(S)+\hat{\nu}(S)} \;\middle|\; \nu(S)+\hat{\nu}(S) > 0\right\} \subseteq \overline{co}\left\{\frac{\boldsymbol{\mu}(S)}{\nu(S)} \;\middle|\; \nu(S) > 0\right\} \cup \overline{co}\left\{\frac{\boldsymbol{\mu}(S)}{\nu(S)} \;\middle|\; \hat{\nu}(S) > 0\right\}$$

each of which is weakly compact by our hypothesis (3) and Phillips' Theorem [9, Corollary 3.2, p. 120]. Thus $\boldsymbol{\mu}$ satisfies (3) with respect to $\nu + \hat{\nu}$ and the existence of a density follows from Theorem 1.

3. **A decomposition of the measure space.** From Theorem 1 we know that the density $\mathbf{m}$ takes values in $\mathcal{P}(H)$. Let $\lambda_n(t)$ denote the eigenvalues of $\mathbf{m}(t)$ repeated according to multiplicity, so that

$$\lambda_1(t) \geq \lambda_2(t) \geq \cdots \geq 0,$$

cf. [5, Corollary 5, p. 905]. It is our objective to show that the eigenvalues $\lambda_n(t)$ are ν-measurable functions, so that there is a partition of R into ν-measurable sets T_n, $n=0, 1, 2, \ldots$ where $\mathbf{m}(t)$ has rank n for $t \in T_n$. This leads us to define a ν-measurable operator valued mapping U which "diagonalises" $\mathbf{m}$ and is compatible with the decomposition T_n in the sense that for $t \in T_n$, $U(t)^*U(t)$ is the orthogonal projector onto a fixed n-dimensional subspace of H. This will be our main tool for handling the spaces $L^p(\boldsymbol{\mu})$.

THEOREM 3. (i) *The eigenvalues λ_n can be chosen ν-measurably.* (ii) *There is a ν-measurable partition of R into $\bigcup_{n=0}^{\infty} T_n$ so that for ν-almost all $t \in T_n$, $\mathbf{m}(t)$ has n-dimensional range.*

Proof. (i) We shall use the ideas of the proof of [5, Lemma 11, p. 1341], which effectively covers the finite dimensional case, in order to reduce our considerations to compact sets on which $\mathbf{m}$ is continuous. The reduction is briefly as follows [5, pp. 1342–3]. If w is a Banach space valued measurable function defined on a set $S \subset R$ with $\nu(S) < \infty$, then for each $\varepsilon > 0$ there is a Borel set $T \subset S$ on which w is continuous and $\nu(S-T) < \varepsilon$. Since ν is σ-finite and regular we may partition R into countably many compact sets S with $\nu(S) < \infty$. Then using again the regularity of ν we may assume that each such S is the union of countably many compacta on each of which $\mathbf{m}$ is continuous. As a consequence of these remarks, it is sufficient for us to show that $\lambda_n(t)$ is measurable for t ranging through a compact set on which $\mathbf{m}$ is continuous.

Now [5, Lemma 5, p. 1091] shows that λ_n is continuous at points of continuity of $\mathbf{m}$. Hence the ν-measurability of the λ_n is established.

(ii) Choose

$$T_0 = \{t \mid \lambda_1(t) = 0\}, \qquad T_n = \{t \mid \lambda_n(t) > 0, \lambda_{n+1}(t) = 0\}, \qquad n > 0.$$

Then these sets form a ν-measurable partition of R since our assumption (4) states that any $t \in R$ must belong to exactly one of these sets.

In case (4) is not satisfied we would define $T_\infty = \{t \mid \lambda_n(t) > 0, n = 1, 2, \ldots\}$ so that the T_n defined above together with T_∞ would form a measurable partition of R. Assumption (4) is equivalent to assuming $\nu(T_\infty) = 0$.

We turn now to the eigenvectors $e_n(t)$ satisfying $\mathbf{m}(t)e_i(t) = \lambda_i(t)e_i(t)$, $t \in T_n$, $i \leq n$. In showing that e_i can be chosen ν-measurably, we shall make use of the reduction to points t of continuity of $\mathbf{m}$ used for Theorem 3.

COROLLARY 1. *$e_i(t)$, $1 \leq i \leq n$, can be chosen ν-measurably on T_n.*

Proof. Let $t \in T_n$ and set $V_1(t)$ equal to the set of unit eigenvectors corresponding to $\lambda_1(t)$. $V_1(t)$ is a closed subset of H. We shall demonstrate that V_1 is

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upper semicontinuous at t which as before we assume to be a point of continuity of $\mathbf{m}$. According to the definition of upper semicontinuity we must show that for a closed subset P of H, the set

$$Q=\{t \mid V_1(t)\cap P\neq \emptyset\}$$

is closed in R.

Let $t_k \in Q$ with $t_k \to t$. We select $v_k \in V_1(t_k)\cap P$. Since $\|v_k\|=1$, we may (by taking a subsequence if necessary) assume the existence of $v \in H$ such that $v_k \to v$ weakly. Now $\mathbf{m}(t_k)$ is a sequence of compact operators with limit (in the uniform operator topology) $\mathbf{m}(t)$. From this it readily follows that $\mathbf{m}(t_k)v_k \to \mathbf{m}(t)v$ strongly. However $\mathbf{m}(t_k)v_k = \lambda_1(t_k)v_k$ which has weak limit $\lambda_1(t)v$. Thus we see that $\lambda_1(t_k)v_k$ has strong limit $\lambda_1(t)v$ and that $\mathbf{m}(t)v=\lambda_1(t)v$. Finally we note that $\lambda_1(t)\neq 0$ since $t\in T_n$ so that $\lambda_1(t_k)v_k \to \lambda_1(t)v$ implies $v_k \to v$ strongly. Note that t being a point of continuity of $\mathbf{m}$, is also a point of continuity of λ_1. Thus $\|v\|=1$ and so $v\in V_1(t)\cap P$. Hence $t \in Q$ and so Q is closed.

We now apply the selection theorem of Kuratowski and Ryll-Nardzewski to give a Baire selector e_1 for V_1 [7, p. 398]. Thus $e_1(t)\in V_1(t)$ and $e_1^{-1}(G)$ is a Borel set for each open $G\subset H$. Hence e_1 is measurable on T_n.

We now repeat the argument using $V_2(t)$ as the set of all unit eigenvectors for $\lambda_2(t)$ perpendicular to $e_1(t)$. This produces a measurable selector $e_2(t)$ with values in $V_2(t)$ and $e_1(t)\perp e_2(t)$. Continuing in this way for at most a finite number of steps if $t\in T_n$ we obtain the desired functions $e_1, e_2, \ldots, e_n$.

Observe that the method holds for $t\in T_\infty$ as well in which case we produce an orthonormal basis $e_1(t)$, $e_2(t), \ldots$ for H.

For the remainder of the paper we fix an orthonormal basis $\{e_i\}_{i=1}^\infty$ of H. We define

$$\begin{aligned} U(t)e_i &= e_i(t) && \text{if} \quad t\in T_n \quad \text{and} \quad i\le n\le\infty, \\ &=0 && \text{otherwise.}\end{aligned}$$

Let $[U_{ij}(t)]$ be the matrix of $U(t)^*$ and $[m_{ij}(t)]$ the matrix of $\mathbf{m}(t)$ relative to the basis $\{e_i\}$.

COROLLARY 2. (i) *U is ν-measurable.*

(ii) *For $t\in T_n$, $U(t)^*U(t)$ is the orthogonal projector onto the subspace of H spanned by $\{e_1, e_2, \ldots, e_n\}$.*

(iii) $U(t)^*\mathbf{m}(t)U(t)=\Lambda(t)$ *where $\Lambda(t)\in\mathcal{P}(H)$ has diagonal matrix* $\operatorname{diag}\{\lambda_1(t), \lambda_2(t), \ldots\}$ *relative to $\{e_i\}$.*

(iv) $\mathbf{m}(t)=U(t)\Lambda(t)U(t)^*$.

Proof. (i) By [4, Proposition 4, p. 392], U is separable valued, so the result follows from [3, Corollary to Proposition 18, p. 103].

(ii) Note that for $t\in T_n$, $\{e_1(t), \ldots, e_n(t)\}$ is an orthonormal basis for the range of $\mathbf{m}(t)$. The result is now trivial.

(iii) The calculation is easy.

(iv) Let $e_j=\sum_{k=1}^n \langle e_k(t), e_j\rangle e_k(t)+x_j$ where $t\in T_n$ and $\mathbf{m}(t)x_j=0$. Then

$$m_{ij}(t)=\langle e_i, \mathbf{m}(t)e_j\rangle=\sum_{k=1}^n \langle e_k(t), e_j\rangle\lambda_k(t)\langle e_i, e_k(t)\rangle$$

$$=\sum_{k=1}^n \overline{U_{ik}(t)}\lambda_k(t)U_{kj}(t)=[U(t)\Lambda(t)U(t)^*]_{ij}.$$

The sets T_n and the function U are our principal tools for the main result of the next section but before proceeding we derive a further result on finite dimensional approximation here.

COROLLARY 3. *There is a sequence of $\mathcal{P}(H)$ valued measures $\boldsymbol{\mu}_n \ll \nu$ with derivatives $\mathbf{m}_n$ (found as per Theorem 1) such that for ν-almost all t $\mathbf{m}_n(t)$ has rank at most n and such that $\boldsymbol{\mu}_n\to\boldsymbol{\mu}$ in variation.*

Proof. Let E_t denote the spectral resolution of $\mathbf{m}(t)$ and set

$$P_n(t)=E_t(\{\lambda_1(t),\ldots,\lambda_n(t)\}),$$

$$\mathbf{m}_n(t)=P_n(t)\mathbf{m}(t)P_n(t),$$

$$\boldsymbol{\mu}_n(S)=\int_S \mathbf{m}_n(s)\,d\nu(s),\qquad S\in\mathcal{B}.$$

It is easy to check that the $\mathbf{m}_n$ are ν-measurable functions and are dominated by $\lambda_1(t)=\|\mathbf{m}(t)\|$, a locally ν-integrable function. Thus $\boldsymbol{\mu}_n$ is a regular measure on $\mathcal{B}$. Now take a bounded Borel set $S\in\mathcal{B}$. Then

$$\mathrm{Var}[\boldsymbol{\mu}(S)-\boldsymbol{\mu}_n(S)]=\sup\left\{\sum_{i=1}^k \|\boldsymbol{\mu}(S_i)-\boldsymbol{\mu}_n(S_i)\| \mid S_i\cap S_j=\varnothing,\, i\neq j;\, S_i\subset S,\, S_i\in\mathcal{B}\right\}$$

$$=\sup\left\{\sum_{i=1}^k \left\|\int_{S_i}(\mathbf{m}(s)-\mathbf{m}_n(s))\,d\nu(s)\right\|\right\}$$

$$\leq\sup\left\{\sum_{i=1}^k\int_{S_i}\lambda_{n+1}(s)\,d\nu(s)\right\}=\int_S\lambda_{n+1}(s)\,d\nu(s).$$

We note that S is a bounded Borel set and so contained in a compact set. Further the $\lambda_{n+1}(s)$ are locally ν-integrable and monotonically decreasing to zero. Thus appealing to Lebesgue's dominated convergence theorem we see that $\boldsymbol{\mu}_n\to\boldsymbol{\mu}$ in variation.

4. **Structure of the spaces $L^p(\boldsymbol{\mu})$.** Following our earlier programme [1], the next step is to use the partition of Theorem 3 to define a sequence of "standard" L^p spaces. Our main result is then that $L^p(\boldsymbol{\mu})$ is either the direct sum of these spaces, or else is incomplete but dense in such a sum.

For $n=1,2,\ldots,L_n^p$ will be defined as the set of (equivalence classes of)

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ν-measurable functions $g=(g_1,\ldots,g_n):T_n\to\mathbb{C}^n$ with norm given by

$$\|g\|_{p,n}=\left[\int_{T_n}\left(\sum_{j=1}^{n}|g_j(t)|^2\right)^{p/2}d\nu(t)\right]^{1/p},\qquad 1\le p<\infty,$$

$$\|g\|_{\infty,n}=\nu-\operatorname*{ess\,sup}_{t\in T_n}\left[\sum_{j=1}^{n}|g_j(t)|^2\right]^{1/2}.$$

L^p_∞ will be defined as the set of (equivalence classes of) ν-measurable functions $g=(g_1,g_2,\ldots):T_\infty\to l^2$ with norm given by

$$\|g\|_{p,\infty}=\left[\int_{T_\infty}\left(\sum_{j=1}^{\infty}|g_j(t)|^2\right)^{p/2}d\nu(t)\right]^{1/p},\qquad 1\le p<\infty$$

$$\|g\|_{\infty,\infty}=\nu-\operatorname*{ess\,sup}_{t\in T_\infty}\left[\sum_{j=1}^{\infty}|g_j(t)|^2\right]^{1/2}.$$

These definitions cover the possibility of hypothesis (4) not being satisfied; i.e. the case in which $\nu(T_\infty)>0$.

If $X_1,X_2,\ldots$ is a sequence of Banach spaces, $\oplus^p X_n$ will denote the Banach space $X\subset \mathsf{X}_{n=1}^{\infty} X_n$ consisting of all x such that

$$\|x\|=\|(x_1,x_2,\ldots)\|=\left[\sum_{n=1}^{\infty}\|x_n\|^p\right]^{1/p},\qquad 1\le p<\infty,$$

$$\|x\|=\|(x_1,x_2,\ldots)\|=\sup_{n}\{\|x_n\|\},\qquad p=\infty.$$

In case we have but two spaces X_1, X_2 we shall write $X_1\oplus^p X_2$.

We are now in a position to state our main result.

THEOREM 4. (i) *If* $\nu(T_\infty)=0$ *then* $L^p(\boldsymbol{\mu})$ *is a Banach space isometrically isomorphic to* $\oplus^p L^p_n$.

(ii) *If* $\nu(T_\infty)>0$ *then* $L^p(\boldsymbol{\mu})$ *is incomplete but dense in* $(\oplus^p L^p_n)\oplus^p L^p_\infty$.

Proof. To prove (i) it suffices to show that for $n=1,2,\ldots$, L^p_n is isometrically isomorphic to $L^p_n(\boldsymbol{\mu})$ which we define as the space of (equivalence classes of) ν-measurable functions $h:T_n\to H$ via the norm

$$\|h\|_{p,n,\mu}=\left[\int_{T_n}\langle h(t),\mathbf{m}(t)h(t)\rangle^{p/2}\,d\nu(t)\right]^{1/p},\qquad 1\le p<\infty,$$

$$\|h\|_{\infty,n,\mu}=\nu-\operatorname*{ess\,sup}_{t\in T_n}\langle h(t),\mathbf{m}(t)h(t)\rangle^{1/2},\qquad p=\infty.$$

We shall treat the case $1\le p<\infty$—the case $p=\infty$ being similar. We shall us the matrix $[U_{ij}(t)]$ defined prior to Corollary 2. For $h\in L^p_n(\boldsymbol{\mu})$ we define

$$(5)\qquad (Ah)_i(t)=\sum_{j=1}^{\infty}U_{ij}(t)\lambda_i(t)^{-1/2}h_j(t),\qquad 1\le i\le n,$$

where $h_j(t)$ is the jth co-ordinate of $h(t) \in H$ relative to our fixed basis $\{e_1, e_2, \ldots\}$.

Clearly $Ah : T_n \to \mathbb{C}^n$ is ν-measurable and

$$\|Ah\|_{p,n}^p = \int_{T_n} \left(\sum_{i=1}^n |(Ah)_i(t)|^2\right)^{p/2} d\nu(t)$$
$$= \int_{T_n} \langle h(t), U(t)\Lambda(t)U(t)^* h(t)\rangle^{p/2} d\nu(t)$$
$$= \|h\|_{p,n,\mu}^p.$$

Note that we have used Corollary 2(iv) in this calculation.

So far $A : L_n^p(\boldsymbol{\mu}) \to L_n^p$ is isometric and obviously linear. To show that A is onto, let $g \in L_n^p$ and put

$$(Bg)_j(t) = \sum_{k=1}^n \overline{U_{kj}(t)}\lambda_k(t)^{-1/2} g_k(t), \qquad j \geq 1.$$

Clearly Bg is ν-measurable and H-valued since

$$\sum_{j=1}^\infty |(Bg)_j(t)|^2 = \sum_{k=1}^n \lambda_k(t)^{-1} |g_k(t)|^2 < \infty, \qquad t \in T_n.$$

Recalling that $\lambda_j(t) = 0$ for $j > n$ we obtain

$$\|Bg\|_{p,n,\mu}^p = \int_{T_n} \left[\sum_{i,j=1}^\infty \sum_{k,l=1}^n \overline{U_{kj}(t)}\lambda_k(t)^{-1/2} g_k(t) m_{ij}(t)\right.$$
$$\left.\cdot\, U_{li}(t)\lambda_l(t)^{-1/2}\overline{g_l(t)}\right]^{p/2} d\nu(t)$$
$$= \int_{T_n} \left[\sum_{k,l=1}^n \lambda_k(t)^{-1/2} g_k(t)\delta_{kl}\lambda_k(t)\lambda_l(t)^{-1/2}\overline{g_l(t)}\right]^{p/2} d\nu(t)$$

(by Corollary 2(iii))

$$= \int_{T_n} \left[\sum_{k=1}^n |g_k(t)|^2\right]^{p/2} d\nu(t) = \|g\|_{p,n}^p < \infty.$$

Thus $Bg \in L_n^p(\boldsymbol{\mu})$. Finally we show $ABg = g$.

$$(ABg)_i(t) = \sum_{j=1}^\infty \sum_{k=1}^n U_{ij}(t)\lambda_i(t)^{1/2}\overline{U_{kj}(t)}\lambda_k(t)^{-1/2} g_k(t)$$
$$= \sum_{k=1}^n \delta_{ik}\lambda_i(t)^{1/2}\lambda_k(t)^{-1/2} g_k(t)$$
$$= g_i(t).$$

This completes the proof of (i).

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To prove (ii) suppose that $S \subset T_\infty$ is compact with positive ν measure and that $\mathbf{m}$, and so each λ_n, is continuous on S. Define

$$f_i^n(t) = \chi_S(t) \sum_{j=1}^{n} U_{ij}(t)\rho_j(t)^{1/2} \qquad (i \geq 1),$$

where χ_S is the characteristic function of S and

$$\begin{aligned} \rho_j(t) &= 1 \quad \text{if} \quad j=1 \quad \text{or} \quad t \notin S \\ &= 1 - \frac{\lambda_j(t)}{\lambda_{j-1}(t)}, \qquad j>1. \quad (\text{Note that } \lambda_j(t) \neq 0 \forall t \in T_\infty). \end{aligned}$$

It is an easy exercise to show that $\sum_{j=1}^{\infty} \rho_j(t)$ is divergent for each t while $\sum_{j=1}^{\infty} \lambda_j(t)\rho_j(t)$ converges. Now $f^n = (f_1^n, f_2^n, \ldots) \in L_\infty^p(\boldsymbol{\mu})$ by the continuity of the λ_j. Further if $n>k$ we have for $1 \leq p < \infty$

$$\|f^n - f^k\|_{p,\infty,\mu}^p = \int_S \left[\sum_{j=k+1}^{n} \lambda_j(t)\rho_j(t)\right]^{p/2} d\nu(t) \to 0 \quad \text{as} \quad n, k \to \infty$$

by the Lebesgue dominated convergence theorem. A similar argument using Dini's theorem holds in the case $p=\infty$. Thus f^n is a Cauchy sequence in $L_\infty^p(\boldsymbol{\mu})$. On the other hand the formal expression

$$\|f^n - f^\infty\|_{p,\infty,\mu}^p = \int_S \left[\sum_{j=n+1}^{\infty} \lambda_j(t)\rho_j(t)\right]^{p/2} d\nu(t) \to 0 \quad \text{as} \quad n \to \infty.$$

Further since $U(t)$ is an isometry for $t \in T_\infty$

$$\|f^\infty(t)\|_H^2 = \sum_{j=1}^{\infty} \rho_j(t)$$

which diverges. Thus for each t, $f^\infty(t) \notin H$ and so $L^p(\boldsymbol{\mu})$ is incomplete. Again, similar arguments hold for the case $p=\infty$.

It remains to show that $L_\infty^p(\boldsymbol{\mu})$ can be isometrically embedded in L_∞^p. Let $g \in L_\infty^p$ and g^n be its truncation after n co-ordinates; i.e.

$$g^n(t) = (g_1(t), g_2(t), \ldots, g_n(t), 0, 0, \ldots).$$

Just as we defined the space $L_n^p(\boldsymbol{\mu})$ relative to the set $T_n \subset R$ we could define $\hat{L}_n^p(\boldsymbol{\mu})$ relative to the set $T_\infty \subset R$. Likewise, using T_∞, we can define $\hat{L}_n^p$ corresponding to L_n^p defined over T_n. Now $g^n \in \hat{L}_n^p$ and so arguing as before we claim the existence of an element $\hat{B}g^n \in \hat{L}_n^p(\boldsymbol{\mu})$. Then we can define

$$Bg^n = ((\hat{B}g^n)_1, \ldots, (\hat{B}g^n)_n, 0, 0, \ldots) \in L_\infty^p.$$

If we now map this element via $\hat{A}$, defined as per (5) but for all i, we reach g^n.

Finally

$$\|g^n - g\|_{p,\infty}^p = \int_{T_\infty} \left(\sum_{j=n+1}^{\infty} |g_j(t)|^2 \right)^{p/2} d\nu(t) \to 0 \quad \text{as} \quad n \to \infty$$

because $g \in L_\infty^p$.

5. **Further properties.** In this section X^* will denote the continuous dual of the normed linear space X. For a non-negative measure ξ on R, $L_X^p(\xi)$ is used in the sense of [3]; that is for the ξ-measurable X-valued functions on R, the pth power of whose norm is ξ-summable.

The decomposition obtained in Theorem 4 enables us to characterize the duals of $L^p(\mathbf{\mu})$ for $1 \le p < \infty$; in particular the dense embedding obtained there for $L^p(\mathbf{\mu})$ is in fact into its second dual, $1 < p < \infty$. We also investigate the separability and uniform convexity and smoothness of these spaces.

As further notation we denote $(\oplus^p L_n^p) \oplus^p L_\infty^p$ by M^p and write $p^{-1} + q^{-1} = 1$. Unless explicitly stated, (4) is not assumed.

COROLLARY 4. (i) *$L^1(\mathbf{\mu})^* \cong M^\infty$ while for $1 < p < \infty$,*

$$L^p(\mathbf{\mu})^* \cong M^q, \qquad L^p(\mathbf{\mu})^{**} \cong M^p.$$

(ii) *If $\nu(T_\infty) = 0$, then $L^1(\mathbf{\mu})^* \cong L^\infty(\mathbf{\mu})$; in particular, $L^\infty(\mathbf{\mu})$ is a Banach space independent of the measure ν used to define it.*

(iii) *If $\nu(T_\infty) = 0$ and $1 < p < \infty$, then $L^p(\mathbf{\mu})$ is a reflexive Banach space and $L^p(\mathbf{\mu})^* \cong L^q(\mathbf{\mu})$.*

(iv) *If $\nu(T_\infty) = 0$ then $L^2(\mathbf{\mu})$ is a Hilbert space with inner product*

$$[f, g] = \int_R \langle f(s), \mathbf{m}(s) g(s) \rangle \, d\nu(s).$$

Proof. (i) In view of Theorem 4(ii) it is sufficient to show $(M^1)^* \cong M^\infty$ and $(M^p)^* \cong M^q$, $1 < p < \infty$. The argument of [1, Corollary 1] carries over as follows. When X^* is a separable Banach space and ξ a Borel measure on R we have [3, Corollary 1, p. 282]

$$L_X^p(\xi)^* \cong L_{X^*}^q(\xi)$$

with $q = \infty$ for the case $p = 1$. Taking X as $\mathbb{C}^n$ or H and ξ as ν or a counting measure we obtain successively

$$(M^p)^* \cong (\oplus^p L_n^p)^* \oplus^q (L_\infty^p)^* \cong \oplus^q (L_n^p)^* \oplus^q L_\infty^q \cong M^q.$$

(ii) and (iii) are ready consequences of (i) and Theorem 4(i) while (iv) is a trivial calculation.

COROLLARY 5. *$L^p(\mathbf{\mu})$ is separable for $1 \le p < \infty$.*

Proof. Let ξ be a finite non-negative measure defined on a σ-field Σ which will be considered as a metric space under the distance

$$d(S, T) = \xi(S \Delta T), \qquad S, T \in \Sigma.$$

It is known that if Σ is countably generated then Σ is separable [12, Theorem 3, p. 69]; and, in particular, if $\xi = \nu$ then $\Sigma = \mathcal{B} \cap [-k, k]$ is separable—recall that ν is regular and cf. [12, Theorem 4, p. 69].

For a separable space X, $L_X^p(\xi)$ is separable if Σ is [5, Exercise 6, p. 169]. This can be proved using the density of the step functions $\sum_{i=1}^{j} x_i \chi_{S_i}$, $x_i \in X$, $S_i \in \Sigma$, [5, Corollary 8, p. 125] and approximating x_i and S_i via the separability of X and Σ.

Thus far we can conclude that each L_n^p and L_∞^p have separable subspaces of functions with support in $[-k, k]$, $k = 1, 2, \ldots$. Forming the union for $k = 1, 2, \ldots$ we obtain separability of L_n^p and L_∞^p. This leads to the separability of M^p since it is no more than an l^p-direct sum of separable spaces. Finally note that $L^p(\boldsymbol{\mu})$ is dense within M^p and so is separable as well.

We turn now to uniform convexity and smoothness of the spaces $L^p(\boldsymbol{\mu})$. Henceforth we take $1 < p < \infty$. If $\gamma: R_+ \to R_+$ satisfies

$$\gamma(\varepsilon) \le \gamma_0(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x+y\| \mid \tfrac{1}{2}\|x-y\| \ge \varepsilon;\ \|x\| = \|y\| = 1;\ x, y \in X\}$$

then γ is called a *modulus of convexity* for X; γ_0 is the optimal modulus. If $\gamma(\varepsilon) > 0$ for $\varepsilon > 0$ then X is said to be *uniformly convex.* Day [2, Theorem 2] has shown that L_X^p is uniformly convex if X is, so the L_n^p are uniformly convex for in this case $X = \mathbb{C}^n$ is a Hilbert space. Day [2, Theorem 3] has shown also that $\oplus^p X_n$ is uniformly convex if the individual spaces X_n are uniformly convex with a common modulus of convexity. With $X_n = L_n^p$ this result will lead to the uniform convexity of $L^p(\boldsymbol{\mu})$ (at least if $\nu(T_\infty) = 0$). We aim, however, to produce optimal moduli and also to consider uniform smoothness and accordingly proceed as follows. A *modulus of smoothness* for X is defined by the inequality

$$\sigma(\varepsilon) \ge \sigma_0(\varepsilon) = \sup\{(1 - \tfrac{1}{2}\|x+y\|)/(\tfrac{1}{2}\|x-y\|) \mid \tfrac{1}{2}\|x-y\| \le \varepsilon;\ \|x\| = \|y\| = 1;\ x, y \in X\}$$

for $\varepsilon \ge 0$. σ_0 is the optimal modulus and X is said to be *uniformly smooth* if it has a modulus σ continuous at 0 with $\sigma(0) = 0$.

THEOREM 5. *Define φ and ψ for $0 < \varepsilon < 2^{-1/q}$ by*

$$1 - \varepsilon^p = [1 - \varphi(\varepsilon)]^p,$$

$$|1 - \psi(\varepsilon) - \varepsilon|^p + [1 - \psi(\varepsilon) + \varepsilon]^p = 2.$$

Then $L^p(\boldsymbol{\mu})$ is uniformly convex and smooth with optimal moduli of convexity and smoothness γ_0, σ_0 respectively where

$$\gamma_0(\varepsilon) = \psi(\varepsilon),\ \varepsilon\sigma_0(\varepsilon) = \varphi(\varepsilon) \quad \textit{if} \quad 1 < p < 2,$$

$$\gamma_0(\varepsilon) = \varphi(\varepsilon),\ \varepsilon\sigma_0(\varepsilon) = \psi(\varepsilon) \quad \textit{if} \quad 2 \le p < \infty.$$

Proof. Only minor changes are needed in the arguments of Hanner [6] who gives γ_0 for l^p and $L^p[0, 1]$. We assume $1<p<2$—the other case is analogous.

For the chosen p range and $x, y \in \mathbb{C}$, Hanner [6, Equations (4), (8)] establishes

$$(\|x\|+\|y\|)^p+|\,\|x\|-\|y\|\,|^p \le \|x+y\|^p+\|x-y\|^p \le 2(\|x\|^p+\|y\|^p). \tag{6}$$

Using the reasoning of [1, Corollary 2], (6) may be extended from $\mathbb{C}$ to any of the spaces $\mathbb{C}^n$, H in which it is isometrically embedded. Integrating over T_n and summing we obtain (6) for $x, y \in M^p$ and finally the isometry A used for Theorem 4 establishes (6) for $L^p(\boldsymbol{\mu})$.

The argument of [6, Theorem 2] can now be taken over directly to give $\gamma_0(\varepsilon) \ge \psi(\varepsilon)$. Equality is established by suitably amending Hanner's example. Choose disjoint sets $S_1, S_2 \subset T_n$ so that $\nu(S_1)=\nu(S_2)=\alpha>0$. Let

$$u(t)=(2\alpha)^{-1/p}[1-\psi(\varepsilon)]\chi_{S_1\cup S_2}(t)e_1(t)\lambda_1(t)^{-1/2},$$
$$v(t)=(2\alpha)^{-1/p}\varepsilon[\chi_{S_1}(t)-\chi_{S_2}(t)]e_1(t)\lambda_1(t)^{-1/2}.$$

Then $x=u+v$ and $y=u-v$ can be easily checked to have norm 1 while $\|u\|=1-\psi(\varepsilon)$ and $\|v\|=\varepsilon$. This completes the discussion of uniform convexity.

Turning to uniform smoothness let

$$\delta=\tfrac{1}{2}\|x+y\|, \qquad \eta=\tfrac{1}{2}\|x-y\|.$$

From the right hand inequality of (6) we have

$$\|\tfrac{1}{2}(x+y)+\tfrac{1}{2}(x-y)\|^p+\|\tfrac{1}{2}(x+y)-\tfrac{1}{2}(x-y)\|^p \le 2(\delta^p+\eta^p).$$

Thus if $\|x\|=\|y\|=1$ we obtain

$$1 \le \delta^p+\eta^p. \tag{7}$$

Consider the problem of maximising $(1-\delta)/\eta$ subject to (7) and $0<\eta\le\varepsilon$. If $\eta=\varepsilon$, then the maximum value is obviously $\varphi(\varepsilon)/\varepsilon$. If $0<\eta<\varepsilon$ then the maximum value is not less than the unconstrained maximum of

$$f(\eta)=[1-(1-\eta^p)^{1/p}]/\eta.$$

Elementary calculus methods show that f is monotonically increasing in $(0, 1]$ so that the maximum of $f(\eta)$ for $0<\eta\le\varepsilon<1$ will be $f(\varepsilon)=[1-(1-\varepsilon^p)^{1/p}]/\varepsilon=\varphi(\varepsilon)/\varepsilon$. Since (7) is but a consequence of the definitions of δ and η, we have so far that $\sigma_0(\varepsilon)\le\varphi(\varepsilon)/\varepsilon$.

To establish equality we again modify one of Hanner's examples [6, p. 243] as follows. Select disjoint sets S_1, $S_2 \subset T_n$ with $\nu(S_1)=\nu(S_2)=\alpha>0$. Let

$$u(t)=\alpha^{-1/p}[1-\varphi(\varepsilon)]\chi_{S_1}(t)e_1(t)\lambda_1(t)^{-1/2}$$
$$v(t)=\alpha^{-1/p}\varepsilon\chi_{S_2}(t)e_1(t)\lambda_1(t)^{-1/2}$$

and $x=u+v$, $y=u-v$. Then $\|x\|=\|y\|=1$ but $\|u\|=1-\varphi(\varepsilon)$, $\|v\|=\varepsilon$.

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