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Rotation intervals of endomorphisms of the circle

R. BAMON, I. P. MALTA, M. J. PACIFICO AND F. TAKENS

Departamento de Matemática, Universidad de Chile, Casilla 653—*Santiago, Chile*; *Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro, Rua Marquês de São Vicente,* 225, *Casa B, CEP* 22.453—*Gávea—Rio de Janeiro RJ—Brazil*; *Instituto de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal* 68530, *CEP* 21.910—*Rio de Janeiro RJ—Brazil*; *Mathematisch Instituut, Postbus* 800, 9700 *AV Groningen, The Netherlands*

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Abstract. The rotation number of a diffeomorphism $f: S^1 \to S^1$, with lift $F: \mathbb{R} \to \mathbb{R}$ is defined as $\lim_{n\to\infty} (F^n(x) - x)/n$. We investigate the case where f is an endomorphism. Then this limit may not exist and may depend on x. We investigate the set of limit points of $(F^n(x) - x)/n$, $n \to \infty$, as a function of x.

1. *Introduction and statement of results*

Let $\text{End}_1^0(S^1)$ be the set of continuous endomorphisms of degree one of the circle. Given $f \in \text{End}_1^0(S^1)$, let F be a lift of f to $\mathbb{R}$, that is, $f \circ \pi = \pi \circ F$, where $\pi: \mathbb{R} \to S^1$ is the natural projection. Since f is of degree one, we have

$$F(x+1) = F(x) + 1 \qquad \text{for all } x \in \mathbb{R}.$$

If f is a homeomorphism it is well known that the limit of $(F^n(x) - x)/n$ for $n \to \infty$, exists, does not depend on x and defines the rotation number of f. In the general case of endomorphisms this limit may not exist. In [2] Newhouse, Palis and Takens introduced the concept of rotation set of endomorphisms. It is defined by

$$\rho(f) = \text{closure } \{\rho^+(f, z),\ z \in S^1\},$$

where

$$\rho^+(f, z) = \limsup_n \frac{F^n(x) - x}{n},$$

$\pi(x) = z$. They also proved that $\rho(f)$ is an interval. Clearly $\rho(f)$ is defined up to translations by integers. In [1] R. Ito proved that each $\alpha \in \rho(f)$ is realized as the rotation number of some point in S^1, in the sense that for some $z \in S^1$, $\lim_{n\to\infty} (F^n(x) - x)/n = \alpha$, $\pi(x) = z$.

Here we define the rotation set $\rho(f, z)$ of $z \in S^1$ as the set of limit points of the sequence $(F^n(x) - x)/n$, where $\pi(x) = z$. Observe that $\rho(f, z) \subseteq \rho(f)$.

The purpose of this paper is to give a complete description of all rotation sets $\rho(f, z)$ in terms of the rotation interval $\rho(f)$.

THEOREM. *If* $f \in \mathrm{End}_1^0(S^1)$ *then*:

(i) $\rho(f, z)$ *is a closed subinterval of* $\rho(f)$ *for all* $z \in S^1$;

(ii) *given* $[\alpha, \beta] \subseteq \rho(f)$, $\alpha \leq \beta$, *there exists* $z \in S^1$ *such that* $\rho(f, z) = [\alpha, \beta]$.

In § 2 we introduce the notion of positive local unstable manifold for periodic points of endomorphisms as well as the concept of fundamental domains which play a key rôle in the proof of the theorem. In § 3 we prove some technical lemmas and finally, the proof of the theorem is given in § 4.

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2. *Preliminaries*

Let $f \in \mathrm{End}_1^0(S^1)$. If z_0 is a fixed point of f, let F be a lift of f with $F(x) = x$ if $\pi(x) = z_0$. Consider the set

$$A^+ = \{z \in S^1;\ F(x) > x,\ \pi(x) = z\}.$$

If there exists a component $U \subset A^+$, $\bar{U} = [z_0, z]$, we define the *local positive unstable manifold* of z_0, $W^{u+}_{\mathrm{loc}}(z_0)$ as the set $\{z_0\} \cup U$. Otherwise we define $W^{u+}_{\mathrm{loc}}(z_0) = \{z_0\}$. Note that even for a fixed point z_0 where f is expanding but orientation reversing, $W^{u+}_{\mathrm{loc}}(z_0) = \{z_0\}$. The (*positive*) *unstable manifold* of z_0, $W^{u+}(z_0)$, is defined as

$$W^{u+}(z_0) = \bigcup_{j \geq 0} f^i(W^{u+}_{\mathrm{loc}}(z_0)).$$

If $\pi(x_0) = z_0$ we define $W^{u+}_{\mathrm{loc}}(x_0)$ as the lift of $W^{u+}_{\mathrm{loc}}(z_0)$ which contains x_0.

If z_0 is a periodic point of f with period q, $W^{u+}_{\mathrm{loc}}(z_0)$ and $W^{u+}(z_0)$ are defined as above considering z_0 as a fixed point of f^q.

In order to define a fundamental domain in $W^{u+}_{\mathrm{loc}}(z_0)$, we proceed as follows. Take a sequence $\{d_i\}_{i \in \mathbb{N}} \subset W^{u+}_{\mathrm{loc}}(z_0)$ such that $f(d_{i+1}) = d_i$; so d_i converges to z_0. We define D_i = interval $[d_i, d_{i-1}]$. We call such an interval D_i, so that f maps the interval $[z_0, d_i]$ inside $W^{u+}_{\mathrm{loc}}(z_0)$, a *fundamental domain* in $W^{u+}_{\mathrm{loc}}(z_0)$. Observe that since f is an endomorphism, there may exist $s \in D_{i+1}$ such that $f(s) \notin D_i$. So the notion of fundamental domain here is weaker than the usual one for diffeomorphisms. In particular, even if $W^{u+}(z_0) = S^1$ and if $D \subset W^{u+}(z)$ is a fundamental domain it may not follow that for some j, $f^j(D) = S^1$. However we shall prove:

PROPOSITION 1. *Let* $f \in \mathrm{End}_1^0(S^1)$ *and* $p/q \in \mathrm{int}\,\rho(f)$, $(p, q) = 1$. *Then there exists a periodic point* z *with rotation number* p/q *and period* q, *and a fundamental domain* $D \subset W^{u+}(z)$ *such that* $f^j(D) = S^1$ *for some* $j \in \mathbb{N}$.

Proof. It is enough to prove the case when $p = 0$ and $q = 1$. Let C be the set of fixed points z with rotation number zero, that is, $F(x) = x$ if $\pi(x) = z$ for $z \in C$. We claim that there exists $z = \pi(x) \in C$ such that

$$F^{i_0}(W^{u+}_{\mathrm{loc}}(x)) \supset [x, x+2] \qquad \text{for some } i_0 \geq 1.$$

Suppose that this is not true. We will prove that this assumption implies $\rho^+(f, w) \leq 0$ for all $w \in S^1$, which is a contradiction. In fact if $y \in W^{u+}_{\mathrm{loc}}(x)$, $\pi(x) \in C$ then $y \geq x$

and so $F^n(y) < y+2$ for all $n \geq 1$. Thus $\rho^+(f, \pi(y)) \leq 0$ for $\pi(y) \in W^{u+}_{\text{loc}}(z)$, $z \in C$. Since $\pi(y) \in W^{u+}_{\text{loc}}(z)$, $z \in C$, if and only if $F(y) \geq y$, it remains to consider y such that $F(y) < y$. In this case we have $F^n(y) < y$ for all $n \geq 1$ or there exists n_0 such that $f^{n_0}(\pi(y)) \in W^{u+}_{\text{loc}}(z)$ for some $z \in C$. So we also have $\rho^+(f, \pi(y)) \leq 0$.

Let $z_0 = \pi(x_0) \in C$ and $i_0 \geq 1$ be as in the claim. Take $y_0 \in W^{u+}_{\text{loc}}(x_0)$ such that $F^{i_0}(y_0) = x_0 + 2$. Let

$$y_1 = \sup \{y \in [x_0, y_0] \text{ such that } F^{i_0}(y) = x_0 + 1\}.$$

Then $F^{i_0}(y) > x_0 + 1$ for $y_1 < y \leq y_0$. Thus if W is any neighbourhood of $w = \pi(y_1)$ then $f^{i_0}(W)$ covers a neighbourhood of z_0 in $W^{u+}_{\text{loc}}(z_0)$. It is possible to choose $\{d_i\}_{i\in\mathbb{N}} \subset W^{u+}_{\text{loc}}(z_0)$ with $f(d_{i+1}) = d_i$ and $w \in [d_2, d_1]$. Clearly, for i big enough, $D = [d_{i+1}, d_i]$ is a fundamental domain with the required property. □

3. *Itineraries and technical lemmas*

Let $f \in \text{End}^0_1(S^1)$ with $\rho(f) = [a, b]$, $a < b$, and $\{p_i/q_i\}_{i\in\mathbb{N}} \subseteq \text{int}(\rho(f))$ be a sequence of rational numbers. For each $i \in \mathbb{N}$, let $z_i \in S^1$ be a periodic point with rotation number p_i/q_i and period q_i such that $W^{u+}(z_i) = S^1$. Let D_i be a fundamental domain for $W^{u+}(z_i)$ and $j_i \in \mathbb{N}$ such that $f^{j_i}(D_i) = S^1$. Given any sequence $(n)_\nu = (n_1, n_2, \ldots)$ of positive integers, where ν is the length of the sequence if it is finite or $\nu = \infty$ if it is infinite, such that $n_i = r_i q_i$ with $r_i \in \mathbb{N}$, we define for $0 \leq i-1 < \nu$

$$N_i = n_1 + j_1 + \cdots + n_{i-1} + j_{i-1} + n_i \quad \text{and} \quad J_i = N_i + j_i.$$

Put $J_0 = 0$. We say that $z = \pi(y) \in W^{u+}_{\text{loc}}(z_1)$ has *itinerary* $(n)_\nu$ with respect to (z_i, j_i) if for each $0 \leq i < \nu$ there exists $x_{i+1} \in \pi^{-1}(z_{i+1})$ such that

$$F^{J_i + kq_{i+1}}(y) - kp_{i+1} \in W^{u+}_{\text{loc}}(x_{i+1}) \qquad \text{for } 0 \leq kq_{i+1} \leq n_{i+1},$$

and

$$f^{N_{i+1}}(z) \in D_{i+1}.$$

Clearly this definition does not depend on the lift y of z. Observe that if z satisfies the itinerary $(n)_\nu$ then the orbit of z is successively n_i iterates near the orbit of z_i.

LEMMA 2. *Given any sequence $(n)_\nu$ as above there exists a point $z \in W^{u+}_{\text{loc}}(z_1)$ with itinerary $(n)_\nu$.*

Proof. For each $0 \leq i-1 < \nu$, let A_i be defined by

$$A_i = \{z \in W^{u+}_{\text{loc}}(z_i) : f^{n_i}(z) \in D_i \text{ and there exist } y \in \pi^{-1}(z), x_i \in \pi^{-1}(z_i)$$
$$\text{such that } F^{kq_i}(y) - kp_i \in W^{u+}_{\text{loc}}(x_i) \text{ for } 0 \leq kq_i \leq n_i\}.$$

Since $f^{n_i}(A_i) = D_i$, A_i is a closed non-empty set. Let $L_i = f^{-j_i}(A_{i+1}) \cap D_i$. Since $f^{j_i}(D_i) = S^1$, we have that L_i is a compact non-empty set. Now define $K_1 = A_1$ and

$$K_i = \{z \in A_1 ; f^{N_m}(z) \in L_m \text{ for } 1 \leq m < i\} \qquad \text{for } i > 1.$$

It follows immediately that each K_i is a compact non-empty set and satisfies $K_{i+1} \subset K_i$. It is also clear that if $z \in K_i$ then z has itinerary $(n)_i = (n_1, \ldots, n_i)$. Thus each $z \in \bigcap_{i=1}^\infty K_i$ has the prescribed itinerary. □

LEMMA 3. *Given $q \in N$ and $\varepsilon > 0$ there exists $n_0 \geq 1$ such that for all $y \in \mathbb{R}$ and all $n \geq n_0$*

$$\left|\frac{F^{n+k}(y)-y}{n+k}-\frac{F^n(y)-y}{n}\right|<\varepsilon \qquad \text{for } 0 \leq k \leq q.$$

Proof. Let $R, S > 0$ be such that $|F^k(y)-y| < R$ for all $y \in \mathbb{R}$ and $0 \leq k \leq q$ and $|(F^n(y)-y)/n| < S$ for all $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Then

$$\left|\frac{F^{n+k}(y)-y}{n+k}-\frac{F^n(y)-y}{n}\right| = \frac{1}{n(n+k)}|n(F^k(F^n(y))-F^n(y))-k(F^n(y)-y)|$$

$$\leq \frac{R}{n+k}+\frac{kS}{n(n+k)}.$$

So it is enough to take $n_0 \in \mathbb{N}$ such that $R/(n+k)+kS/n(n+k) < \varepsilon$ for $n \geq n_0$ and $0 \leq k \leq q$. □

LEMMA 4. *Let $z \in S^1$ be a periodic point of f with rotation number p/q and $m \in \mathbb{N}$. Then given $\varepsilon > 0$ there exists $n_0 \geq 1$ such that for all $y \in \mathbb{R}$ and $x \in \pi^{-1}(z)$, if $n \geq n_0$ and*

$$|(F^{n+m}(y)-F^m(y))-(F^n(x)-x)| \leq 2$$

then

$$\left|\frac{F^{n+m}(y)-y}{n+m}-p/q\right|<\varepsilon.$$

Proof. Let $R > 0$ be such that $|F^m(y)-y| < R$ for all $y \in \mathbb{R}$. Take $n_0 \geq 1$ such that $(2+R)/n_0 < \varepsilon/2$ and $|(F^n(x)-x)/(n+m)-p/q| < \varepsilon/2$ for $n \geq n_0$. Note that n_0 does not depend on the lift x of z. Then, if $n \geq n_0$ we have

$$\left|\frac{F^{n+m}(y)-y}{n+m}-p/q\right| \leq \left|\frac{(F^{n+m}(y)-F^m(y))-(F^n(x)-x)}{n+m}\right|$$

$$+\left|\frac{F^m(y)-y}{n+m}\right|+\left|\frac{F^n(x)-x}{n+m}-p/q\right|$$

$$\leq \frac{2+R}{n+m}+\varepsilon/2 < \varepsilon.$$ □

4. *Proof of the theorem*

To prove (i), since

$$\frac{F^n(x)-x}{n}=\frac{\sum_{i=1}^n F^i(x)-F^{i-1}(x)}{n}$$

and $\{F^i(x)-F^{i-1}(x)=F(F^{i-1}(x))-F^{i-1}(x)\}_{i\in\mathbb{N}}$ is uniformly bounded, it is enough to prove the following lemma:

LEMMA. *Let $\{a_i\}_{i\in\mathbb{N}}$ be a bounded sequence. Then the set of limit points of $(1/n)\sum_{i=1}^n a_i$, as $n \to \infty$, is a closed interval.*

Proof. Let a_+, a_- be the lim sup and the lim inf of $(1/n)\sum_{i=1}^n a_i$, as $n \to \infty$. The set of limit points is clearly contained in $[a_-, a_+]$ and contains a_- and a_+. We choose

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a subsequence n_j such that

$$\lim_{j\to\infty}\frac{1}{n_{2j}}\sum_{i=1}^{n_{2j}} a_i = a_-, \qquad \lim_{j\to\infty}\frac{1}{n_{2j+1}}\sum_{i=1}^{n_{2j+1}} a_i = a_+.$$

For an arbitrary point $a\in[a_-, a_+]$ we choose a subsequence n_j^a so that $n_j\le n_j^a\le n_{j+1}$ and so that, with this restriction, $(1/n_j^a)\sum_{i=1}^{n_j^a} a_i$ is as near as possible to a. The fact that $(1/n_j^a)\sum_{i=1}^{n_j^a} a_i$ converges to a follows from the observation that

$$\frac{1}{n}\sum_{i=1}^{n} a_i - \frac{1}{n+1}\sum_{i=1}^{n+1} a_i = \frac{1}{n(n+1)}\sum_{i=1}^{n} a_i - \frac{1}{n+1}a_{n+1}$$

goes to zero for $n\to\infty$. □

Proof of (ii). We may assume that $\rho(f)$ is an interval of positive length. Otherwise (ii) is trivial.

Let $[\alpha,\beta]\subseteq\rho(f)$ be a subinterval. Choose $\{\rho_i=p_i/q_i\}_{i\in\mathbb{N}}$ so that $\alpha<\rho_i<\beta$, $\lim_{i\to\infty}\rho_{2i-1}=\alpha$ and $\lim_{i\to\infty}\rho_{2i}=\beta$.

Let $\{\varepsilon_i\}_{i\in\mathbb{N}}$ be a sequence of positive real numbers such that $\alpha<\rho_i-\varepsilon_i$ and $\rho_i+\varepsilon_i<\beta$, $i\ge1$.

For each $i\ge1$ let $z_i\in S^1$, $D_i\subset S^1$ and $j_i\in\mathbb{N}$ be given by proposition 1; that is:

(1) z_i is a periodic point of f with rotation number ρ_i and period q_i.

(2) D_i is a fundamental domain for $W^{u+}(z_i)$ such that $f^{j_i}(D_i)=S^1$.

(3) for each $i\ge1$ there exists $k_i\ge1$ such that

$$\left|\frac{F^k(y)-y}{k}-\rho_i\right|<\varepsilon_i$$

if $k\ge k_i$, y and $F^{k'q_i}(y)-k'p_i$ belong to $W^{u+}_{\mathrm{loc}}(x_i)$ for some $x_i\in\pi^{-1}(z_i)$, where $k=k'q_i+r$ with $0\le r<q_i$.

This result is obtained using lemma 4 with $m=0$, $n=k'q_i$, and lemma 3 with $q=q_i$.

Now we will construct, by induction, an itinerary $(n)_\infty$ such that a point $z\in S^1$ with this itinerary with respect to (z_i, j_i) will satisfy $\rho(f,z)=[\alpha,\beta]$. We claim that there exists a sequence $(n)_\infty=(n_1, n_2,\dots)$ such that if $z=\pi(y)$ has itinerary $(n)_\infty$ with respect to (z_i, j_i), then for $i\ge1$

(a) $$\left|\frac{F^{N_i+k}(y)-y}{N_i+k}-\rho_i\right|<\varepsilon_i \qquad \text{for } 0\le k\le k_i+j_{i-1}, \text{ where } j_0=0.$$

From lemma 4 with $m=0$ and lemma 3 with $q=k_1$ we can obtain an integer r_1 such that $n_1=r_1q_1$ and condition (a) is satisfied with $i=1$ for all $y\in\mathbb{R}$ such that $z=\pi(y)$ has itinerary $(n)_1=(n_1)$. Suppose we have $(n)_i=(n_1, n_2,\dots, n_i)$, $i\ge1$, such that all $z=\pi(y)$ with itinerary $(n)_i$ satisfy condition (a) for $1\le l\le i$. As before, we can obtain, from lemma 4 with $m=J_i$ and lemma 3 with $q=k_{i+1}+j_i$, an integer $r_{i+1}\ge1$ such that $n_{i+1}=r_{i+1}q_{i+1}$ and if

$$|(F^{n_{i+1}+J_i}(y)-F^{J_i}(y))-(F^{n_{i+1}}(x_{i+1})-x_{i+1})|\le2$$

for some $x_{i+1}\in\pi^{-1}(z_{i+1})$ then

$$\left|\frac{F^{n_{i+1}+J_i+k}(y)-y}{n_{i+1}+J_i+k}-\rho_{i+1}\right|<\varepsilon_{i+1} \qquad \text{for } 0\le k\le k_{i+1}+j_i.$$

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Since $N_{i+1}=n_{i+1}+J_i$ it is clear that if $z=\pi(y)$ has itinerary $(n)_{i+1}=(n_1,\ldots,n_{i+1})$ then y satisfies (a) for $1\le l\le i+1$. This proves the claim.

Thus if $z=\pi(y)$ has itinerary $(n)_\infty$ with respect to (z_i, j_i) then y satisfies condition (a) for all $i\ge 1$. We also have

$$\lim_{k\to\infty}\frac{F^{N_{2k-1}}(y)-y}{N_{2k-1}}=\alpha \quad\text{and}\quad \lim_{k\to\infty}\frac{F^{n_{2k}}(y)-y}{N_{2k}}=\beta.$$

We now prove that $\rho(f,z)=[\alpha,\beta]$. For this it is enough to prove that $\alpha\le(F^n(y)-y)/n\le\beta$ for all $n\ge n_1$. Let $n>n_1$. Since $N_i\to\infty$ as $i\to\infty$ and $N_{i+1}>N_i$ there exist $i\ge 1$ and $0\le k<n_i+j_{i-1}$ such that $n=N_{i+1}+k$. The result is clear if $k\le k_i+j_{i-1}$. If $k_i+j_{i-1}<k<n_i+j_{i-1}$ we have

$$\frac{F^{N_{i-1}+k}(y)-y}{N_{i-1}+k}=\left(\frac{F^{N_{i-1}+j_{i-1}}(y)-y}{N_{i-1}+j_{i-1}}\right)\cdot\frac{N_{i-1}+j_{i-1}}{N_{i-1}+k}$$
$$+\left(\frac{F^{k-j_{i-1}}(F^{N_{i-1}+j_{i-1}}(y))-F^{N_{i-1}+j_{i-1}}(y)}{k-j_{i-1}}\right)\cdot\frac{k-j_{i-1}}{N_{i-1}+k},$$

which is a convex combination of two numbers. Clearly the first one is in an ε_{i-1}-neighbourhood of ρ_{i-1} and so it is in $[\alpha,\beta]$. Since $n_i>k-j_{i-1}>k_i$ we can apply condition (3) to conclude that the second one is in an ε_i-neighbourhood of ρ_i and so it also belongs to $[\alpha,\beta]$. This completes the proof. □

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