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Algebraicity of some Weil Hodge Classes

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Abstract. We show that the Prym map for 4-th cyclic étale covers of curves of genus 4 is a dominant morphism to a Shimura variety for a family of Abelian 6-folds of Weil type. According to the result of Schoen, this implies algebraicity of Weil classes for this family.

1 Introduction

The Hodge conjecture is still open even for Abelian varieties. Because the conjecture is true for all projective 3-folds, the first targets are 4-folds. In this dimension, we have non-trivial examples of Hodge classes for special Abelian varieties that are called of Weil type.

In [7] and [8], Schoen constructed algebraic cycles on generalized Prym varieties for cyclic covers that give Weil classes, and he proved algebraicity of Weil classes for a family of Abelian 6-folds of Weil type for $k = \mathbb{Q}(\sqrt{-3})$ with some δ by showing the denseness of Prym varieties in the family.

This method works also for Weil 4-folds and 6-folds for $\mathbb{Q}(\sqrt{-1})$ with $\delta = 1$. For the 4-dimensional case, van Geemen gave another proof in [4].

In this note, we consider the 6-dimensional case. The problem is to show the dominantness of the associated Prym map. We construct genus 13 curves C_{13} in $(\mathbb{P}^1)^4$ that are invariant under a cyclic permutation σ of factors of $(\mathbb{P}^1)^4$. So σ acts on C_{13}, and we show that this action is fixed point free. For the covering $C_{13} \to C_{13}/\langle\sigma\rangle$, we compute the codifferential map of the Prym map explicitly.

2 The Prym Construction of Abelian Varieties of Weil Type

In this section, we explain our problem and state the main theorem.

Let us recall the definition of Abelian varieties of Weil type (see [3, 9]). Let A be an Abelian $2n$-fold ($n \geq 2$) with a polarization E and let $\phi\colon k \to \mathrm{End}^0(A)$ be an inclusion of an imaginary quadratic field $k = \mathbb{Q}(\sqrt{-d})$. We assume that

$$E\big(\phi(\sqrt{-d})_* x, \phi(\sqrt{-d})_* y\big) = dE(x, y).$$

Then we say that A is of Weil type if the multiplicities of eigenvalues $\sqrt{-d}$ and $-\sqrt{-d}$ of the action of $\phi(\sqrt{-d})_*$ on the tangent space $T_0 A$ are equal to n. Then $H_1(A, \mathbb{Q})$ has a structure of k-module and

$$H(x, y) = E\big(x, \phi(\sqrt{-d})_* y\big) + \sqrt{-d}E(x, y)$$

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gives a Hermitian form on $H_1(A,\mathbb{Q})$ of the signature (n,n). The class $\delta = \det H \bmod (k^*)^2$ gives an isogenus invariant, and we call this the discriminant of (A, E, ϕ). For these Abelian $2n$-folds, Weil constructed a subspace

$$W(A) = \bigwedge_k^n H^1(A,\mathbb{Q}) \subset H^{n,n}(A) \cap H^{2n}(A,\mathbb{Q})$$

of the Hodge classes ($\dim_{\mathbb{Q}} W(A) = 2$), and elements in $W(A)$ are called Weil classes. The special Munford-Tate group for a general A is a special unitary group of the signature (n,n), and in this case we have

$$H^{n,n}(A) \cap H^{2n}(A,\mathbb{Q}) = D^n(A) \oplus W(A),$$

where $D^\bullet(A)$ is the subspace of classes generated by divisors. To prove the Hodge conjecture, therefore, we need algebraic cycles that never come from divisors.

In special cases, the setting up of the problem is established in [7] (see also [4]). We consider a curve C_{13} of genus 13 which is a 4-th cyclic étale cover of a curve C_4 of genus 4. Then we have étale double cover $C_{13} \to C_7$ of a curve C_7 of genus 7 as the intermediate cover. Let us consider the Prym variety $P = Prym(C_{13}/C_7)$ that is the connected component of the kernel of the norm map $Nm\colon J(C_{13}) \to J(C_7)$ including $0 \in J(C_{13})$. This is a principally polarized 6-dimensional Abelian variety. The Galois group $Gal(C_{13}/C_4) \cong \mathbb{Z}/4\mathbb{Z}$ acts on P, and P becomes Abelian variety of Weil type for $\mathbb{Q}(\sqrt{-1})$ by this action. It is known that the discriminant δ of P is 1 (see [4]), and the Weil classes $W(P)$ are generated by algebraic cycles (see [7]).

Let $\mathcal{M}_{13/4}$ be the moduli space of 4-th cyclic covers $C_{13} \to C_4$ and $\mathcal{A}_6$ be the moduli space of principally polarized Abelian varieties of dimension 6. Then we have the Prym map

$$Pr\colon \mathcal{M}_{13/4} \to \mathcal{H}_6 \subset \mathcal{A}_6, \quad \{C_{13} \to C_7 \to C_4\} \mapsto Prym(C_{13}/C_7),$$

where $\mathcal{H}_6$ is the Shimura variety of dimension 9 given in [4]. Therefore the Weil classes $W(A)$ are generated by algebraic cycles for a general $A \in \mathcal{H}_6$ if the image of Pr is Zariski dense in $\mathcal{H}_6$. Because the moduli space $\mathcal{M}_{13/4}$ is finite over the moduli space $\mathcal{M}_4$ of genus 4 curves and $\dim\mathcal{M}_4 = 9$, it is enough to prove the dominantness of Pr if we show that the differential (equivalently, the codifferential) of Pr at some point of $\mathcal{M}_{13/4}$ is an isomorphism.

Let us take $\Pi = \{X \to Y \to Z\} \in \mathcal{M}_{13/4}$ and the line bundle L on Z which gives $\pi\colon X \to Z$. Then we have the decomposition

$$\pi_*\omega_X = \omega_Z \oplus (\omega_Z \otimes L) \oplus (\omega_Z \otimes L^2) \oplus (\omega_Z \otimes L^3).$$

Replacing moduli spaces by them with a level structure if necessary, cotangent spaces at Π and at the intermediate Prym variety P of Π are represented by

$$T_P^*\mathcal{H}_6 = (T_P^*\mathcal{A}_6)^{Gal(X/Z)} = H^0(Z, \omega_Z \otimes L) \otimes H^0(Z, \omega_Z \otimes L^3),$$
$$T_\Pi^*\mathcal{M}_{13/4} = T_Z^*\mathcal{M}_4 = H^0(Z, \omega_Z \otimes \omega_Z),$$

and the codifferential map of Pr at Π is given the multiplication map

$$\mu\colon H^0(Z,\omega_Z\otimes L)\otimes H^0(Z,\omega_Z\otimes L^3)\to H^0(Z,\omega_Z\otimes\omega_Z). \tag{1}$$

Now we state our result:

Theorem 2.1 *There exists a 4-th cyclic étale cover $\pi\colon X\to Z$ of genus 4 curve Z such that the multiplication map μ in (1) is an isomorphism, and therefore the Prym map Pr is dominant.*

By the specialization argument in [7], we know that

Corollary 2.1 *The Weil classes are generated by algebraic cycles for Abelian 6-folds of Weil type for $\mathbb{Q}(\sqrt{-1})$ with $\delta=1$.*

Remark 2.1 By the Proposition 10 in [8], we see that the Weil classes are generated by algebraic cycles for all Abelian 4-folds of Weil type for $\mathbb{Q}(\sqrt{-1})$.

3 Complete Intersections in $(\mathbb{P}^1)^4$ and a Cyclic Permutation

In this section we construct a genus 13 curve C_{13} in $(\mathbb{P}^1)^4$ with a fixed point free automorphism σ of order 4, and we show that the natural projection $C_{13}\to C_{13}/\langle\sigma\rangle$ satisfies the required condition.

Let $[s_0:s_1]\times[t_0:t_1]\times[x_0:x_1]\times[y_0:y_1]$ be the coordinate of $(\mathbb{P}^1)^4$. For simplicity, we denote this by (s,t,x,y) with $s=s_1/s_0$ and so on. Let σ be a cyclic permutation $(s,t,x,y)\mapsto(y,s,t,x)$ on $(\mathbb{P}^1)^4$. Then σ acts on the vector space

$$V=H^0((\mathbb{P}^1)^4,\otimes_{i=1}^4 p_i^*\mathcal{O}_{\mathbb{P}^1}(1))=\oplus\,\mathbb{C}s_it_jx_ky_l,\quad i,j,k,l\in\{0,1\},$$

where $p_i\colon(\mathbb{P}^1)^4\to\mathbb{P}^1$ is the i-th projection. Let $V(\alpha)$ be the eigenspace for the eigenvalue α. We have the following basis of $V(\alpha)$:

$$\begin{aligned}
V(1):\ & a_1=s+t+x+y,\quad a_2=st+tx+xy+ys,\\
& a_3=txy+sxy+sty+stx,\quad a_4=sx+ty,\\
& a_5=stxy,\quad a_6=1,\\
V(-1):\ & b_1=s-t+x-y,\quad b_2=st-tx+xy-ys,\\
& b_3=txy-sxy+sty-stx,\quad b_4=sx-ty,\\
V(i):\ & c_1=s-it-x+iy,\quad c_2=st-itx-xy+iys,\\
& c_3=txy-isxy-sty+istx,\\
V(-i):\ & d_1=s+it-x-iy,\quad d_2=st+itx-xy-iys,\\
& d_3=txy+isxy-sty-istx,
\end{aligned}\tag{2}$$

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where we identified a multi-homogeneous polynomial $f \in V$ with the polynomial $f/(s_0 t_0 x_0 y_0)$ in affine coordinates. Note that every element in $V(1)$ is invariant also for the involution

$$\tau\colon (\mathbb{P}^1)^4 \to (\mathbb{P}^1)^4, \quad (s,t,x,y) \mapsto (x,t,s,y),$$

namely, they are invariant under the dihedral group $G = \langle \sigma, \tau \rangle$.

Lemma 3.1 *The linear system $V(1)$ gives a morphism $\varphi\colon (\mathbb{P}^1)^4 \to F \subset \mathbb{P}^5$ of generic degree 8, where F is the cubic hypersurface*

$$a_1^2 a_4 - a_1 a_3 a_4 + a_2 a_4^2 - 4 a_2 a_5 a_6 + a_3^2 a_6 = 0. \tag{3}$$

So the fiber $\varphi^{-1}(P)$ of a generic point $P \in F$ is a G-orbit.

Proof The morphism

$$(\mathbb{P}^1)^4 \to \mathbb{P}^4, \quad (s,t,x,y) \mapsto [a_1 : a_2 + a_4 : a_3 : a_5 : a_6]$$

induces an isomorphism $(\mathbb{P}^1)^4/S_4 \to \mathbb{P}^4$ because they are the fundamental symmetric polynomials. Hence $V(1)$ has no base points, and φ is a finite morphism onto the image. We can check that the a_is satisfy the above cubic equation, and that the morphism $F \to \mathbb{P}^4$ is of generic degree 3. Now we see that φ is of degree 8 since $|S_4| = 24$. ■

Lemma 3.2 *Let C be a hyperelliptic curve of genus 4, and L be a non-trivial line bundle such that $L^2 = \mathcal{O}_C$. Then the associated map $\varphi_{\omega_C \otimes L}\colon C \to \mathbb{P}^2$ defined by $H^0(C, \omega_C \otimes L)$ satisfies one of the following conditions.*

(1) *$\varphi_{\omega_C \otimes L}$ is a rational map to a conic in $\mathbb{P}^2$.*
(2) *$\varphi_{\omega_C \otimes L}$ is a birational map.*

Proof Because L is isomorphic to $\mathcal{O}_C(P_1 - P_2)$ or to $\mathcal{O}_C(P_1 + P_2 - P_3 - P_4)$ where $\pi(P_i)$'s are distinct blanch points of the double cover $\pi\colon C \to \mathbb{P}^1$, it is enough if we consider these two cases.

Let $y^2 = f(x)$ be the equation of C and $\pi(P_i) = \lambda_i$. Then a basis of $H^0(C, \omega_C \otimes L)$ is given by

$$(x - \lambda_2)\frac{dx}{y}, \quad (x - \lambda_2)^2 \frac{dx}{y}, \quad (x - \lambda_2)^3 \frac{dx}{y},$$

for $L = \mathcal{O}_C(P_1 - P_2)$, and this is the case (1) in the assertion. In the case of $L = \mathcal{O}_C(P_1 + P_2 - P_3 - P_4)$, we can take the following basis of $H^0(C, \omega_C \otimes L)$

$$\eta_1 = (x - \lambda_3)(x - \lambda_4)\frac{dx}{y}, \quad \eta_2 = (x - \lambda_3)(x - \lambda_4)^2 \frac{dx}{y}, \quad \eta_3 = \frac{dx}{(x - \lambda_1)(x - \lambda_2)}.$$

Because $\eta_2/\eta_1 = x - \lambda_4$ and $\eta_3/\eta_1 = g(x)y$ with a rational function $g(x)$, we see that $\varphi_{\omega_C \otimes L}$ is birational in this case. ■

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Proposition 3.1 *For general elements $f_1, f_2, f_3 \in V(1)$, the complete intersection $f_1 = f_2 = f_3 = 0$ defines a smooth curve X of genus 13, and we have an isomorphism of vector spaces*

$$H^0(X, \omega_X) \cong V(1) \bigm/ (\mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3) \oplus V(-1) \oplus V(i) \oplus V(-i).$$

The cyclic permutation σ acts on X without fixed point. Hence we have an étale cyclic cover $X \to Y = X/\langle\sigma^2\rangle \to Z = X/\langle\sigma\rangle$, and an isomorphism $H^0(Z, \omega_Z) \cong V(-1)$. The genus 4 curve Z is not hyperelliptic.

Proof Because $V(1)$ is base point free and the divisor given by a general $f \in V(1)$ is reduced, the curve $X = \{f_1 = f_2 = f_3 = 0\}$ is smooth for general $f_1, f_2, f_3 \in V(1)$. By the Adjunction formula, we know that the restriction of V on X gives the canonical class and the genus of X is 13.

Obviously σ acts on X, and the fixed points of σ^2 on $(\mathbb{P}^1)^4$ is

$$\Delta = \{(s, t, s, t) \in (\mathbb{P}^1)^4\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

The restriction of the basis of $V(1)$ in (2) to Δ is given by

$$\begin{aligned} &a_1 = s_0t_0(s_1t_0 + s_0t_1), \quad a_2 = s_0t_0s_1t_1, \quad a_3 = s_1t_1(s_1t_0 + s_0t_1), \\ &a_4 = s_1^2t_0^2 + s_0^2t_1^2, \quad a_5 = s_1^2t_1^2, \quad a_6 = s_0^2t_0^2, \end{aligned} \tag{4}$$

up to constant, with coordinates $[s_0 : s_1] \times [t_0 : t_1]$. They have no base point and we see that $C \cap \Delta = \phi$ for general f_i's.

Because $V(-1)$ is the unique 4-dimensional eigenspace for the action of σ, we may identify $V(-1)$ with $H^0(Z, \omega_Z)$.

Now let L be the line bundle corresponding to $\pi\colon Y \to Z$. We see that

$$H^0(Z, \omega_Z \otimes L) \cong V(1)/(\mathbb{C}f_1 \oplus \mathbb{C}f_2 \oplus \mathbb{C}f_3).$$

By Lemma (3.1), the image of $\varphi_{\omega_Z \otimes L}$ is a cubic curve E which is a section of F by $\mathbb{P}^3 \subset \mathbb{P}^5$, and this is not any case in Lemma (3.2). Therefore Z is not hyperelliptic. ∎

Remark 3.1 We can check that the singular locus of F is 1-dimensional, and that the section of F by a generic $\mathbb{P}^3$ in $\mathbb{P}^5$ is a smooth cubic curve E. The Prym canonical map $\varphi_{\omega_Z \otimes L}$ is just the natural map $Z \to E = Z/\langle\tau\rangle$, so our curve Z is bi-elliptic.

Let f_1, f_2, f_3 and X be as in Proposition (3.1), $\pi\colon X \to Z = X/\langle\sigma\rangle$ be the quotient map and L be the line bundle on Z corresponding to π. Because Z is not hyperelliptic, the multiplication map

$$Sym^2 H^0(Z, \omega_Z) \to H^0(Z, \omega_Z \otimes \omega_Z)$$

is a surjection by Max Noether's Theorem (see [1]). By Riemann-Roch Theorem, we see that this map has the 1-dimensional kernel. We denote a generator of this kernel by Q, and we use the same symbol for the corresponding element in $Sym^2 V(-1)$. Namely, we have isomorphisms

$$Sym^2 V(-1)/\mathbb{C}Q \cong H^0(Z, \omega_Z \otimes \omega_Z),$$
$$V(i) \otimes V(-i) \cong H^0(Z, \omega_Z \otimes L) \otimes H^0(Z, \omega_Z \otimes L^3).$$

Therefore the map μ in (1) defines the induced multiplication map

$$m\colon V(i) \otimes V(-i) \to Sym^2 V(-1)/\mathbb{C}Q, \quad c \otimes d \mapsto cd \bmod (f_1, f_2, f_3) \tag{5}$$

by the above identification. (The map m is well-defined only modulo $f_1 = f_2 = f_3 = 0$.) Now the bijectivity of μ is equivalent to the linear independence of $\{c_i d_j\}_{1 \le i,j \le 3}$ modulo $f_1 = f_2 = f_3 = 0$.

Let us show the linear independence. First of all, we have the following quadric equations (these are a part of the Segre relations, and we can check them with a computer)

$$\begin{gathered} b_1^2 = a_1^2 - 4a_2a_6, \quad b_2^2 = a_2^2 - 4(a_1a_3 - a_2a_4) + 16a_5a_6, \\ b_3^2 = a_3^2 - 4a_2a_5, \quad b_1b_3 = a_1a_3 - 2a_2a_4, \quad b_1b_4 = a_1a_4 - 2a_3a_6, \\ b_3b_4 = -a_3a_4 + 2a_1a_5, \quad b_4^2 = a_4^2 - 4a_5a_6, \end{gathered} \tag{6}$$

and

$$\begin{gathered} c_1d_1 = a_1^2 - 2a_2a_6 - 4a_4a_6, \quad c_3d_3 = a_3^2 - 2a_2a_5 - 4a_4a_5, \\ c_2d_2 = a_2^2 - 2a_1a_3 + 2a_2a_4, \\ \frac{1}{2}(c_1d_3 + c_3d_1) = -a_1a_3 + a_2a_4 + 8a_5a_6, \quad \frac{i}{2}(c_1d_3 - c_3d_1) = b_2b_4, \\ \frac{1}{1-i}(c_1d_2 - ic_2d_1) = a_1a_2 - 4a_3a_6, \quad \frac{1}{1+i}(c_1d_2 + ic_2d_1) = b_1b_2, \\ \frac{1}{1+i}(c_2d_3 + ic_3d_2) = -a_2a_3 + 4a_1a_5, \quad \frac{-i}{1+i}(c_2d_3 - ic_3d_2) = b_2b_3. \end{gathered} \tag{7}$$

Without a loss of generality, we may assume that our equations $f_1 = 0$, $f_2 = 0$ and $f_3 = 0$ are given by

$$\begin{gathered} a_4 = A_1a_1 + A_2a_2 + A_3a_3, \quad a_5 = B_1a_1 + B_2a_2 + B_3a_3, \\ a_6 = C_1a_1 + C_2a_2 + C_3a_3, \end{gathered} \tag{8}$$

with coefficients $A_i, B_i, C_i \in \mathbb{C}$. Substituting them, we can eliminate a_4, a_5, a_6 in equations in 6). Then each product b_ib_j in (6) is a linear combination of six elements

$$a_1^2, \quad a_2^2, \quad a_3^2, \quad a_1a_2, \quad a_1a_3, \quad a_2a_3. \tag{9}$$

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Since there are seven elements in (6), we have a non-trivial linear relation of b_ib_j's in (6) which gives the unique vanishing quadric Q. Therefore a_ia_j's in (9) and b_1b_2, b_2b_3, b_2b_4 give a base of the vector space $Sym^2 V(-1)/\mathbb{C}Q$ (modulo equations in (8)).

Now let us consider a base of the vector space $V(i) \otimes V(-i)$ given in (7). Eliminating a_4, a_5, a_6, we obtain a vector equation $\gamma = M\alpha$, where

$$\gamma = {}^t\big(c_1d_1,\ c_2d_2,\ c_3d_3,\ \frac{1}{1-i}(c_1d_2 - ic_2d_1),\ \frac{1}{2}(c_1d_3 + c_3d_1),\ \frac{1}{1+i}(c_2d_3 + ic_3d_2)\big),$$
$$\alpha = {}^t(a_1^2,\ a_2^2,\ a_3^2,\ a_1a_2,\ a_1a_3,\ a_2a_3),$$

and M is a matrix of polynomials in A_i, B_i, C_i. If we have $\det M \neq 0$, we can conclude that c_id_j's form a base of $Sym^2 V(-1)/\mathbb{C}Q$, and we finish the proof of theorem.

Let us consider the case that all A_i, B_i, C_i are 0. Then $a_4 = a_5 = a_6 = 0$, and obviously we have $\det M = 1$. Therefore $\det M$ is not identically zero as a polynomial of A_i, B_i, C_i.

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