Canad. Math. Bull. Vol. **43** (3), 2000 pp. 320–329

On Classification of Certain C^*-Algebras

George Elliott and Igor Fulman

Abstract. We consider C^*-algebras which are inductive limits of finite direct sums of copies of $C([0,1]) \otimes \mathcal{O}_2$. For such algebras, the lattice of closed two-sided ideals is proved to be a complete invariant.

1 The Problem and the Result

We consider the following class of C^*-algebras: inductive limits of finite direct sums of copies of $C([0,1]) \otimes \mathcal{O}_2$, where $\mathcal{O}_2$ denotes the Cuntz algebra with two generators. Therefore, an algebra A from this class can be represented as the limit

$$A_1 \to A_2 \to \cdots \to A$$

with each A_i being isomorphic to

$$\bigoplus_{j=1}^{n_i} C([0,1]) \otimes \mathcal{O}_2.$$

We prove that a complete invariant for this class of C^*-algebras is the lattice of closed two-sided ideals of the algebra. More precisely, we have proved the following theorem.

Theorem 1 *Let A and B be two C^*-algebras as above. If their lattices of closed two-sided ideals $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are isomorphic as lattices, then the C^*-algebras A and B are isomorphic.*

2 Partial Case Considered by J. Mortensen

Jacob Mortensen [3] has solved the above problem in a particular case. The invariant is the same, namely the lattice of closed two-sided ideals $\mathcal{I}(A)$ of the algebra A. The problem was solved only for the algebras with totally ordered ideals.

Theorem 2 **(Mortensen's Classification Theorem) (See [3, Theorem 5.1.1])** *Let A and B be two C^*-algebras as above, and assume that $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are totally ordered. If the lattices $\mathcal{I}(B)$ and $\mathcal{I}(A)$ are isomorphic, then the algebras A and B are isomorphic.*

Sketch of Mortensen's Proof Suppose A and B are two algebras as above, and assume that $\mathcal{I}(A) \cong \mathcal{I}(B)$.

Received by the editors August 20, 1998; revised January 21, 1999.
AMS subject classification: 46L05, 46L35.

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

Main "tool": for a homomorphism of C^*-algebras $\varphi\colon C \to D$ one can construct a map $\hat{\varphi}\colon \mathcal{I}(D) \to \mathcal{I}(C)$, namely: for each $I \in \mathcal{I}(D)$, one puts $\hat{\varphi}(I) \equiv \varphi^{-1}(I)$.

(Remark: the map $\hat{\varphi}$ is not in general a homomorphism of lattices, it is only infimum-preserving.)

Then, one obtains the following diagram:

$$\begin{array}{ccccccc}
\mathcal{I}(A_1) & \longleftarrow & \mathcal{I}(A_2) & \longleftarrow & \cdots & \longleftarrow & \mathcal{I}(A) \\
 & & & & & & \updownarrow \\
\mathcal{I}(B_1) & \longleftarrow & \mathcal{I}(B_2) & \longleftarrow & \cdots & \longleftarrow & \mathcal{I}(B)
\end{array}$$

where the horizontal arrows come from the construction of A and B, and the vertical arrow represents the given isomorphism of lattices.

Mortensen completes this diagram as follows:

$$\begin{array}{ccccccc}
\mathcal{I}(A_1) & \longleftarrow & \mathcal{I}(A_2) & \longleftarrow & \cdots & \longleftarrow & \mathcal{I}(A) \\
\uparrow & \swarrow & \uparrow & \swarrow & & & \updownarrow \\
\mathcal{I}(B_1) & \longleftarrow & \mathcal{I}(B_2) & \longleftarrow & \cdots & \longleftarrow & \mathcal{I}(B)
\end{array}$$

to get an approximately commuting diagram (after passing to a subsequence and renumbering). In this process, he strongly relays on the condition of total ordering of $\mathcal{I}(A)$ and $\mathcal{I}(B)$.

Then, he uses his existence and uniqueness theorems to "lift" each map of the type $\mathcal{I}(D) \to \mathcal{I}(C)$ to the corresponding C^*-algebra homomorphism $C \to D$. The whole diagram above can be lifted to the corresponding diagram for C^*-algebras:

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

Mortensen manages to do this in such a way that the resulting diagram is the approximate intertwining in the sense of Elliott [2]. Therefore, there exists an isomorphism ρ between the limit C^*-algebras A and B completing the above diagram:

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \updownarrow \rho \\
B_! & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

Finally, one can prove that the map $\hat{\rho}\colon \mathcal{I}(B) \to \mathcal{I}(A)$ coming from ρ as above coincides with the given map. ∎

3 General Case

Our goal in this section is to prove Theorem 1 which generalizes Mortensen's Theorem 2.

Remark We don't assume anymore that $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are totally ordered.

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

3.1 New "Tool"

To eliminate the condition of total ordering in the Mortensen's setting, we add another "tool". For a C^*-algebra homomorphism $\varphi\colon C \to D$ we consider the map between the lattices of ideals $\check{\varphi}\colon \mathcal{I}(C) \to \mathcal{I}(D)$ acting in forward direction (while $\hat{\varphi}$ is acting in backward direction). The map $\check{\varphi}$ is defined naturally: for each $I \in \mathcal{I}(C)$, $\check{\varphi}(I)$ is the ideal in $\mathcal{I}(D)$ generated by the image $\varphi(I)$.

Remarks 1. The map $\check{\varphi}$ is supremum-preserving.

2. The maps $\hat{\varphi}$ and $\check{\varphi}$ are not (in general) inverses of each other, but they determine each other by simple formulas. Namely, for $I \in \mathcal{I}(C)$:

$$\check{\varphi}(I) = \inf\{J \in \mathcal{I}(D) \mid I \subseteq \hat{\varphi}(J)\}.$$

Analogously, for $J \in \mathcal{I}(D)$:

$$\hat{\varphi}(J) = \sup\{I \in \mathcal{I}(C) \mid \check{\varphi}(I) \subseteq J\}.$$

Also, the connection between $\hat{\varphi}$ and $\check{\varphi}$ can be expressed in the following formula: for $I \in \mathcal{I}(C)$, $J \in \mathcal{I}(D)$:

$$I \subseteq \hat{\varphi}(J) \iff \check{\varphi}(I) \subseteq J$$

3. Mortensen gives the intrinsic description of the maps $\hat{\varphi}$: these are the infimum-preserving maps from $\mathcal{I}(D)$ to $\mathcal{I}(C)$ which are continuous in the Hausdorff metric on subsets of $[0, 1]$. We don't know such an intrinsic definition for the maps $\check{\varphi}$.

Suppose that A and B are as above. We use the following notation: φ_{ij} denotes the (given) homomorphism between the finite stage algebras A_i and A_j, φ_i denotes the homomorphism from A_i to the limit algebra A, ψ_{ij} and ψ_i have the same meaning for B_i and B.

Assume that there is a lattice isomorphism $\Psi\colon \mathcal{I}(A) \to \mathcal{I}(B)$. From these data we get the following diagram for the lattices:

$$\begin{array}{ccccccc}
\mathcal{I}(A_1) & \longrightarrow & \mathcal{I}(A_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}(A) \\
 & & & & & & \big\updownarrow{\scriptstyle\Psi} \\
\mathcal{I}(B_1) & \longrightarrow & \mathcal{I}(B_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}(B)
\end{array} \tag{1}$$

where again the horizontal arrows come from the structure of A and B, while the vertical arrow represents the given isomorphism.

3.2 New Metrics on the Lattices $\mathcal{I}(A_n)$, $\mathcal{I}(A)$, $\mathcal{I}(B_m)$, $\mathcal{I}(B)$

We choose the new metrics as follows. Find a countable dense set $D_n = \{d_{n,1}, d_{n,2}, \dots\}$ in the unit ball of each algebra A_n, so that the union D of images of all D_n's in A is dense in the unit ball of A.

Let $l\colon \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be the function of "counting by diagonals", defined as follows: $l(n, m) = (n + m - 1)(n + m - 2)/2 + n$.

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

For $I, J \in \mathcal{I}(A)$ and $d \in D$ let $\|d\|_I = \|d + I\|$ in A/I. Let $D_d(I, J) = |\,\|d\|_I - \|d\|_J|$. Finally, let $D(X, Y) = \sum_{n,m} D_{d_{n,m}}(I, J) \cdot 2^{-l(n,m)}$.

Analogously, for $I, J \in \mathcal{I}(A_k)$ and $d \in D_k$ let $\|I\|_d = \|d + I\|$ in A/I and $D_d(I, J) = |\,\|I\|_d - \|J\|_d|$. Then, let $D(X, Y) = \sum_{n \le k,\ \text{all}\ m} D_{d_{n,m}}(I, J) \cdot 2^{-l(n,m)}$. Here, the elements from $D(n)$ with $n < k$ are identified with their images in A_k.

We choose the metrics for $\mathcal{I}(B_n)$ and $\mathcal{I}(B)$ in an analogous way.

3.3 Building a "Forwards" Intertwining Map

We will complete the diagram (1) to get the following intertwining diagram:

$$\begin{array}{ccccccc}
\mathcal{I}(A_1) & \longrightarrow & \mathcal{I}(A_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}(A) \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \updownarrow \\
\mathcal{I}(B_1) & \longrightarrow & \mathcal{I}(B_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}(B)
\end{array}$$

where the intertwining maps are being built inductively in a special way.

For simplicity we will always assume that all "finite stage" algebras A_i and B_i are isomorphic to $C([0, 1], \mathcal{O}_2)$.

We begin with building a single intertwining map.

For a given finite stage A_i and a given positive number δ we will choose a certain finite subset $F \subset \mathcal{I}(A_i)$ as follows. Elements of F correspond to the open intervals in the spectrum, such that the union of all the intervals is the whole segment $[0, 1]$, every interval has the length of δ, every interval is contained in the union of its neighbors, and the length of the intersection of any two neighboring intervals is at most $2\delta/3$.

Elements of F has a natural order, we will define them by $f_1, f_2, \ldots, f_k$.

It follows that every interval is contained in a compact set (denoted by K_i) which is contained in the union of its neighbors.

Proposition 3 *Let F be a finite subset in $\mathcal{I}(A_n)$ as above and ε be a positive number. Let G be another finite set in $\mathcal{I}(B_{m_0})$ (see the diagram below).*

There exist $m > m_0$ and a map $\Phi \colon F \to \mathcal{I}(B_m)$ satisfying the following properties:

1. *In the the following diagram*

$$\begin{array}{ccccc}
 & & \mathcal{I}(A_n) & \xrightarrow{\check{\varphi}_n} & \mathcal{I}(A) \\
 & & \Phi\downarrow & & \updownarrow\Psi \\
\mathcal{I}(B_{m_0}) & \xrightarrow{\check{\psi}_{m_0,m}} & \mathcal{I}(B_m) & \xrightarrow{\check{\psi}_m} & \mathcal{I}(B)
\end{array}$$

the square is commutative on elements from F up to ε;

2. *for every $i \le k$, there exists a compact set M_i such that*

$$\Phi(f_i) \subset M_i \subset \sup\{\Phi(f_{i-1}), \Phi(f_{i+1})\}$$

where the ideals are identified with the corresponding open subsets of $[0, 1]$*;*

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

3. *for every* $f \in F$ *and every* $g \in G$ *such that* $\Psi \circ \check{\varphi}_n(f) \subseteq \check{\psi}_{m_0}(g)$ *one has:* $\Phi(f) \subseteq \check{\psi}_{m_0,m}(g)$ *(i.e., if the image of* f *at infinity is contained in the image of* g*, then the same inclusion holds at the* m*-th stage).*

Proof We will construct the images of the elements from F by several successive "adjustments".

One may assume that G consists of a single element g. To satisfy condition (3.) it's enough to construct the image of f within the image of g. To satisfy also condition (1.), it's enough to choose m sufficiently large, so that for all $f \in F$ the distance between $\check{\psi}_m \circ \hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f)$ and $\check{\varphi}_n(f)$ is smaller than $\varepsilon/2$, and the distance between $\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f)$ and $\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f) \cap \check{\psi}_{m_0,m}(g)$ is also smaller than $\varepsilon/2$. So, the "first" approximation for $\Psi(f)$ will be $\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f) \cap \check{\psi}_{m_0,m}(g)$. All successive approximations will be made within it, to preserve condition (3.).

Now we will choose the images of the elements of F to satisfy condition (2.). Here and further $\|x\|_I$ stands for the norm of $x + I$ in A/I, as before. For each i, let x_i be a positive element in $\sup\{f_{i-1}, f_{i+1}\}$ and a_i be a positive number such that $K_i = \{I \in \mathcal{I}(A_n) \mid \|x\|_I \geq a_i\}$. (Such x_i and a_i evidently exist.) Let $x_i' = \varphi_n(x_i)$. Let K_i' be the subset of the primitive spectrum of A defined by $K_i' = \{I \in \operatorname{Prim} A \mid \|x_i'\|_I \geq a_i\}$. By [1, 3.3.7], K_i' is compact in the Jacobson topology. Moreover, one checks immediately that $\check{\varphi}_m(f_i) \subset K_i' \subset \sup\{\check{\varphi}_m(f_{i-1}), \check{\varphi}_m(f_{i+1})\}$. (Here again the ideals are identified with the corresponding open subsets in the spectrum of A.)

The lattice isomorphism Ψ is a homeomorphism on the level of the primitive spectra with the Jacobson topology. Therefore, the images of all K_i' under Ψ are also compact. Moreover, as the function $I \mapsto \|x\|_I$ is lower semi-continuous (see [1, 3.3.2]), every compact set is contained in a compact set of the above type, *i.e.*, there exist positive elements $z_i' \in \sup\{\Psi \circ \check{\varphi}_m(f_{i-1}), \Psi \circ \check{\varphi}_m(f_{i+1})\}$ and positive numbers b_i such that each $\Psi(K_i')$ is contained in $L_i' = \{I \in \operatorname{Prim} B \mid \|z_i'\|_I \geq b_i\}$. One may assume that all z_i' are the images of some elements of some finite stage algebra B_m. (Denote these latter elements by z_i.) Moreover, one may assume that each z_i belongs to the respective "first approximation" for the image of f_i.

Let g_i be a continuous function such that $g_i(\lambda) = 0$ if $\lambda \geq b_i/2$ and $g_i(\lambda) > 0$ if $\lambda > b_i/2$. Let $y_i = g_i(z_i)$ and $y_i' = \psi_m(y_i)$. (Of course $y_i' = g_i(z_i')$). The ideal in B_m generated by y_i (denote this ideal by Y_i) corresponds to the open set $\{I \in \operatorname{Prim} B_m \mid \|z_i\|_I > b_i/2\}$. Let $Y_i' = \check{\psi}_m(Y)$. The ideal Y_i' is generated by y_i'. One can also check that Y_i' corresponds to the open set $\{I \in \operatorname{Prim} B \mid \|z_i'\|_I > b_i/2\}$. Hence, Y_i' contains $\Psi \circ \check{\varphi}_n(f_i)$.

Therefore, each z_i' is contained in the ideal generated by y_{i-1}' and y_{i+1}'. Hence, by choosing m large enough, one can achieve that each z_i is approximately contained in the ideal generated by y_{i-1} and y_{i+1}, and the discrepancy is less than the smallest of the numbers $b_i/4$. Then, for each z_i there exists an approximation $\tilde{z}_i$ which belongs to the ideal generated by y_{i-1} and y_{i+1}.

Let $\tilde{g}_i$ be a continuous function such that $\tilde{g}_i(\lambda) = 0$ if $\lambda \geq b_i/4$ and $g_i(\lambda) > 0$ if $\lambda > b_i/4$. Let $\tilde{y}_i = \tilde{g}_i(\tilde{z}_i)$ and $\tilde{y}_i' = \psi_m(\tilde{y}_i)$. (Then again $\tilde{y}_i' = \tilde{g}_i(\tilde{z}_i')$.) The ideal generated by y_i is contained in the ideal generated by $\tilde{y}_i$. (Denote the latter ideal by $\tilde{Y}_i$.) It follows that each $\tilde{z}_i$ is contained in the ideal generated by $\tilde{y}_{i-1}$ and $\tilde{y}_{i+1}$. Now, for each i we define $\Phi(f_i)$

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

to be $\big(\hat{\psi}_m \circ \Psi \circ \check{\varphi}_n(f_i)\big) \cap \tilde{Y}_i$. Then condition (2.) is satisfied, with M_i defined as follows:

$$M_i = \{I \in \mathcal{I}(B_m) \mid \|\tilde{z}_i\|_I \geq b_i/4\} \qquad \blacksquare$$

Corollary 4 *There exists a map $\tilde{\Phi}\colon F \to \mathcal{I}(B_m)$ satisfying all the conditions for Φ in Proposition 3, and in addition such that all the open subsets corresponding to all $\tilde{\Phi}(f_i)$ satisfy the following conditions:*

1. *every such subset is a union of a* finite *number of intervals;*
2. *endpoints of different subsets don't coincide.*

Proof Every open set corresponding to $\Phi(F_i)$ is the union of countably many intervals. One can choose finitely many of them whose union still covers the compact set M_i. Moreover, one can decrease some of the intervals if necessary to make their endpoints different. Take the ideal obtained this way for $\tilde{\Phi}(f_i)$. If the approximations made are close enough, the diagram (3) with $\tilde{\Phi}$ instead of Φ is still approximately commutative. $\blacksquare$

3.4 Building the Whole "Forwards" Intertwining Diagram

Starting with $\varepsilon = 1/2$ we get $F_1 \subset \mathcal{I}(A_1)$, as before. Then we choose B_m as in Lemma 3 and renumber it as B_1. (On this stage, we take $G = \varnothing$.)

Then we choose a finite set (denote it by G_1) in $\mathcal{I}(B_1)$ in the same way as F_1, but in addition so that for every $I \in F_1$, the ideal $\Phi_1(I)$ is the supremum of some elements of G_1.

Then we apply the same procedure to $\mathcal{I}(B_1)$ with $\varepsilon = 1/4$. Now we take F_1 for the set G in Proposition 3. We get the following diagram:

$$\begin{array}{ccccc} \mathcal{I}(A_1) & \longrightarrow & \mathcal{I}(A_2) & \longrightarrow & \mathcal{I}(A) \\ \downarrow & \nearrow & & & \updownarrow \\ \mathcal{I}(B_1) & & \longrightarrow & & \mathcal{I}(B) \end{array} \tag{2}$$

In this diagram, the "horizontal" map $\mathcal{I}(A_1) \to \mathcal{I}(A_2) \to \mathcal{I}(A)$ is approximately equal to the map $\mathcal{I}(A_1) \to \mathcal{I}(B_1) \to \mathcal{I}(B) \to \mathcal{I}(A)$, which is approximately equal to the map $\mathcal{I}(A_1) \to \mathcal{I}(B_1) \to \mathcal{I}(A_2) \to \mathcal{I}(A)$. Therefore, the map $\mathcal{I}(A_1) \to \mathcal{I}(A_2) \to \mathcal{I}(A)$ is approximatively equal to the map $\mathcal{I}(A_1) \to \mathcal{I}(B_1) \to \mathcal{I}(A_2) \to \mathcal{I}(A)$. Also, the image of every $f \in F_1$ under the latter map is contained in its image under the former map. By construction of the metric on $\mathcal{I}(A)$ and also because of finite domains of all maps in question, these two maps are approximately equal on some finite stage, *i.e.*, there exists an integer n such that the map $\mathcal{I}(A_1) \to \mathcal{I}(A_2) \to \mathcal{I}(A_n)$ is approximately equal to $\mathcal{I}(A_1) \to \mathcal{I}(B_1) \to \mathcal{I}(A_2) \to \mathcal{I}(A_n)$, with the same condition of inclusion. We renumber A_n as A_2.

Lemma 5 *The triangle in diagram (2) satisfy the following condition: for every $f \in F_1$, the image of f under the map $\mathcal{I}(A_1) \to \mathcal{I}(B_1) \to \mathcal{I}(A_2)$ is contained in the image of f under the map $\check{\varphi}_{1,2}\colon \mathcal{I}(A_1) \to \mathcal{I}(A_2)$.*

Proof For $f \in F_1$, let its image in $\mathcal{I}(B_1)$ be the supremum of $g_1, g_2, \dots, g_k \in \mathcal{I}(B_1)$. By the construction, the images of all $g_1, g_2, \dots, g_k$ under $\Psi^{-1} \circ \check{\psi}_1$ are contained in $\check{\varphi}_1(f)$. By Proposition 3, their images under the map $\mathcal{I}(B_1) \to \mathcal{I}(A_2)$ are contained in $\check{\varphi}_{1,2}(f)$. But the image of f under the map $\mathcal{I}(A_1) \to \mathcal{I}(B_1) \to \mathcal{I}(A_2)$ is their supremum. ■

This procedure can be repeated with ε's summing up to a finite sum, to get the following intertwining diagram:

$$\begin{array}{ccccccc} \mathcal{I}(A_1) & \longrightarrow & \mathcal{I}(A_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}(A) \\ \downarrow & \nearrow & \downarrow & \nearrow & & & \updownarrow \\ \mathcal{I}(B_1) & \longrightarrow & \mathcal{I}(B_2) & \longrightarrow & \cdots & \longrightarrow & \mathcal{I}(B) \end{array} \tag{3}$$

3.5 Building a Single "Backwards" Map

Let $C, D \in (A_n)_{n=1}^{\infty} \cup (B_n)_{n=1}^{\infty}$. Through this Subsection, we will identify ideals in C or D with the corresponding open subsets of $[0, 1]$.

Let $\varepsilon > 0$, and let $F \subset \mathcal{I}(C)$ be a finite subset as chosen above. This set has a natural order; let $F = \{f_i\}_{i=1}^{k}$. Let $\Phi \colon F \to \mathcal{I}(D)$ be an arbitrary map.

We will build the corresponding "backwards" everywhere defined map $\Psi \colon \mathcal{I}(D) \to \mathcal{I}(C)$. Everywhere we identify the ideals with the corresponding open sets—their open supports.

Elements from the image $\Phi(F)$ correspond to open subsets of $[0, 1]$. By the conditions above, these open sets consist of finite number of open intervals with different endpoints. These intervals break the whole segment $[0, 1]$ into the disjoint union of a finite number of intervals which may be open or closed or half-open. Denote the set of these intervals by R. For each interval $r \in R$, denote the middle point of r by m_r. Let P be the set of all these middle points.

It's enough to define Ψ only on maximal ideals corresponding to the open subsets of the type $S_t = [0, t) \cup (t, 1]$ and make sure it is continuous in the Hausdorff metric. (See [3, Proof of Theorem 4.3.1].)

For every $p \in P$, we put $\Psi(S_p)$ to be the union of those elements of F whose images do not contain the point p. Then, $\Psi(S_p)$ is a certain open set.

Moreover, for neighboring points $p, q \in P$, the images $\Psi(S_p)$ and $\Psi(S_q)$ are at most ε apart in the Hausdorff metric in $\mathcal{I}(C)$. Indeed, these two images are different by exactly one small interval from F, say f_i. This interval can bring to a large jump with respect to the Hausdorff metric only in one case: namely, if the interval f_i covers a gap. In any other case, the jump would be small. But if this case happens, it means that both $\Psi(S_p)$ and $\Psi(S_q)$ don't contain at least one of the neighbors of f_i. (Because if they contained both of them, the gap wouldn't exist.) Suppose these sets don't contain f_{i-1}. Then, they must contain f_{i-2} (unless we are doing near the left border) because otherwise the gap would be too large to be covered by f_i. But this means that after adding (or before subtracting) f_i, the union of the intervals would contain both f_{i-2} and f_i but not contain f_{i-1}. This is a contradiction: if the images of both f_{i-2} and f_i don't cover a certain point (p or q), the image of f_{i-1} shouldn't do either.

Finally, we will define Ψ on all remaining S_t's by interpolation, making it continuous. We will perform the interpolation as follows. Let $p, q \in P$ be two neighboring points, corresponding to the neighboring intervals $p', q' \in R$. Let a be the common endpoint of p' and q'. Assume that $a \in p'$, that $\Psi(S_q) = \Psi(S_p) \cup (b, d)$, and that $\Psi(S_p) \cap (b, d) = (b, c)$. (All other cases are considered analogously.) For all $t \in (p, a]$ we put $\Psi(S_t) \equiv \Psi(S_p)$, and for $t \in (a, q)$ we define $\Psi(S_t) \equiv \Psi(S_p) \cup \big(b, c + (d - c)(t - a)/(q - a)\big)$. One checks that this is a continuous interpolation such that the resulting backwards map Ψ satisfies the following property: for every $f \in F$: $\Phi(f) = \inf\{I \mid I \subset \Psi(f)\}$. In other words, the "forwards" map $\mathcal{I}(C) \to \mathcal{I}(D)$ derived from Ψ as described in Subsection 3.1 extends the map Φ.

3.6 Building the Whole "Backwards" Intertwining Diagram

Proposition 6 *Let $C, D \in (A_n)_{n=1}^{\infty} \cup (B_n)_{n=1}^{\infty}$. Suppose that the lattice $\mathcal{I}(C)$ is equipped with the Hausdorff metric, while the lattice $\mathcal{I}(D)$ is equipped with an arbitrary metric, in which it is a compact space. Let $\varepsilon > 0$. Let F be the finite subset of $\mathcal{I}(C)$ representing covering of $[0, 1]$ by segments of length ε. Let Ψ_1 and Ψ_2 be two continuous infimum-preserving maps from $\mathcal{I}(D)$ to $\mathcal{I}(C)$. Let δ be the modulus of uniform continuity of the map Ψ_1 corresponding to $\varepsilon/2$. Let Θ_1 and Θ_2 be the maps from $\mathcal{I}(C)$ to $\mathcal{I}(D)$ corresponding to Ψ_1 and Ψ_2 respectively as in Subsection 3.1. Suppose that for every $f \in F$: $\Theta_2(f) \subseteq \Theta_1(f)$, and the distance between $\Theta_2(f)$ and $\Theta_1(f)$ is not more than δ. Then for every $I \in \mathcal{I}(D)$, the distance between $\Psi_1(I)$ and $\Psi_2(I)$ is not more than 2ε.*

Proof Let $I \in \mathcal{I}(D)$. Let $J = \Psi_1(I)$ and $K = \Psi_2(I)$. Let $F_J = \{f \in F \mid f \nsubseteq J\}$ and $F_K = \{f \in F \mid f \nsubseteq K\}$. If $F_J = F_K$ then by definition of the Hausdorff metric, the distance between J and K is not more than ε. By the condition of inclusion above, we always have that $F_K \subseteq F_J$. Indeed, if an interval f isn't contained in K, then $\Theta_2(f)$ isn't contained in I, therefore $\Theta_1(f)$ (which is larger) isn't contained in I either, so f isn't contained in J.

Now suppose that $f \in F_J \setminus F_K$. If for each such f the set F_K includes at least one of the neighbors of f, then the distance between J and K is not more than 2ε. So, we can assume that F_K includes neither f nor its neighbors. Let g be the supremum of f and its neighbor(s). Then $\Theta_2(g) \subseteq I$. Let $L = \Theta_1(g)$ and $M = \Theta_2(g)$. Then the distance in $\mathcal{I}(D)$ between L and M is not more than δ. Therefore, the distance in $\mathcal{I}(C)$ between $\Psi_1(L)$ and $\Psi_1(M)$ should be no more than ε. But $M \subseteq I$, therefore $\Psi_1(M) \subseteq \Psi_1(I) = J$. In particular, f isn't contained in $\Psi_1(M)$. On the other side, g is contained in $\Psi_1\big(\Theta_1(g)\big) = \Psi_1(L)$. Therefore, the distance in Hausdorff metric between $\Psi_1(L)$ and $\Psi_1(M)$ is at least $2\varepsilon/3$. This is a contradiction. ∎

Proposition 7 *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every n and every two ideals $I, J \in \mathcal{I}(A_n)$ lying at the distance less than δ from each other in terms of the metric defined in Subsection 3.2, the Hausdorff distance between the preimages of I and J in $\mathcal{I}(A_1)$ is less than ε. (In other words, all the maps $\mathcal{I}(A_n) \to \mathcal{I}(A_1)$ have the common modulus of uniform continuity.)*

Proof First, suppose that both $\mathcal{I}(A_n)$ and $\mathcal{I}(A_1)$ are equipped with the metric defined in Subsection 3.2. Then the map $\mathcal{I}(A_n) \to \mathcal{I}(A_1)$ mapping every ideal to its preimage is a

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

contraction. Indeed, for each $I \in \mathcal{I}(A_n)$, let J be the preimage of I in $\mathcal{I}(A_1)$. The homomorphism $A_1/J \to A_n/I$ (which is induced from the given homomorphism $A_1 \to A_n$) is one-to-one, therefore isometric. Therefore for every $d \in A_1$: $\|d\|_J = \|d\|_I$. (Here d is identified with its image in A_n.) Therefore, the distance between I and J is larger than the distance between their preimages, as the former contains the same terms as the latter does, plus some additional terms.

Now, it suffices to let δ be the modulus of uniform continuity of the identity map from $\mathcal{I}(A_1)$ with the metrics coming from elements to itself with the Hausdorff metric, corresponding to ε. ■

Now we will build the "backwards" intertwining diagram analogous to the "forwards" diagram (3). Letting ε be subsequently equal to $1/2$, $1/4$, $1/8$ *etc.*, we find the corresponding values of δ in accordance with Proposition 7. By passing to an appropriate sub-diagram in (3), we can achieve that the tolerances of the triangles are not more than these values of δ. Then, subsequently applying Proposition 6, we obtain the intertwining backwards diagram like in the Mortensen's case:

$$\begin{array}{ccccccc}
\mathcal{I}(A_1) & \longleftarrow & \mathcal{I}(A_2) & \longleftarrow & \cdots & \longleftarrow & \mathcal{I}(A) \\
\uparrow & \swarrow & \uparrow & \swarrow & & & \updownarrow \\
\mathcal{I}(B_1) & \longleftarrow & \mathcal{I}(B_2) & \longleftarrow & \cdots & \longleftarrow & \mathcal{I}(B)
\end{array}$$

Applying Mortensen's existence and uniqueness theorems to every intertwining map in the above diagram, we can build the corresponding approximate intertwining of the C^*-algebras:

$$\begin{array}{ccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \\
B_! & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B
\end{array}$$

This gives the isomorphism $\rho\colon A \to B$.

3.7 $\check{\rho} = \Psi$

Proposition 8 *The map $\check{\rho}\colon \mathcal{I}(A) \to \mathcal{I}(B)$ arising from the isomorphism ρ as above coincides with the given isomorphism Ψ.*

Proof First, we prove that for $I \in \mathcal{I}(A)$, $\check{\rho}(I) \subseteq \Psi(I)$. For this, it's enough to check that $\rho(I) \subseteq \Psi(I)$. Let $x \in I$. Then $\rho(x) \in \check{\rho}(I)$.

May suppose that x is the image of some $y \in A_n$. Moreover, up to arbitrarily small ε, $\rho(x)$ is the image of the same y. Denoting all the images of the element y in all A_m by the same letter y, and denoting all the preimages of I by the same letter Y, we have:

$$\|Y\|_y = 0$$

Passing to the images of y and Y in B_m and denoting them again by the same letters y and Y, we have: $\|Y\|_y$ is arbitrarily small in B_m for sufficiently large m. On the other

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press

hand, by construction of the intertwining we have: $\|Y\|_y \to \|\Psi(I)\|_{\rho(y)}$ as $m \to \infty$. So, $\|\Psi(I)\|_{\rho(y)} = 0$ and $\rho(y) \in \Psi(I)$. Therefore, $\check{\rho}(I) \subseteq \Psi(I)$.

Now we have:

$$\rho\big(\rho^{-1}(I)\big) = I$$
$$\rho\big(\hat{\rho}(I)\big) = I$$
$$\check{\rho}\big(\hat{\rho}(I)\big) = I$$

So, $\check{\rho}(J) = (\hat{\rho})^{-1}(J)$ for $J \in \mathcal{I}(B)$. In addition, $\widehat{\rho^{-1}}(I) = \rho(I) = \check{\rho}(I)$. Therefore, $\widehat{\rho^{-1}}(I) \subseteq \Psi(I)$.

Exchanging the places of A and B we get the same results with ρ^{-1} instead of ρ and Ψ^{-1} instead of Ψ. Hence, for $J \in \mathcal{I}(B)$: $(\check{\rho})^{-1}(J) = \hat{\rho}(J) \subseteq \Psi^{-1}(J)$.

Therefore, all the four maps: $\check{\rho}$, $(\check{\rho})^{-1}$, Ψ, and Ψ^{-1} preserve inclusions. Let $J = \Psi(I)$, $K = \check{\rho}(I)$, $L = \Psi^{-1}(J)$, $M = \Psi^{-1}(K)$, and $N = (\check{\rho})^{-1}(K)$. We have:

1. $L = \Psi^{-1}\big(\Psi(I)\big) = I$ and $N = (\check{\rho})^{-1}\big(\check{\rho}(I)\big) = I$;
2. $N = (\check{\rho})^{-1}(K) \subseteq \Psi^{-1}(K) = M$;
3. $M = \Psi^{-1}(K) \subseteq \Psi^{-1}(J) = L$.

Therefore, $N = M = L = I$ and hence $J = K$. ■

References

[1] J. Dixmier, *Les C^*-Algèbres et leurs Représentations*. Gauthier-Villars, Paris, 1964.
[2] G. A. Elliott, *On the classification of C^*-algebras of real rank zero*. J. Reine Angew. Math. **443**(1993), 179–219.
[3] J. Mortensen, *Classification of certain non-simple C^*-algebras*. Preprints of Odense Universitet, Denmark, **9**, 1996.

Department of Mathematics
University of Toronto
Toronto, Ontario
M5S 3G3
email: elliott@math.toronto.edu

Department of Mathematics and Statistics
University of Calgary
2500 University Drive NW
Calgary, Alberta
T2N 1N4
email: ifulman@math.ucalgary.ca

https://doi.org/10.4153/CMB-2000-039-4 Published online by Cambridge University Press