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ON CONTINUOUS SOLUTIONS OF AN EQUATION OF THE GOŁĄB–SCHINZEL TYPE

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Abstract

We characterise solutions $f, g : \mathbb{R} \to \mathbb{R}$ of the functional equation $f(x + g(x)y) = f(x)f(y)$ under the assumption that f is continuous. Our considerations refer mainly to a paper by Chudziak ['Semigroup-valued solutions of the Gołąb–Schinzel functional equation', *Abh. Math. Semin. Univ. Hambg.* **76**, (2006), 91–98], in which the author studied the same equation assuming that g is continuous.

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In 1959 Gołąb and Schinzel [7] introduced the functional equation

$$f(x + f(x)y) = f(x)f(y) \tag{1}$$

in connection with looking for subgroups of the centroaffine group of a field. It turned out that the equation and its generalisations are one of the most important composite type functional equations because of their applications in the determination of substructures of algebraic structures [3], in the theory of geometric objects [1], in classification of near-rings [4] and quasialgebras [11], as well as in differential equations in meteorology and fluid mechanics [10]. That is why for over fifty years so many papers devoted to various generalisations of (1) have been published. (An extensive bibliography concerning the Gołąb–Schinzel type functional equations and their applications can be found in the survey paper [5].)

In [9], the author considered the most general equation of the Gołąb–Schinzel type, that is, the Pexiderised Gołąb–Schinzel equation

$$f(x + g(x)y) = h(x)k(y) \tag{2}$$

in the class of functions f, g, h, k mapping a linear space over a field $\mathbb{K}$ into $\mathbb{K}$. It has been proved that solutions of (2) can be described by solutions of the equation

$$f(x + g(x)y) = f(x)f(y). \tag{3}$$

Therefore, (3) plays one of the most important roles among equations of the Gołąb–Schinzel type.

This equation was considered for the first time by Chudziak [6]. He determined, among other things, solutions $f, g : \mathbb{R} \to \mathbb{R}$ of (3) under the assumption that the function g is continuous. Here we characterise solutions $f, g : \mathbb{R} \to \mathbb{R}$ of (3), where f is continuous. In fact, we prove that the continuity of f implies the continuity of g, provided f is not constant.

Equation (3) also generalises the well-known exponential equation

$$f(x + y) = f(x)f(y) \tag{4}$$

(for information on which we refer the reader to [2, pp. 25–33, 52–57]), as well as two other equations of the Gołąb–Schinzel type:

$$f(x + f(x)^n y) = f(x)f(y) \quad \text{for some } n \in \mathbb{N}$$

and

$$f(x + M(f(x))y) = f(x)f(y)$$

(which have been studied mainly by the author and by Brzdęk).

Throughout this paper we use the following notation:

$$A = f^{-1}(\{1\}), \quad B = g^{-1}(\{1\}), \quad W = f(\mathbb{R}) \setminus \{0\},$$
$$F = \{x \in \mathbb{R} : f(x) \neq 0\}, \quad G = \mathbb{R} \setminus F.$$

Let us recall some basic properties of functions satisfying (3).

Lemma 1 [8, Lemma 1]. *Let X be a real linear space, $f, g : X \to \mathbb{R}$, $f \neq 1$ and $f \neq 0$. If f and g satisfy (3), then the following hold.*

(i) $f(0) = 1$ *and* $g(0) \neq 0$.
(ii) $F = \{x \in X : g(x) \neq 0\}$.
(iii) $f(g(x)^{-1}(z - x)) = f(x)^{-1} f(z)$ *for every* $x \in F$ *and* $z \in X$.
(iv) $(y - x)/g(x) \in A$ *for every* $x, y \in F$ *with* $f(x) = f(y)$.
(v) f *and* $\bar{g}$ *satisfy (3), where*

$$\bar{g}(x) = \frac{g(x)}{g(0)} \quad \text{for each } x \in X. \tag{5}$$

Lemma 2 [8, Lemma 2]. *Let X be a real linear space, $f, g : X \to \mathbb{R}$, $f(X) \setminus \{0, 1\} \neq \emptyset$ and $g(X) \setminus \{0, 1\} \neq \emptyset$. If f and g satisfy (3) and A is a linear space, then there exists an $x_0 \in X \setminus A$ such that*

$$f^{-1}(\{f(x)\}) = (g(x) - 1)x_0 + A \quad \text{for each } x \in F.$$

First we prove a proposition which plays a very important role in the proof of the main theorem.

Proposition 3. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be nonconstant solutions of (3) such that f is continuous and $g(0) = 1$. Then there exist $c \in \mathbb{R} \setminus \{0\}$ and $r > 0$ such that f, g have*

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one of the following forms:

$$\begin{cases} g(x) = cx + 1 & \text{for } x \in \mathbb{R}, \\ f(x) = |cx + 1|^r & \text{for } x \in \mathbb{R}, \end{cases} \tag{6}$$

$$\begin{cases} g(x) = cx + 1 & \text{for } x \in \mathbb{R}, \\ f(x) = |cx + 1|^r \operatorname{sgn}(cx + 1) & \text{for } x \in \mathbb{R}, \end{cases} \tag{7}$$

$$\begin{cases} g(x) = \max\{0, cx + 1\} & \text{for } x \in \mathbb{R}, \\ f(x) = (\max\{0, cx + 1\})^r & \text{for } x \in \mathbb{R}. \end{cases} \tag{8}$$

PROOF. Setting $z = 0$ in Lemma 1(iii) and using Lemma 1(i), we obtain that W is a multiplicative group. Since f is nonconstant and continuous, there are $a, a^{-1} \in W \setminus \{1\}$ and the set W contains a closed interval I such that $1 \in \operatorname{int} I$. Hence, by the multiplicativity of W, $(0, \infty) \subset W$.

Observe that $A \cap B = \{0\}$. Clearly, by Lemma 1(i), $0 \in A \cap B$. Suppose that there is an $x \in (A \cap B) \setminus \{0\}$. Then, by (3), $f(x + y) = f(y)$ for each $y \in \mathbb{R}$. This means that f is continuous and periodic. Hence W is bounded, which is a contradiction.

In the next step we prove that $|g(x)| = 1$ for each $x \in A$. To this end, we consider two cases.

First, suppose that there is an $x \in A$ such that $|g(x)| < 1$. According to Lemma 1(ii), $g(x) \neq 0$. Then, by (3), using induction,

$$\begin{aligned} f(y) &= f(x)^n f(y) = f(x)^{n-1} f(x + g(x)y) \\ &= f(x)^{n-2} f(x + g(x)(x + g(x)y)) = f(x)^{n-2} f(x(1 + g(x)) + g(x)^2 y) \\ &= \cdots = f(x(1 + g(x) + \cdots + g(x)^{n-1}) + g(x)^n y) \\ &= f\Big(x\frac{1 - g(x)^n}{1 - g(x)} + g(x)^n y\Big) \end{aligned}$$

for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus, by the continuity of f,

$$f(y) = \lim_{n\to\infty} f\Big(x\frac{1 - g(x)^n}{1 - g(x)} + g(x)^n y\Big) = f\Big(\frac{x}{1 - g(x)}\Big) \quad \text{for each } y \in \mathbb{R}.$$

This is a contradiction, because f is not constant.

Next, suppose that there is an $x \in A$ such that $|g(x)| > 1$. Then, by Lemma 1(iii), using induction,

$$\begin{aligned} f(y) &= f(x)^{-n} f(y) = f(x)^{-n+1} f\Big(\frac{y - x}{g(x)}\Big) \\ &= f(x)^{-n+2} f\Big(\frac{(y - x)/g(x) - x}{g(x)}\Big) = f(x)^{-n+2} f\Big(\frac{y - x(1 + g(x))}{g(x)^2}\Big) \\ &= \cdots = f\Big(\frac{y - x(1 + g(x) + \cdots + g(x)^{n-1})}{g(x)^n}\Big) \end{aligned}$$

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$$= f\Big(g(x)^{-n}y - x\frac{1-g(x)^n}{1-g(x)}g(x)^{-n}\Big)$$
$$= f\Big(g(x)^{-n}y - x\frac{g(x)^{-n}-1}{1-g(x)}\Big)$$

for every $y \in \mathbb{R}$ and $n \in \mathbb{N}$. Using the continuity of f, we get

$$f(y) = \lim_{n\to\infty} f\Big(g(x)^{-n}y - x\frac{g(x)^{-n}-1}{1-g(x)}\Big) = f\Big(\frac{x}{1-g(x)}\Big) \quad \text{for } y \in \mathbb{R},$$

which is a contradiction.

In this way, we have proved that $g(x) = 1$ or $g(x) = -1$ for each $x \in A$. Moreover, according to (3), $f(z - x) = f(x)$ for every $z \in A$ with $g(z) = -1$ and $x \in \mathbb{R}$. This means that f is symmetric in a line $x = z/2$ for each $z \in A$ with $g(z) = -1$. Suppose that there are $z_1, z_2 \in A \cap g^{-1}(\{-1\})$ with $z_1 \neq z_2$. Then $f(z_1 - x) = f(x) = f(z_2 - x)$ for $x \in \mathbb{R}$ and thus $f(z_1 - z_2 + x) = f(x)$ for $x \in \mathbb{R}$. Since f is periodic and continuous, f has to be bounded, which contradicts $(0, \infty) \subset W$. Consequently, $A \cap g^{-1}(\{-1\}) \in \{\emptyset, \{x_0\}\}$ for an $x_0 \neq 0$.

Since $A \cap B = \{0\}$, we obtain that either $A = \{0\}$, or $A = \{0, x_0\}$ for an $x_0 \neq 0$ with $g(x_0) = -1$.

Case 1. First consider the case in which $A = \{0, x_0\}$ with $x_0 \neq 0$ such that $g(x_0) = -1$. Then f is symmetric in a line $x = x_0/2$. Assume that $x_0 > 0$. (If $x_0 < 0$, the proof is analogous.) According to (3), $f(x_0 - x) = f(x) \neq 0$ for each $x \in F$. Hence, by Lemma 1(iv),

$$\frac{x_0 - 2x}{g(x)} \in A = \{0, x_0\} \quad \text{for } x \in F.$$

Thus, either $x = x_0/2$, or $x \neq x_0/2$ and $(x_0 - 2x)/g(x) = x_0$ for $x \in F$. Consequently,

$$g(x) = -\frac{2}{x_0}x + 1 \quad \text{for } x \in F \setminus \{x_0/2\}.$$

Suppose that $x_0/2 \in F$. Then, in view of (3), for each $x \in F \setminus \{x_0/2\}$,

$$f\Big(\frac{x_0}{2}\Big)f(x) = f\Big(x + g(x)\frac{x_0}{2}\Big) = f\Big(x + \Big(-\frac{2}{x_0}x + 1\Big)\frac{x_0}{2}\Big) = f\Big(\frac{x_0}{2}\Big) \neq 0.$$

Thus $f(x) = 1$ for $x \in F \setminus \{x_0/2\}$ and, consequently, by the continuity of f, $f = 1$. But f cannot be constant. This contradiction proves that $f(x_0/2) = 0$. Hence

$$g(x) = -\frac{2}{x_0}x + 1 \quad \text{for each } x \in F. \tag{9}$$

In the next step, we prove that f is one-to-one on the set $F \cap (-\infty, x_0/2)$. To the contrary, suppose that there are $x_1 < x_2 < x_0/2$ such that $f(x_1) = f(x_2) \neq 0$.

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Then, since f is symmetric in the line $x = x_0/2$, there is an $x_3 > x_0/2$ such that $f(x_1) = f(x_2) = f(x_3)$. Thus, in view of Lemma 1(iv),

$$\frac{x_1 - x_2}{g(x_2)}, \frac{x_3 - x_2}{g(x_2)} \in A = \{0, x_0\}.$$

Moreover, $x_1 - x_2 < 0$, $x_3 - x_2 > 0$ and hence

$$\frac{x_1 - x_2}{g(x_2)} = x_0 = \frac{x_3 - x_2}{g(x_2)}.$$

This means that $g(x_2) > 0$ and $g(x_2) < 0$, which is a contradiction.

Since f is continuous, $f|_{F\cap(-\infty,x_0/2)}$ is one-to-one, $x_0/2 > 0$, $f(0) = 1$ and $f(x_0/2) = 0$, we obtain that $(-\infty, 0] \subset F$. Hence $G \subset (0, \infty)$. Moreover, the set G is closed because of the continuity of f. Let $z_0 = \min G$. Suppose that $z_0 < x_0/2$. Since $f|_{F\cap(-\infty,x_0/2)}$ is one-to-one, $f|_{[z_0,x_0/2]} = 0$. Hence, according to (9),

$$0 = f(y)f(z_0) = f(y + g(y)z_0) = f\Big(y - \frac{2}{x_0}yz_0 + z_0\Big) \quad \text{for each } y \in F.$$

This means that $y(1 - (2/x_0)z_0) + z_0 \geq z_0$ for $y \in F$. Since $1 - (2/x_0)z_0 > 0$, we have $y \geq 0$ for $y \in F$. But $(-\infty, 0] \subset F$. This contradiction proves that $z_0 = x_0/2$. Thus, using the symmetry of f in the line $x = x_0/2$, $G = \{x_0/2\}$. Hence, according to (9), setting $c = -2/x_0$,

$$g(x) = cx + 1 \quad \text{for each } x \in \mathbb{R}. \tag{10}$$

Case 2. Now we consider the second case, in which $A = \{0\}$. Then, by Lemma 1(iv), $(x - y)/g(y) \in A = \{0\}$ for every $x, y \in F$ with $f(x) = f(y)$. Hence $f|_F$ is one-to-one. Thus, in view of Lemma 2, there is an $x_0 \in \mathbb{R} \setminus \{0\}$ such that $x = (g(x) - 1)x_0$ for each $x \in F$. This means that

$$g(x) = \frac{x}{x_0} + 1 \quad \text{for each } x \in F. \tag{11}$$

Assume that $x_0 < 0$. (The case where $x_0 > 0$ is analogous.) In view of (11) and Lemma 1(ii) we have $f(-x_0) = 0$.

Since f is continuous, $f|_F$ is one-to-one, $-x_0 > 0$, $f(0) = 1$ and $f(-x_0) = 0$, we obtain that $(-\infty, 0] \subset F$ and $G \subset (0, \infty)$. The set G is closed because of the continuity of f, so there exists a $z_0 = \min G$. Suppose that $z_0 < -x_0$. Then $f|_{[z_0,-x_0]} = 0$ because $f|_F$ is one-to-one. Hence, according to (11),

$$0 = f(x)f(z_0) = f(x + g(x)z_0) = f\Big(x + \frac{x}{x_0}z_0 + z_0\Big) \quad \text{for each } x \in F.$$

This means that $x(1 + z_0/x_0) + z_0 \geq z_0$ for $x \in F$. Since $1 + z_0/x_0 > 0$, we have $x \geq 0$ for $x \in F$, which contradicts $(-\infty, 0] \subset F$. Consequently, $-x_0 = z_0 = \min G$.

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Now we prove that $G \in \{\{-x_0\}, [-x_0, \infty)\}$. To the contrary, suppose that there is a $z_1 = \max G$ such that $z_1 > -x_0$. Since $f|_F$ is one-to-one, $G = [-x_0, z_1]$. In view of (11),

$$0 = f(x)f(z_1) = f(x + g(x)z_1) = f\Big(x + \frac{x}{x_0}z_1 + z_1\Big) \quad \text{for each } x \in F.$$

Thus $x(1 + z_1/x_0) + z_1 \le z_1$ for $x \in F$. Moreover, $1 + z_1/x_0 < 0$, so $x \ge 0$ for $x \in F$. This contradicts $(-\infty, 0] \subset F$. Hence either $-x_0 = z_1 = \max G$, or $\max G$ does not exist. Consequently, $G = \{-x_0\}$ or $G = [-x_0, \infty)$. Setting $c = 1/x_0$, in view of (11), if $G = \{-x_0\}$, then g is given by (10); otherwise,

$$g(x) = \max\{cx + 1, 0\} \quad \text{for } x \in \mathbb{R}. \tag{12}$$

In this way we have proved that g has one of the forms (10) and (12). Now define a function $\phi : \mathbb{R} \to \mathbb{R}$ as follows:

$$\phi(x) = f\Big(\frac{x-1}{c}\Big) \quad \text{for each } x \in \mathbb{R}.$$

Since f is continuous and nonconstant, so is ϕ. Moreover, according to (3), for every $x, y \in \mathbb{R}$,

$$\begin{aligned}\phi(x)\phi(y) &= f\Big(\frac{x-1}{c}\Big)f\Big(\frac{y-1}{c}\Big)\\ &= f\Big(\frac{x-1}{c} + x\frac{y-1}{c}\Big) = f\Big(\frac{xy-1}{c}\Big) = \phi(xy).\end{aligned}$$

This means that ϕ is multiplicative. Hence, by [2, p. 31, Corollary 9], either

$$\phi(x) = |x|^r \quad \text{for } x \in \mathbb{R},$$

or

$$\phi(x) = |x|^r \operatorname{sgn} x \quad \text{for } x \in \mathbb{R}$$

with some $r > 0$. Thus, either

$$f(x) = \phi(cx + 1) = |cx + 1|^r \quad \text{for } x \in \mathbb{R}$$

and (6) holds, or

$$f(x) = \phi(cx + 1) = |cx + 1|^r \operatorname{sgn}(cx + 1) \quad \text{for } x \in \mathbb{R}$$

and (7) holds.

Finally, if g is given by (12), then we define a function $\phi : [0, \infty) \to \mathbb{R}$ by the formula

$$\phi(x) = f\Big(\frac{x-1}{c}\Big) \quad \text{for each } x \ge 0.$$

As before, we obtain that ϕ is continuous, nonconstant and multiplicative. Hence, by [2, p. 30, Proposition 6],

$$\phi(x) = x^r \quad \text{for } x \ge 0$$

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with some $r > 0$. Thus

$$f(x) = \phi(cx + 1) = (cx + 1)^r \quad \text{for } x \in \mathbb{R} \text{ with } cx + 1 \geq 0$$

and, in view of (12) and Lemma 1(ii), $f(x) = 0$ for $x \in \mathbb{R}$ with $cx + 1 < 0$. Consequently

$$f(x) = (\max\{0, cx + 1\})^r \quad \text{for } x \in \mathbb{R},$$

which ends the proof of (8). □

Now we are in a position to prove our main result.

Theorem 4. *Functions $f, g : \mathbb{R} \to \mathbb{R}$ satisfy (3) and f is continuous if and only if one of the following conditions holds.*

(i) *g is arbitrary and either $f = 0$, or $f = 1$.*
(ii) *$g = 1$ and there is a $c \in \mathbb{R} \setminus \{0\}$ such that $f(x) = \exp(cx)$ for $x \in \mathbb{R}$.*
(iii) *There are $c \in \mathbb{R} \setminus \{0\}$ and $r > 0$ such that f and g have one of the forms (6)–(8).*

Proof. Let f and g satisfy (3). Clearly, if f is constant, then either $f = 1$, or $f = 0$. So assume that f is not constant. Then, by Lemma 1(vi), f with $\bar{g}$ given by (5) also fulfill (3) and $\bar{g}(0) = 1$.

If $\bar{g} = 1$, using (3) we find that f is a nonconstant continuous solution of (4) and hence, according to [2, p. 29, Theorem 5], there is a $c \in \mathbb{R} \setminus \{0\}$ such that $f(x) = \exp(cx)$ for $x \in \mathbb{R}$. Hence, by (3),

$$c(x + g(x)y) = cx + cy \quad \text{for every } x, y \in \mathbb{R}.$$

Thus $g = 1$. In this way we have proved that (ii) holds.

Now assume that both functions f and $\bar{g}$ are not constant. Then, in view of Proposition 3, f and $\bar{g}$ satisfy one of conditions (6)–(8). We prove that $g(0) = 1$. Write $g(0) = a$. Since f, g as well as $f, \bar{g}$ satisfy (3),

$$f(x + g(x)y) = f(x)f(y) = f(x + \bar{g}(x)y) \quad \text{for } x, y \in \mathbb{R}.$$

Hence, setting $x = 0$ and using $\bar{g}(0) = 1$,

$$f(ay) = f(y) \quad \text{for each } y \in \mathbb{R}.$$

If f and $\bar{g}$ are given by (6), then

$$|ca + 1|^r = f(a) = f(1) = |c + 1|^r,$$
$$|2ca + 1|^r = f(2a) = f(2) = |2c + 1|^r.$$

Hence either $a = 1$, or $a \neq 1$ and $a = -1 - 2/c = -1 - 1/c$. Thus $a = 1$.

If f and $\bar{g}$ are given by (7), then f is injective. Hence $f(a) = f(1)$ implies $a = 1$.

Finally, let f and $\bar{g}$ be given by (7). Then

$$(\max\{0, ca + 1\})^r = f(a) = f(1) = (\max\{0, c + 1\})^r$$

implies that

$$\text{either } \begin{cases} ca+1>0, \\ c+1>0 \end{cases} \quad \text{or} \quad \begin{cases} ca+1\le 0, \\ c+1\le 0. \end{cases}$$

If $ca+1>0$ and $c+1>0$, then

$$(ca+1)^r = f(a) = f(1) = (c+1)^r;$$

otherwise, $ca+1\le 0$ and $c+1\le 0$, which implies $-ca+1>0$, $-c+1>0$ and

$$(-ca+1)^r = f(-a) = f(-1) = (-c+1)^r.$$

In both cases $a=1$.

In this way we have proved that $g(0)=1$. Then, according to (5), $g=\bar{g}$ and consequently f and g have one of the forms (6)–(8).

It may be checked that functions f and g which have one of the forms (6)–(8) satisfy (3). This ends the proof. □

References

[1] J. Aczél, 'Beiträge zur Theorie der geometrischen Objekte', *Acta Math. Acad. Sci. Hungar.* **8** (1957), 19–52.

[2] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, 31 (Cambridge University Press, Cambridge, 1989).

[3] J. Aczél and S. Gołąb, 'Remarks on one–parameter subsemigroups of the affine group and their homo– and isomorphism', *Aequationes Math.* **4** (1970), 1–10.

[4] E. Aichinger and M. Farag, 'On when the multiplicative center of a near-ring is a subnear-ring', *Aequationes Math.* **68** (2004), 46–59.

[5] J. Brzdęk, 'The Gołąb–Schinzel equation and its generalizations', *Aequationes Math.* **70** (2005), 14–24.

[6] J. Chudziak, 'Semigroup–valued solutions of the Gołąb–Schinzel functional equation', *Abh. Math. Semin. Univ. Hambg.* **76** (2006), 91–98.

[7] S. Gołąb and A. Schinzel, 'Sur l'équation fonctionnelle $f(x+f(x)y)=f(x)f(y)$', *Publ. Math. Debrecen* **6** (1959), 113–125.

[8] E. Jabłońska, 'Continuous on rays solutions of an equation of the Gołąb–Schinzel type', *J. Math. Anal. Appl.* **375** (2011), 223–229.

[9] E. Jabłońska, 'Remarks concerning the pexiderized Gołąb-Schinzel functional equation'. *J. Math. Appl.*, in press.

[10] P. Kahlig and J. Matkowski, 'A modified Gołąb–Schinzel equation on a restricted domain (with applications to meteorology and fluid mechanics)', *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **211** (2002), 117–136.

[11] P. Plaumann and S. Strambach, 'Zweidimensionale Quasialgebren mit Nullteilern', *Aequationes Math.* **15** (1977), 249–264.

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