2

What Time Reversal Means

Précis. *The meaning of time reversal can be reconstructed from the structure of time translations. 'Instantaneous' time reversal is just its proxy in a state space representation.*

The Russian physicist Vladimir Fock is rumoured to have remarked:

> Physics is essentially a simple science. The main problem is to understand which symbol means what.[1]

What does time reversal mean? Its textbook presentation is a little mysterious. On the one hand, time reversal is presented as a simple mapping $t \mapsto -t$, with the explanation that it 'reverses the arrow of time'. On the other hand, it is treated as a richly structured concept that reverses all sorts of quantities, like momenta, magnetic fields, and spin. It even conjugates wave functions. Where does all this structure come from? Received opinion is that it comes from 'motion reversal', not 'time reversal'. But 'motion reversal' is rarely defined, and the relationship between the map $t \mapsto -t$ and all of that rich structure is rarely made clear. Philosophers and physicists have made some progress in clearing the fog, but disagreement remains.

This chapter will show how elementary considerations from representation theory explain the mystery. I will argue that it is firstly a matter of determining which symbol means what: the time reversal transformation $t \mapsto -t$ is not a map on time coordinates like $t =$ two o'clock, but rather a map on time *translations* like $t =$ time-shift by two hours. All the rich

[1] Khriplovich and Lamoreaux (1997, p.53).

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structure of time reversal that gets referred to as 'motion reversal' arises out of how time translations are represented on a state space, which I will derive in Chapter 3. All that structure is not just motion reversal: it is a representation of time reversal.

The central idea that I will present is a general perspective on how spacetime symmetries appear in state space, which I call the *Representation View*. In short, I will propose that we separate our discussions of time into two components: time translations in spacetime and time translations in state space, which are related by a representation. This simple idea is so powerful that it will carry us through several of the arguments in this book. In this chapter, I will use it to clarify how time reversal gets its meaning.

I begin by setting out the problem of defining time reversal in Section 2.1 and then recounting in Section 2.2 how two competing camps have responded. Section 2.3 then presents a first look at the Representation View, which I claim mediates between these camps in a way that dissolves the debate. Section 2.4 presents the philosophy of time that will underlie my response: that time is not a mere ordered set but has 'relational' or 'structural' properties like a Lie group of time translations. Drawing on structuralist and functionalist perspectives for inspiration, Section 2.5 argues that time reversal is a reversal of time translations; and Section 2.6 recovers its structure as a group element from the structure of time translations. Finally, the meaning of time reversal in state space is presented in Section 2.7, as the transformation corresponding to $\tau : t \mapsto -t$ in a state space representation.

2.1 The Definability Problem

Some favourite interpretative strategies in the philosophy of physics appear unhelpful when applied to time reversal. For example, take Percy Bridgman's operationalism. Bridgman performed experiments at higher pressures than had ever been achieved before; in 1922, one exploded, killing two people and destroying his basement laboratory in the Jefferson physics building at Harvard – even shattering the windows of the nearby law school.[2] In high-pressure environments where existing gauges blow, it wasn't clear to Bridgman how 'pressure' should be defined. His response, inspired by Einstein's (1905) account of simultaneity, was to interpret all meaning as deriving from physical operations:

[2] As recounted by Walter (1990, pp.37–8). Sadly, Bridgman's research assistant Atherton Dunbar and carpenter William Connell were both killed by the explosion, and eight graduate students in the room above were injured.

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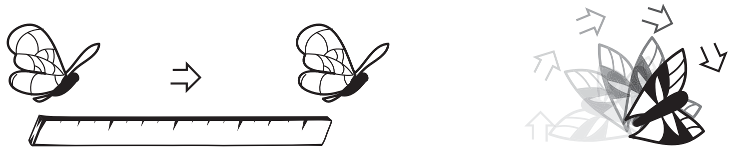

Figure 2.1 Operational definitions of spatial translation and rotation.

> We may illustrate by considering the concept of length: what do we mean by the length of an object? . . . To find the length of an object, we have to perform certain physical operations. The concept of length is therefore fixed when the operations by which length is measured are fixed: that is, the concept of length involves as much as and nothing more than the set of operations by which length is determined. (Bridgman 1927, p.5)

Setting aside the radical implications of operationalism for physics,[3] it appears that spatial translation and rotation can be defined operationally: a spatially translated subsystem is one resulting from the operation of rigid motion along a ruler, while a spatially rotated one results from rigid rotation about an axis, as shown in Figure 2.1. Time translation can be defined operationally too, by comparison of a subsystem to the state of a clock.

But, operationalism seems at best unhelpful when it comes to time reversal. What operation, short of science fiction, would reverse time? Physicists and philosophers tend to reply with analogies to films played in reverse. But this strategy is imprecise, and its relevance is questionable. This is especially true for phenomena like the time reversal violating neutral kaon: apart from its inconvenient invisibility in a spark chamber, the typical lifetime of even the 'long-lived' neutral kaon state K_L is around 5×10^{-8} seconds. In this case, the film analogy would require an impossibly high frame-rate of over 20 million frames per second.

As a matter of convention, one can give the phrase 'time reversal' whatever definition one wishes. Many physicists do. But, if no justification were given, there would be no connection between time reversal and any non-conventional arrow of time. We would lose the physics community's poetic conclusion that, following the experimental discovery of time reversal violation, we learned that "at least one theme is played more slowly" backwards in time than forwards.[4] Call this the *definability problem:* How can one

[3] As Bridgman himself notes, operationalism complicates the meaning of simple concepts, since (for example) measuring stellar lengths in light years and measuring laboratory lengths with rulers require different operations, resulting in multiple concepts of 'length'.

[4] See Section 1.5.

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justify a definition of time reversal in a way that makes it relevant to the arrow of time in our world?

2.2 Two Competing Camps

Philosophers and physicists have responded to the definability problem by separating it into two parts: an order reversing part and an instantaneous part. David Albert (2000, §1) explains these two parts using the term *instantaneous state* to refer to a point in a typical state space, which describes the world at an instant: a distribution of particles, a probability distribution, a description of an electric field on a spacelike surface, and so on.[5] Examples of a state space include a Hilbert space or a phase space, as discussed in more detail in Chapter 3. Albert uses *temporal sequence* to refer to any one-parameter set of these states, parametrised by time. Then the definition of 'time reversal' could include one or both of the following:

1. *Time order reversal:* transformation of a temporal sequence of instantaneous states to a sequence with the very same elements, but with the reverse ordering.
2. *Instantaneous reversal:* transformation of each instantaneous state to a 'reversed' instantaneous state.

Viewing a sequence of instantaneous states as a stack of pancakes, the first transformation reverses the order of items in the stack, while the second transformation turns around the individual pancakes, as shown in Figure 2.2.

Although standard textbook treatments take time reversal to involve both, the first is much easier to understand. Sachs even begins his classic book, *The Physics of Time Reversal*, with the proposal that "[t]he qualitative meaning of the time variable is that of an ordering parameter in one-to-one

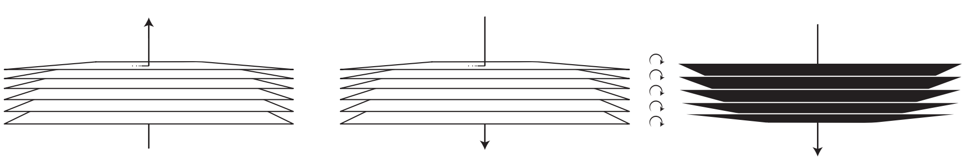

Figure 2.2 A time ordering (left), its time order reversal (middle), and time order reversal together with instantaneous reversal (right).

[5] Albert further asks that these descriptions be 'complete' and 'genuinely instantaneous', as I discuss in Sections 2.5.1 and 3.1.1. See also Butterfield (2006a,b) for a similar view. I view my account as compatible with both.

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correspondence to a sequence of events" (Sachs 1987, p.4). Defining 'time' to be an instance of an ordering makes the statement 'Time reversal is ordering reversal' into a trivial analytic truth, analogous to 'A bachelor is unmarried'. But, this definition is plausible whether or not we share Sachs' quoted view on time. Time order reversal can even be operationally defined: take two clocks, one that counts down and the other that counts up, and associate each clock-time with a physical event, as in Figure 2.3. The melting ice cubes might suggest that time 'goes up the page'. But, operationally, each clock determines a sequence of states ordered by increasing clock-times, and each sequence can be defined as the time order reverse of the other.

In contrast, instantaneous reversal appears a bit more mysterious. Of course, a time order reversal of the form $t \mapsto -t$ induces a transformation of velocity $v = dx/dt$ to its negative, $v \mapsto -v$. But, textbooks treat many other quantities as transforming under time reversal as well: spin, momentum, and magnetic fields all reverse sign under time reversal, even though none of these are typically written as a rate of change of some parameter. Even a magisterial treatise on time reversal like the book *CP Violation* by Bigi and Sanda (2009, p.5), for all its virtues, leaves room for confusion when they say that time reversal "can be viewed as reversal of motion", and then – in the same paragraph – that it provides evidence that "nature makes a difference between past and future". What then is the link between these two?

Physicists and philosophers have responded to this situation in two ways. The first camp, containing a majority of physicists and philosophers,

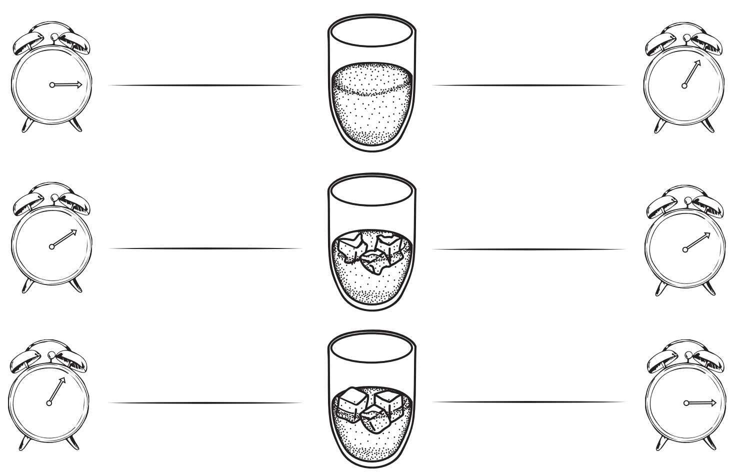

Figure 2.3 Operational definition: the orderings determined by increasing clock times for the left and right clocks are the time order reverse of each other.

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emphasises the first claim of Bigi and Sanda: they accept instantaneous reversal, while still insisting that time reversal is 'really' motion reversal. This tradition goes right back to Wigner's founding discussion of time reversal introduced in Section 1.4, although Wigner uses the phrase 'time inversion' in place of time reversal: "'reversal of the direction of motion' ['Bewegungsumkehr'] is perhaps a more felicitous, though longer, expression than 'time inversion"' (Wigner 1931, p.325).[6] The second camp rejects instantaneous reversal: as Albert (2000, p.18) exclaims, "What can it possibly mean for a single instantaneous physical situation to be happening '*backward*'?" He answers: "gibberish", and Callender (2000, p.254) agrees. They conclude that time reversal can only mean the conceptually less-problematic concept of time order reversal.[7]

The first camp has been defended in a variety of ways. Sachs (1987, p.23) suggests that the hidden velocities associated with "microscopic current structure" explain the reversal of the magnetic field. For other quantities, such as those in quantum theory, it is common to appeal to rules about a classical-to-quantum correspondence: if classical momentum transforms as $p \mapsto -p$, then it is not so far-fetched to imagine that quantum momentum transforms in the same way. John Earman gives an argument like this, in his explanation of why time reversal ought to conjugate position wavefunctions:

> Since $\psi(\mathbf{x}, 0)$ encodes information about the momentum of the particle it must be 'turned around' or non-trivially transformed by the time reversal operation so that the reversed state at $t = 0$ describes the wave packet propagating in the opposite direction. So instead of making armchair philosophical pronouncements about how the state cannot transform, one should instead be asking: How can the information about the direction of motion of the wave packet be encoded in $\psi(\mathbf{x}, 0)$? Well (when you think about it) the information has to reside in the phase relations of the components of the superposition that make up the wave packet. And from this it follows that the time reversal operation must change the phase relations. (Earman 2002b, p.248)

Although some find it intuitive to motivate time reversal on the basis of classical 'motion reversal', it's hard not to see this as a change of subject, from

[6] After Wigner (1931), two prominent examples are the wonderful textbooks by Sakurai and by Ballentine: "*time reversal* is a misnomer; it reminds us of science fiction. Actually what we do in this section can be more appropriately characterized by the term *reversal of motion*" (Sakurai 1994, p.266); and, "the term 'time reversal' is misleading, and the operation that is the subject of this section would be more accurately described as *motion reversal*" (Ballentine 1998, p.377). Among philosophers, see Earman (2002b, p.247): "the reversal of motion prescription for particles forces out the standard account of the time reversal behaviour of electromagnetic fields"; and Uffink (2001, p.314): "the term 'time reversal' is not meant literally. That is to say, we consider processes whose reversal is or is not allowed by a physical law, not a reversal of time itself".

[7] See Albert (2000, §1) and Callender (2000) for statements of this view, which both physicists and philosophers have endorsed; cf. Allori (2015), Castellani and Ismael (2016), and Rachidi, Rubinstein, and Paolone (2017, §§1.3.6 and 1.3.7).

'time reversal' to 'motion reversal'. It is also not always clear how to apply it. For example, the term 'motion reversal' alone offers little advice about how internal degrees of freedom transform under time reversal. However, not all such defences appeal to motion reversal: Malament (2004) and Roberts (2017) give derivations of time reversal in electromagnetism and quantum theory, respectively, which appeal directly to facts about time.

Still, all this does seem to support Albert's summary of the textbook account:

> (1) that in the case of Newtonian mechanics the procedure is 'obviously' to reverse the velocities of all the particles, and to leave everything else untouched; and (2) that the question needs to be approached afresh (but with the Newtonian case always somehow in the back of one's head) in each new theory one comes across; and (3) that what it is *in all generality* for one physical situation to be the time reverse of another is (not surprisingly!) an obscure and difficult business. (Albert 2000, p.18)

To those who would conclude that time reversal is obscure or difficult, Albert responds: "It isn't, really". A conceptually clearer alternative, according to him, is to say that time reversal is just the reflection of time, $t \mapsto -t$.

I agree with this alternative, although I will soon argue that this does not settle the debate in favour of the first camp. We should all agree that, at the end of the day, it must be possible to view time reversal as a conceptually simple operation, which does nothing more than reverse time. If we are to continue referring to it as 'time reversal' with any honesty, then all its bells and whistles must ultimately derive from this touchstone. And yet, as Callender (2000, p.262) says, "many arguments attempt to blur the difference" between temporal reflection $t \mapsto -t$ and the transformation of other degrees of freedom associated with motion. In an effort to restore clarity, Callender even refers to the former as 'time reversal' and the latter 'Wigner reversal'; may each be worthy of the name.

I will argue that Callender's distinction is unnecessary, because the two amount to the same thing. The first camp's beautifully simple intuition is correct, that time reversal is a map that transforms time to its negative. However, more is needed to understand it in state space, where there is room for confusion about what Albert and Callender say. Referring to the highly structured state space of quantum mechanics, associated with a Hilbert space $\mathcal{H}$, Callender considers a curve $\psi(t)$ indexed by a parameter t associated with time, and says:

> It does not logically follow, as it does in classical mechanics, that the momentum or spin must change signs when $t \mapsto -t$. Nor does it logically follow from $t \mapsto -t$ that one must change $\psi \mapsto \psi^*$. (Callender 2000, p.23)

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Here I disagree. Similarly, I think Albert goes too far when he writes about electromagnetism that,

> Magnetic fields are *not* the sorts of things that any proper time-reversal transformation can possibly turn around. Magnetic fields are not – either logically or conceptually – the *rates of change* of anything. (Albert 2000, p.20)

On the face of it, these authors seem to reject the standard dogma that time reversal conjugates wavefunctions and reverses magnetic fields. Some physicists have even followed suit in this observation, distinguishing the 'strict' time reversal of Albert and Callender, which is not a symmetry of most physical theories, from the 'soft' time reversal of most physicists which often is a symmetry (cf. Rachidi, Rubinstein, and Paolone 2017, §1.3.4).

That sort of conclusion goes too far, and these statements have correspondingly been met with widespread criticism.[8] But, this should not distract from the basic insight of Albert and Callender, which I will argue is correct: time reversal simply reverses time. Albert and Callender are even right to say that the transformation rules for time reversal on state space do not logically follow from this alone; however, once the meaning of 'time's passage' in state space has been specified, I will argue that they do.

Thus, to be clear: I also endorse the instantaneous reversal transformation of the second camp. However, I would like to offer a path of reconciliation. Let me characterise their essential claims as follows:

Time Reflection Camp: Time reversal is just time order reversal: a reflection of time's arrow.
Instantaneous Camp: Time reversal is time order reversal, together with a non-trivial transformation of instantaneous states in a space of states.

In this chapter and the following, I will show a sense in which these two views are in fact one and the same. This has only been obscured because our perspective on time has been too simple. In the next section, I will outline a more careful view that corrects this.

2.3 The Representation View

The Time Reflection Camp describes time reversal as a reflection of time, $t \mapsto -t$. The Instantaneous Camp describes it as including a transformation

[8] Cf. Arntzenius (2004), Arntzenius and Greaves (2009), Butterfield (2006b, §3.3), Earman (2002b), Malament (2004), North (2008), Peterson (2015), and Roberts (2012, 2017, 2021).

T on state space. In this section, I will present a general view of spacetime symmetries that mediates between the two camps and argue that it settles the debate. My discussion here will be relatively informal; the details will then follow in the remainder of this chapter and the next.

In physics and its philosophy, we often speak loosely of a symmetry as if its meaning were fixed in all contexts. For example, we speak of rotations and time translations in spacetime, and then again of rotations and time translations in a state space, like a Hilbert space. But, mathematically and conceptually, spacetime and state space are two different things. When we define a concrete symmetry transformation, we must specify what it transforms: Is it a transformation of spacetime, or a transformation of a state space? We cannot have it both ways.

When we do use the same language for symmetries in two contexts, it is justified only because there is a systematic relationship between them. To make that explicit, we can imagine taking the concept of a symmetry and 'pulling it apart' into (at least) two separate concepts: spacetime symmetries on the one hand and state space symmetries on the other. What is the relationship between these concepts?

This question is answered by a framework for characterising dynamical systems that is well-known in physics, originally appearing in the great work of Wigner (1939) on the representation of Minkowski spacetime symmetries on Hilbert space.[9] The framework has two components: first, there is a structure characterising a set of symmetry transformations, such as those of spacetime or of a gauge theory; second, there is a structure describing the symmetries of a space of states. I will discuss the meaning of both in more detail in what follows. The relationship between them is sometimes called a *group action*, but I will call it a *representation:* a homomorphism (or 'structure-preserving map') from the first structure to the other. This ensures that a 'structure-preserving copy' of the first symmetries are living inside the symmetries of state space.

All that I would like to add to this well-known view is an interpretive proposal, that a representation is what gives meaning to the concept of a

[9] Although Bargmann (1954) and Wigner (1939) together form the *locus classicus* for this framework, the basic idea is implicit in Sophus Lie's (1893) analysis of differential equations using what are now called Lie groups. Wigner himself traces the idea to the work of Ettore Majorana (1932) on representing rotations of fermions; Wigner's paper was submitted just months before Majorana's mysterious disappearance in March 1938 at age 32, on a boat leaving Palermo, never to be seen again (Recami 2019).

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'spacetime symmetry' in the context of a state space. This simple, guiding symmetry principle is what underpins the argument of virtually every chapter in this book. So, let me set it out clearly for reference:

The Representation View: A symmetry of a state space can be interpreted as a 'spacetime symmetry' only if it is an element of a *representation* of a spacetime symmetry structure.

The Representation View is a wide-ranging perspective, which can be applied to virtually any spacetime symmetries and any state space structure. Among other things, the spacetime symmetries can be Newtonian or Minkowski; state space can be Newtonian configuration space or quantum theory. The view can even be given a local expression, using the concept of a local Lie group, in order to be applied in general relativity.[10] However, for most purposes in this book, and especially this chapter, the particular symmetries of interest will be the *time translations*.

In the special case of time translations, the Representation View says that a symmetry of state space only deserves to be called a 'time translation' if it is a representation of the time translation group. This requires us to 'pull apart' the concept of a time translation and identify two separate concepts: time translations in spacetime and time translations as state space symmetries. The connection between them is then given by a representation, which ensures there is a homomorphic copy of the spacetime time translations in state space, as in the state space diagram for a harmonic oscillator illustrated in Figure 2.4.

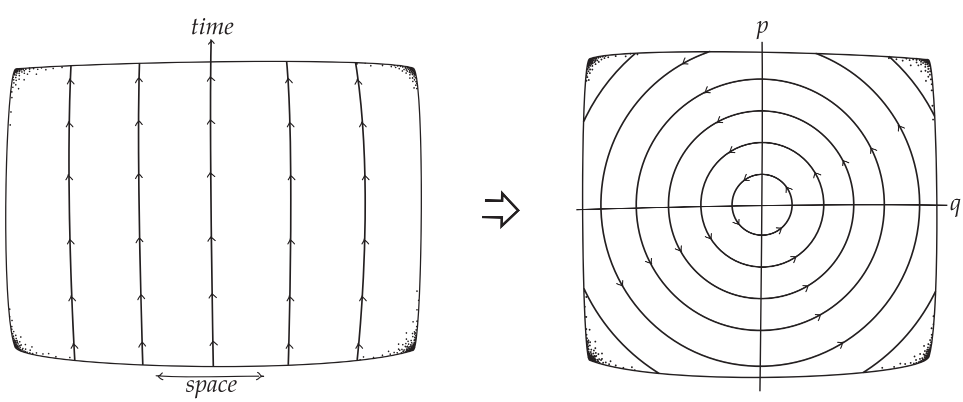

Figure 2.4 A group of time translations on spacetime (left) represented amongst the symmetries of a state space (right) for the harmonic oscillator.

[10] See Olver (1993, p.18) and also Section 2.7.

Sometimes we make the Representation View explicit, as when one derives the dynamics of quantum theory from a group describing time translations like $G = (R, +)$, which are then associated with a strongly continuous representation amongst the symmetries of Hilbert space.[11] Other times it is left implicit, as when one says things like: "Let time evolution be represented by the solutions to this differential equation on a state space". There is nothing wrong with such a statement. However, it is important to notice what it leaves out: nothing is said about why these solutions represent changes over time, as compared to changes with respect to any other parameter.

For example, in the path of a particle traced out along a line in Figure 2.5, the picture alone says nothing about why the line represents a time translation as opposed to a spatial translation. A sociologist might say that a description of time in state space is one in which the physicist uses the letter 't'. A philosopher of physics should say that a description of time in state space is a representation of time translations.

The Representation View provides an elegant solution to the debate about the meaning of time reversal in dynamical systems, described in Section 2.2. Instead of thinking of time and time reversal as a single concept, we can pull it apart into two: time reversal is a spacetime symmetry of the form $\tau : t \mapsto -t$, but it is also a state space symmetry T, where the two are related by a representation associating the former with the latter: $\tau \mapsto T$. After all, it is hard to justify calling a theory 'dynamical' without a representation of time's passage! So, Albert (2000) and Callender (2000) are right to view time reversal as a transformation that simply reverses time. And, when Earman (2002b) proposes that time reversal is a state space transformation that "turns around" the phase relations of a wave packet at an instant, he is right too: this is the effect of the state space transformation T. The two views are not inconsistent; they simply describe two different sides of the same representation relation.

In the remaining sections of this chapter, I will develop the philosophical and mathematical understandings of time that underpin the Representation

Figure 2.5 A translation in time, or space?

[11] I will give this argument explicitly in Section 3.4.2; see also Jauch (1968, §10-1 and 10-2) or Landsman (2017, §5.12), and Landau and Lifshitz (1981, §6) in the context of mechanics. A similar argument can be also given in the context of classical Hamiltonian physics (Section 3.3.1).

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View. Of course, as the physics of time advances, we can expect our understanding of time to change. But, whatever structure turns out to more accurately describe temporal relations, we will now have a strategy for determining the associated meaning of time reversal, by viewing time in physics as a representation of that more accurate structure instead. What is perhaps surprising is that, when one adopts this view, the structure of time translations essentially determines the meaning of the time reversal operator T on state space, with all its bells and whistles. To see this, one must first recognise various senses in which time is a highly structured concept, which includes group theoretic and continuity properties. That is the subject of the next section.

2.4 Temporal Structure

2.4.1 Structuralist and Functionalist Inspiration

The composer Hector Berlioz once wrote, "Time is a great master, it is said; the misfortune is that it is an inhumane master who kills the pupils".[12] Like Berlioz, I would like to identify some relevant properties of time without settling the question of what time actually is. Bracketing this question is an important part of understanding the broader nature of time, and it is the strategy I will adopt for the analysis of time reversal.

There are signs that neither physics nor philosophy is currently prepared to say what time is. The spacetimes of general relativity and later twentieth-century physics are widely believed to be an approximation to reality, which emerge on familiar scales from some hidden underlying substrate governed by quantum theory. There is currently little agreement about what that substrate is. However, abstract properties of time, like its symmetries, may hold some insight into its nature.

Philosophical debates about spacetime realism and antirealism have similarly led to little agreement on what time is. This is in spite of many important clarifications following classic works like Sklar (1974) and Earman (1989), and following the debate over Einstein's 'hole argument'.[13] The *substantivalist* takes spacetime to exist independently of its contents, as when Newton characterises Absolute Time and Absolute Space without reference to anything 'external'. Antirealism denies this, usually coupled with some statement of how facts about spacetime are reducible to something else, as

[12] "Le temps est un grand maître, dit-on; le malheur est qu'il soit un maître inhumain qui tue ses éléves" (Berlioz 1989, p.390, 27 November 1856 letter to the playwright Saint-George).

[13] For an introduction, see Norton (2019) and the references therein.

when Leibniz suggests that temporal relations reduce to causal relations.[14] A modern relationist account was articulated and defended by Brown (2005) and by Brown and Pooley (2006), and remains the subject of much debate.[15]

However, recent developments have shown that it is possible to set aside the question of realism and still say something interesting. My discussion takes two developments as inspiration. In the first place, following John Worrall (1989), I find that a pleasant way to face controversy is to focus on our points of agreement: in the absence of a complete account of time, we can instead focus our attention on the more abstract structure of time, which is better-confirmed by experiment and less philosophically controversial.[16] In the second place, following Eleanor Knox (2013, 2019), I find that a natural way to identify that structure is through the functional role that time plays in more widely-agreed contexts. Butterfield and Gomes (2020) have placed this approach in its correct historical context: functionalism about a problematic concept like 'time' is an interpretation that identifies the concept with the occupant of a pattern of relations (a 'functional role') in a less problematic context, such as the behaviour of matter-energy – and, if we can, we also establish that the occupier of this role is unique.[17]

I will not take a position on either of these debates. Besides inspiration, the lesson that I would like to take from them is that the concept of 'time' in physics, whatever it actually refers to, must include rich relational or structural properties. As Callender (2017) points out, one simply cannot develop a physical theory without it:

> This demands scores of decisions about time. Are there instants? Ordered? Partially or totally? What is the topology . . .? Is time continuous, dense, or discrete? Open or closed? One-dimensional? (Callender 2017, p.20)

I will shortly make this more concrete and identify particular 'structural patterns' expressed by a Lie group as providing the relevant 'functional

[14] This is a common reading of Leibniz 1715, p.18, expanded by Grünbaum (1973, §12), Reichenbach (1928, §43), and Winnie (1977), among others. Newton's statement of substantivalism can be found in his Scholium to the Definitions in the second edition of the *Principia* (Newton 1999, pp.54–5) and Clarke's defence of Newton's position in the famous Leibniz–Clarke correspondence (Leibniz and Clarke 2000).

[15] Cf. Gryb and Thébault (2016), Myrvold (2019), Norton (2008b), and Pooley (2013).

[16] Group structure has indeed been proposed as a foundation for the ontology of physics, following work on the ontology of physics by French (2014), Ladyman (1998), French and Ladyman (2003), and Ladyman and Ross (2007). I will not take a position on that debate here; for my position, see Roberts (2011).

[17] Butterfield and Gomes locate the origins of functionalism in the theories of mind due to Armstrong (1968), Lewis (1966, 1972), and Putnam (1960). They conclude that spacetime functionalism should be viewed as "a species of reduction (in particular: reduction of chronogeometry to the physics of matter and radiation)", where reduction is interpreted in a Nagelian sense (Butterfield and Gomes 2020, p.1).

role' for time. The pay-off for insisting on this, I will argue, is a clear account of where time reversal gets its meaning: the standard textbook definitions of time reversal can be viewed as fundamentally arising out of the Lie group structure of time, viewed here as the unique occupant of a well-understood pattern of relations (to use the language of Butterfield and Gomes 2020).

2.4.2 The Lie Group of Time Translations

How should one characterise temporal structure? A substantivalist about spacetime might identify one structural property as given by a 'time translation', described as a function from the set of all temporal instants to itself; in contrast, a relationist might take that same structural property to be a function of matter-energy states. I would like to postulate something that both sides should agree on, that the temporal structure at least includes a structure describing time translations, illustrated in Figure 2.6.[18] In many applications this forms a 'Lie group': roughly speaking, a group that is also a differentiable manifold, associated with a collection of coordinate charts called an 'atlas'. More precisely, I mean the following (cf. Landsman 2021; Olver 1993):

Definition 2.1 A *Lie group* is a set G, a binary operation $+$, and an atlas such that:

(i) (*group*) G is a group with respect to the binary operation; and
(ii) (*manifold*) G is a smooth, connected, paracompact real manifold with respect to the atlas, and the group operation $(t, t') \mapsto t + t'$ and the group inverse map $t \mapsto -t$ are both smooth maps on G.

Postulate (i) (*group*) captures the sense in which time translations can be composed to form new time translations, through an associative map $+ : (t, t') \mapsto t + t'$. The identity element $0 \in G$ can be interpreted as 'no translation'; the existence of an inverse translation $-t$ such that $t - t = -t + t = 0$ can be interpreted as meaning that translations can 'cancel each other out' and produce no translation. The postulate that time translations satisfy (ii) (*manifold*) captures the claim that they form a connected continuum. Of course, time translations might not have these properties, for example in models that treat time as discrete. I see no reason why my proposal cannot be generalised to these contexts, following my comments at the end of the previous section. Moreover, on scales where theories like general relativity and quantum theory are well-confirmed, both (i) and (ii) are ubiquitous.

[18] Belot (2007, p.171) proposes an interpretation of time in the spirit of this view.

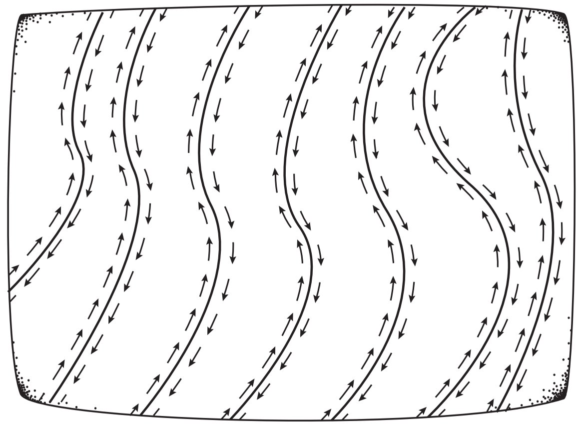

Figure 2.6 A group of time translations along curves in spacetime.

The following postulate is often true of time translations as well, and we will generally adopt it:

(iii) (*one-parameter*) As a manifold, G is one-dimensional.

When (iii) (*one-parameter*) is true, one may treat temporal relations like those of a one-dimensional timeline. This is of course not generally true in physics: even time translations in Minkowski spacetime fail to form a one-parameter group, owing to the absence of absolute time. It also fails in branching-time scenarios associated with multiverse and Everettian scenarios.[19] However, neither context poses a problem for us. In a relativistic spacetime with a reference frame associated with a timelike vector field, time translations threading the vector field form a one-parameter Lie group; this is a standard strategy in $(3 + 1)$ approaches to relativity theory. Similarly, in Everettian branching time, the emergent experience of the measuring physicist is still described by the one-parameter Lie group translating along a 'thin red line' through the 'tree', whose branches are the worlds of a multiverse.[20]

Reverting to the language of Section 2.4.1, suppose that the 'pattern of relations' or 'functional role' of time is associated with these properties: time translations are a one-parameter Lie group. Then the occupier of this role is nearly unique: there are only two (connected, inextendible, paracompact in the standard topology) one-parameter Lie groups, the translations of the reals $(\mathbb{R}, +)$ and the group of rotations of the circle $SO(2)$ (Olver 1993, §1.2).

[19] Compare e.g. Wallace (2012), as well as Belnap (1992), Belnap, Perloff, and Xu (2001), and Belnap, Müller, and Placek (2021a,b).
[20] I take the 'thin red line' metaphor from Belnap and Green (1994).

If we speak 'locally', then we get uniqueness exactly: these two groups are isomorphic (as 'local' Lie groups) in a neighbourhood of every point. And, speaking globally, since modelling time translations as the circle group is appropriate only for exotic scenarios like closed timelike curves and eternal recurrence, I will set this possibility aside here.[21] Thus, now that we have more clarity on the philosophical underpinnings of this hypothesis, we can say in many applications: *let time translations be characterised by the Lie group* $(\mathbb{R}, +)$.

2.5 Time Translation Reversal

2.5.1 *Not Just Reversal of a Set of Instants*

Whatever time is, its structure includes time translations. The basic postulate of this section is that time reversal is the reversal of those time translations, in a sense that I will soon make precise. It will soon become clear that lack of clarity about this point is perhaps the single biggest source of confusion in discussions of the meaning of time reversal.

The postulate means, for example, that if $t \mapsto \Sigma_t$ is a sequence of spacelike hypersurfaces ordered by the real line, each representing an instant t, then time reversal cannot just be an order reversal of that set, such as,

$$\Sigma_t \mapsto \Sigma_{-t}. \tag{2.1}$$

Focusing on this transformation alone would ignore the rich structure of time. Time is not just a set of instants but also a set of translations describing temporal structure, like the relations between instants, and in particular how one can 'slide' each instant forward or backward by a duration t, via the transformation of each instant Σ_{t_0} by a map, $\varphi_t : \Sigma_{t_0} \mapsto \Sigma_{t_0+t}$.

This structure can go some way towards clarifying the debate over the meaning of time reversal. The account of Albert and Callender, which I have called the Time Reflection Camp (Section 2.2), begins with a description of how a "complete description" of a physical situation would characterise instants:

> There would seem to be two things you want from a description like that:
> a. that it be genuinely *instantaneous* ...; and b. that it be *complete* (which is to say, that all the physical facts about the world can be read off from the full temporal set of its descriptions). (Albert 2000, pp.9–10)

[21] The analysis that I give in this chapter and the next would suggest the possibility of a theory of 'time travel' dynamical systems, given by unitary or symplectic representations of $SO(2)$; however, I am not aware of any such analysis.

But, there is an ambiguity in what could be meant here by a "full temporal set". A conservative reading might take it to mean a collection of instantaneous descriptions having nothing more than set structure. However, a more liberal interpretation might also include structure on the set of instants, such as the Lie group of time translations. Which one is it? Albert appears to adopt the conservative reading in his presentation of time reversal: "any physical process is necessarily just some infinite sequence $S_I, \ldots, S_F$ of instantaneous states. . . . And what it is for that process to happen backward is just for the sequence $S_F, \ldots, S_I$" (Albert 2000, p.111). But, on a more liberal reading, this might not be all there is to say about it: Albert has, after all, equipped time with an ordering structure to make it a sequence; so, this transformation might plausibly include a transformation of the structure of time translations too. In what follows, I will adopt the latter interpretation.

The more liberal interpretation appears more clearly in the account of Callender, which on the surface has a similar ambiguity:

> Relative to a co-ordinisation of spacetime, the time reversal operator takes the objects in spacetime and moves them so that if their old co-ordinates were t, their new ones are $-t$, assuming the axis of reflection is the co-ordinate origin. . . . As I understand it here, 'T' switches the temporal order by switching the sign of t. It also switches the sign of *anything logically supervenient* [my emphasis] upon switching the sign of t, e.g., the velocity $d\mathbf{x}/dt$. But that is all 'T' does. (Callender 2000, pp.253–4)

On a conservative interpretation, the structures that are supervenient[22] on time might not include the structural facts like the time translations; but on a more liberal reading, they do. There is also a sense in which the liberal reading is already implicit in Callender's remark: time translation by t is the map $\varphi_t : \Sigma_{t_0} \mapsto \Sigma_{t_0+t}$, where each Σ_{t_0} is a spacelike surface. As a result, the transformation $t_0 \mapsto -t_0$ on instants induces a transformation on time translations, $\varphi_t \mapsto \varphi_{-t}$. In particular, if we write 'flip' to denote Callender's time reversal operation, then the succession of transformations 'flip-translate-flip' is equivalent to translating in the opposite direction,[23] as shown in Figure 2.7. In what follows, I will adopt a reading of both Albert and Callender according to which time reversal is a reversal of time translations, $\varphi_t \mapsto \varphi_{-t}$.

[22] Supervenience is a philosophical term of art: a set of properties M supervenes on a set of properties S if and only if a difference in M-properties implies a difference in S-properties (cf. McLaughlin and Bennett 2018). For example, statistical macro-states would usually be said to supervene on the underlying micro-states, but not conversely. As the philosopher Nicholas Rescher once told me: "Supervenience is a gift horse that one should stare in the mouth" (private communication).

[23] More formally, if T is Callender's time reversal transformation defined by $T(\mathbf{x}, t) = (\mathbf{x}, -t)$, then the induced transformation on time translations given by $\varphi_t \mapsto \varphi_t^T := T \circ \varphi_t \circ T^{-1}$ is just equal to $\varphi_t^T = \varphi_{-t}$ for all t.

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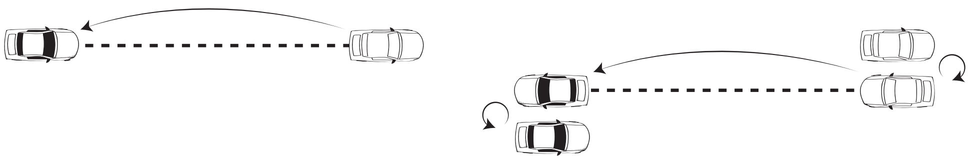

Figure 2.7 'Flip-translate-flip' (right) is equivalent to translating in the reverse direction (left).

Of course, one could insist on ignoring the structural properties of time when defining time reversal. This would be plausible in a naïve theory of time that consists entirely of an unstructured set of sentences or models describing what's happening at each instant Σ_t. But, philosophers have recently argued convincingly that this naïve perspective on theories is disastrous (cf. Dewar 2022; Halvorson 2012, 2019).

Thus, it is wrong to describe time as nothing more than an unstructured set of instants Σ_t. When physicists seem to suggest otherwise, I would like to interpret this as shorthand – as when Von Neumann (1932, p.354) says that in a dynamical theory, "there corresponds to the time an ordinary number-parameter t", or when Sachs (1987, p.4) says that, "[t]he qualitative meaning of the time variable is that of an ordering parameter in one-to-one correspondence to a sequence of events". Both immediately go on to attribute a great deal of further structure to time, including statements about its group of time translations, and thus they do not really mean that time is an ordered set and nothing more. Similarly, when philosophers like Albert (2000, p.11) write,[24] "any physical process is necessarily just some infinite sequence of states", I would like to read this as shorthand, so as not to deny the rich structural properties of time.

2.5.2 *Reversing Structure Too*

Maudlin (2007, §4) has noted an essential connection between time translations and time reversal in his defence of the passage of time:

> The passage of time is deeply connected to the problem of the direction of time, or time's arrow. If all one means by a 'direction of time' is an irreducible intrinsic asymmetry in the temporal structure of the universe, then the passage of time

[24] See also Callender (2017), who writes of a semiclassical approach to canonical quantum gravity: "The claim may have other problems . . . but one of them is not the identification of time with a metaphysically rich batch of properties. Instead here the role time plays couldn't be more spare. Time is simply identified as that parameter with respect to which the quantum matter fields evolve". (Callender 2017, p.110)

implies a direction of time. But the passage of time connotes more than just an intrinsic asymmetry: not just any asymmetry would produce passing. (Maudlin 2007, p.109)

Let me focus on Maudlin's observation that time translations are 'directed'. Each time translation is indeed associated with another time translation in the 'reverse' direction. From this perspective, it is natural to view time reversal as a transformation relating oppositely-directed time translations. Confusingly, one typically writes $t \mapsto -t$ for both the transformation of time coordinates and the automorphism of a group of time translations; but, unless otherwise specified, I will always use the latter notation and interpret time reversal as an automorphism that reverses time translations.

Why is time reversal associated with the particular map $t \mapsto -t$ and not some other, such as $t \mapsto -t + t_0$? North has pressed this objection,[25] that there may be some freedom in "choice of temporal origin" (North 2008, p.218). But we avoid this issue here by defining time reversal as a map on time translations rather than on time coordinates: $t = 0$ is not a point on a time axis but rather the identity translation by a duration of zero time. No choice of temporal origin is needed to express this.

Perhaps a more challenging concern is: even viewing time reversal as a map on time translations, there are many order reversing transformations besides $t \mapsto -t$. For example, the two transformations of a line depicted in Figure 2.8 are two different order reversing transformations. One could impose the restriction that time reversal τ be a 'reversal', understood mathematically as an *involution;* this means that applying it twice produces the identity transformation, τ^2 = identity.[26] But, there are many involutions besides $f : t \mapsto -t$, such as the map $g : t \mapsto (1 - t^3)^{1/3}$; both satisfy $f \circ f = g \circ g = \mathbb{1}$, where $\mathbb{1}$ is the identity transformation $\mathbb{1}(t) = t$.

The inspiration of structuralists and functionalists helps here, as it did in Section 2.4.1. To use the nomenclature of Butterfield and Gomes (2020): although time reversal is a 'problematic' concept, in that it is not obvious how to define it, we can still identify its functional role and hope for a uniqueness result. That would break the underdetermination as to which order reversing involution is time reversal.

[25] Compare Sachs (1987, p.8), who states that time reversal should be a linear transformation of t. Roberts (2017, p.325) similarly uses linearity to express that time reversal does not change durations and observes that all non-trivial linear involutions have the form $\tau(t) = -t + t_0$. But neither of these proposals solves the problem pointed out by North.

[26] This assumption is proposed to help determine the meaning of time reversal (for example) in Peterson (2015) and Roberts (2012, §2.4.1).

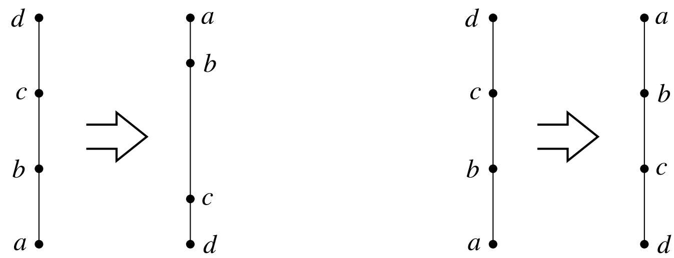

Figure 2.8 Two ways to reverse order.

I propose we make three such observations about its typical functional role in physical theorising:

1. (*Involution*) Time reversal is a 'reversal', in that applying it twice to the group of time translations must produce the identity transformation.
2. (*Automorphism*) Time reversal does nothing more than reverse time translations, and in particular it does not stretch, scale, or discontinuously transform the Lie group structure. This is enforced by requiring it to be an automorphism of $(\mathbb{R}, +)$.
3. (*Non-triviality*) Time reversal does something interesting, in that it is not the identity transformation.

These three properties can be shown to provide a functional definition of time reversal, in that they uniquely determine a transformation of the Lie group $(\mathbb{R}, +)$:

Proposition 2.1 *If τ is a non-trivial automorphism of the one-parameter Lie group $(\mathbb{R}, +)$ satisfying $\tau(\tau(t)) = t$, then $\tau(t) = -t$.*

Proof First note that every continuous automorphism τ of $(\mathbb{R}, +)$ satisfies $\tau(t) = \tau(1)t$ for all $t \in \mathbb{R}$, where 1 is the element $t = 1$ and $\tau(1)t$ is ordinary multiplication. This is proved for natural numbers t by induction: the base case is obvious, and if $\tau(t) = \tau(1)t$, then since τ is an automorphism of $(\mathbb{R}, +)$,

$$\tau(t + 1) = \tau(t) + \tau(1) = \tau(1)t + \tau(1) = \tau(1)(t + 1). \tag{2.2}$$

A second induction proves it for the rationals, and so it holds of the unique continuous extension of τ to $\mathbb{R}$. Applying $\tau(\tau(t)) = t$, we now get $t = \tau(\tau(t)) = \tau(\tau(1)t) = \tau(1)^2 t$, so $\tau(1) = \pm 1$. Thus $\tau(t) = \tau(1)t = \pm t$, and so by non-triviality we get $\tau(t) = -t$. ∎

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I take this fact to settle the meaning of time reversal as a transformation of the Lie group of time translations $(\mathbb{R}, +)$: time reversal is the map $\tau : t \mapsto -t$. Notably, the proof immediately generalises to discrete time translations, such as those given by the group of integers or rationals under addition. However, one should make no mistake – the result that time translations admit a non-trivial automorphism is substantial. Chapter 4 will show that it implies every representation of this structure in state space is time reversal invariant.

2.5.3 Generalising Malament's Picture

Viewing time reversal as a map on the Lie group of time translations given by $t \mapsto -t$ is a natural generalisation of the account of time reversal given by Malament (2004). Showing this requires the language of relativity theory, although I will soon return to a more general discussion.

Spacetime in relativistic physics is represented by a Lorentzian manifold (M, g_{ab}), where M is a smooth real manifold and g_{ab} is a Lorentz-signature metric. A *temporal orientation* is an equivalence class of timelike vector fields defined by the equivalence relation of being co-directed.[27] If a temporal orientation exists, then there are exactly two. Co-directed vector fields 'point' into the same light cone lobe at every point, as illustrated in Figure 2.9, and so it is common practice to treat a temporal orientation as defining a temporal asymmetry (cf. Earman 1974). In describing a temporal orientation, one typically selects a representative vector field from the equivalence class, while keeping in mind that any other in the same equivalence class would serve equally well.

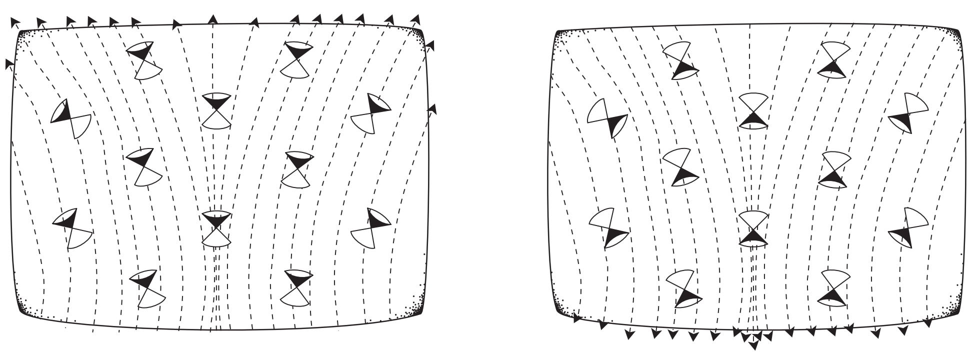

Figure 2.9 Time translations are associated with a temporal orientation.

[27] A pair of smooth timelike vector fields ξ^a and ϕ^a are *co-directed* if and only if $\xi^a \phi_a > 0$.

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Malament's basic proposal is then to view time reversal as the reversal of a temporal orientation:[28]

> The time reversal operation is naturally understood as one taking fields on M as determined relative to one temporal orientation to corresponding fields on M as determined relative to the other. (Malament 2004, p.306)

I agree: this is a special case of the account I have given above. A spacetime (M, g_{ab}) can be equipped with the structure of a Lie group, given a smooth timelike vector field ξ^a: in some local region[29] around a point $p \in M$, there is a one-parameter group of diffeomorphisms 'threading' the vector field that is locally isomorphic to $(\mathbb{R}, +)$ in a neighbourhood of $0 \in \mathbb{R}$. This group is naturally interpreted as a group of 'time translations' in a reference frame associated with the timelike vector field ξ^a. The resulting set of integral curves $\gamma(t) := \varphi_t(p)$ has ξ^a as its tangent vector field. Thus, the transformation $t \mapsto -t$ of the group $(\mathbb{R}, +)$ gives rise to a reversal of the tangent vector field $\xi^a \mapsto -\xi^a$ at every point in the region, and vice versa. In short: in the context of Malament's argument, treating time reversal as reversing a local Lie group of time translations is equivalent to treating it as reversing time orientation.

There are two advantages to viewing Malament's proposal as a special case of the reversal of time translations. First, time translation reversal can be expressed in any spacetime theory, and in any interpretation of that theory, so long as we agree on the (e.g. group) structure of time translations. This includes non-standard representations of time translation on a spacetime (M, g_{ab}), for example with exotic tachyon motion that cannot be described by a smooth timelike vector field, or in alternative spacetime theories. Second, as we will see in Chapter 3, time translation reversal allows one to more directly interpret time reversal in dynamical theories, when the latter admit a representation of time translations.

Malament views his proposal as a critical response to Albert (2000), and in particular Albert's claim that classical electromagnetism (and most other physical theories) are not time reversal invariant. I agree with this critique and that Albert has not given any reason to think that classical electromagnetism is time reversal violating.

28 A similar but less-detailed proposal is found in Earman (1974, p.25) and Wald (1984, p.60).
29 This 'local region' requirement allows one to avoid incomplete timelike vector fields, whose integral curves cannot be parametrised by all of $\mathbb{R}$, for example in black hole spacetimes that are (timelike) geodesically incomplete.

However, this conclusion does not necessarily follow from Albert's account if one includes relational or structural properties in his description of time reversal. Recall that the idea motivating both Albert and Callender is that time reversal has the simple form

$$t \mapsto -t. \tag{2.3}$$

As I have suggested, by interpreting this t as a time *translation* rather than a time *coordinate*, we are led to the very same specific definition of time reversal that Malament has formulated in this context. All these accounts of time reversal and time reversal invariance amount to the same thing on the Representation View. To make this precise, let me now illustrate how the group theoretic structure of time reversal arises directly out of the structure of time translations.

2.6 Constructing the Group Element

When one postulates a concrete group of spacetime symmetries, such as the Lorentz group or the Galilei group, one often postulates a set of 'discrete' group elements that include time reversal. This group structure turns out to be crucial for the Representation View, in which we will take the meaning of time reversal on state space to be determined by such a group element. So far, we have only been viewing time reversal as an automorphism of a group, and not a group element itself. In this section I will show that, remarkably, the time reversal group element can be recovered directly from the structure of those time translations, or more generally of the group of spacetime symmetries.

In the discussion above, I argued that time translations are an essential part of the structure of time. We have seen in particular that, when that group is $(\mathbb{R}, +)$, the reversal of time translations $t \mapsto -t$ is the unique non-trivial, involutive automorphism. In this section I will show that it is always possible to extend $(\mathbb{R}, +)$ in a way that 'adds in' time reversal as a group element τ, which was hidden all along in the structure of time translations. More generally, when time translations form part of a continuous spacetime symmetry group like the restricted Poincaré group, the same technique can be applied to construct the complete Poincaré group of continuous and discrete symmetries. This will allow us to apply the Representation View and to interpret the instantaneous time reversal operator on state space as the representative of this time reversal element τ.

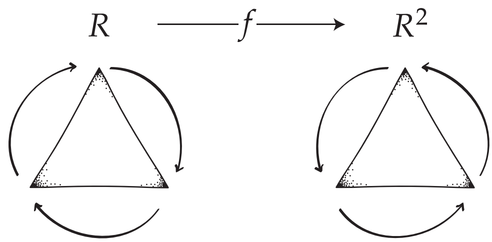

Figure 2.10 The flip automorphism exchanges the rotations of C_3, or equivalently reverses the direction of the rotations.

To understand how time translations can be extended to include time reversal, we will need to make use of the elementary group theory of semidirect products.[30]

Definition 2.2 Given a group $(G, \cdot)$ together with a group S of automorphisms $s : G \to G$, the *semidirect product operation* is the binary operation on $(G \times S)$ given by

$$(g, s)(g', s') := (g \cdot s(g'), s \circ s'). \tag{2.4}$$

The resulting group, denoted $G \rtimes S$, is called the *semidirect product* of G with S.

As a warm-up for how we will use semidirect products to understand time reversal, consider the example of the cyclic group of order three, $C_3 = \{1, R, R^2\}$. It describes the symmetries of an equilateral triangle with $1 = R \cdot R^2$. This group has two automorphisms: the identity transformation ι and the exchange of the two rotations defined by $f : R \mapsto R^2$ and $f : R^2 \mapsto R$. The automorphism f is equivalent to applying a 'flip' to the triangle before each rotation, which is again equivalent to a rotation in the reverse direction (Figure 2.10).

Now, suppose we want to create a larger group, which 'adds in' the flip automorphism f as a group element. A semidirect product allows us to do this: writing the automorphism group as $S = \{\iota, f\}$, we take our new group elements to be the pairs (g, s) for each $g \in C_3$ and $s \in S$ and define our new group operation using Eq. (2.4). This new group now has six elements. Three of them form a subgroup that just implements the original rotations, which we can write in shorthand as

$$1 := (1, \iota) \qquad R := (R, \iota) \qquad R^2 := (R^2, \iota). \tag{2.5}$$

[30] For an introduction to semidirect products, see Robinson (1996, §1.5); for a related group theoretic analysis of the Poincaré group, see Varadarajan (2007, §IX.2), and for applications to quantization theory, see Landsman (1998, esp. §2.2 and Part IV).

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But, we now have three more elements, $(1, f)$, (R, f), and (R^2, f), which apply a flip. Adopting the shorthand that $f := (1, f)$, one can check[31] that this implies that $fRf^{-1} = R^2$ and $fR^2f^{-1} = R$. That is, conjugation by the 'flip element' $g \mapsto fgf^{-1}$ is equivalent to an application of our automorphism f. The result is a larger group describing both the rotational and flip symmetries of the triangle, isomorphic to the symmetric group S_3.

As the 'flip' analogy suggests, a semidirect product is exactly what we need to construct a time reversal group element. We begin with the group of time translations $G = (\mathbb{R}, +)$, which we have shown has a unique automorphism that is appropriate for time reversal, $\tau : t \mapsto -t$ (Proposition 2.1). Let $S = \{\iota, \tau\}$ be our automorphism group, with ι the identity transformation on $\mathbb{R}$. We can now construct a new, larger group with elements of the form (t, s) for each $t \in \mathbb{R}$ and $s \in S$. Adopting additive notation for the binary operation $+$, a semidirect product operation is given by $(t, s)(t', s') := (t + s(t'), s \circ s')$.

As in our example of C_3, this new group contains a subgroup of elements of the form (t, ι) with $t \in \mathbb{R}$, which is isomorphic to the original group of time translations. So, as before, we can write $t := (t, \iota)$ as shorthand for these group elements. But, we now have a further set of elements (t, τ) with $t \in \mathbb{R}$ that includes time reversal. In particular, we adopt the shorthand $\tau := (0, \tau)$ and refer to this special element as the *time reversal group element*.

To check that this element behaves like time reversal, we can simply observe that it implements the automorphism $t \mapsto -t$ when it acts on time translations by conjugation, $\tau t \tau^{-1} = -t$. That is, by our definitions,

$$\tau t \tau^{-1} = (0, \tau)(t, \iota)(0, \tau^{-1}) = (0, \tau)(t, \tau^{-1}) = (-t, \iota) = -t. \tag{2.6}$$

The resulting semidirect product group is denoted $(\mathbb{R}, +) \rtimes S$, which I will refer to as the *group of time translations with time reversal*. It has very little structure beyond the structure of time translations, except for the inclusion of a time reversal group element implementing the automorphism $t \mapsto -t$. In this sense, the group of time translations already contains the structure of a time reversal group element, through the extension of those time translations to one with time reversal as well.

Although I have restricted attention here to the simple time translation group $G = (\mathbb{R}, +)$, this procedure for introducing a time reversal element is much more general. Suppose we begin with the group $G = \mathcal{P}_+^\uparrow$ of rigid translations, rotations, and Lorentz boosts on Minkowski spacetime, called the (restricted) Poincaré group. This group has four automorphisms,

[31] By our definitions, $fRf^{-1} = (1, f)(\iota, R)(1, f^{-1}) = (1, f)(f(R), f^{-1}) = (f(R), \iota) = (R^2, \iota) = R^2$. The remaining arguments are similar.

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$D = \{\iota, p, \tau, p\tau\}$, representing the identity, spatial translation reversal, time translation reversal, and the composition of the last two. Just as above, one can extend $P_+^\uparrow$ to a group that 'adds in' these transformations as group elements using a semidirect product in exactly the same way. The result is what is commonly known as the (complete) Poincaré group $\mathcal{P} := \mathcal{P}_+^\uparrow \rtimes D$.

This procedure is indeed a standard technique in the construction of the Poincaré group (cf. Varadarajan 2007, §IX.2). It is perhaps not so widely studied. However, the discussion here shows that it provides a useful technique for establishing the group structure of discrete symmetries. We will return to its broader applications in Chapter 8, and especially Section 8.3, on the meaning of CPT symmetry.

2.7 The Representation View of Time Reversal

The last two sections show that the structural properties of time, and in particular the time translations, give rise to a group structure for time translations with time reversal. This provides a foundation for the meaning of time reversal on spacetime. To carry its meaning over to state space, we now apply the Representation View, set out in Section 2.3: a symmetry of state space is called 'time reversal' only if it is a representative of the time reversal group element. That representative, I claim, is just what is commonly referred to as T, the 'instantaneous time reversal transformation'. Remarkably, the Representation View allows one to more or less determine the meaning of this transformation. In this section I will outline how that works. Chapter 3 is then devoted to deriving its meaning in a variety of physical state spaces.

In the debate between the two camps, Albert (2000) and Callender (2000) pointed out that time reversal should be characterised by the simple property that $\tau : t \mapsto -t$. We have seen how that is true, and indeed that time reversal is an element of the spacetime symmetry group, when each t is interpreted as a time translation. However, this does not imply that time reversal on the state space of a dynamical theory just 'reverses the little-t' parameter in each trajectory. A dynamical theory is hardly deserving of the name 'dynamical' unless it admits a representation of time translations. Once such a representation is chosen, 'time reversal' on state space simply refers to the representation of τ, the time reversal group element.[32] Let me

[32] Struyve (2020) has made a similar observation about this very debate, diagnosing the disagreement about the meaning of time reversal as arising from different choices of 'ontology'. I agree, insofar as each 'ontology' is associated with a representation.

set out this aspect of the Representation View in more formal terms, so that we can apply it in physics.

Definition 2.3 Let G contain a substructure representing time translations (typically a Lie group) as well as a time reversal element $\tau \in G$. Let $\mathcal{A}$ be state space structure (typically an object in some category), with $\mathrm{Aut}(\mathcal{A})$ its automorphism group. A *representation of* G is a homomorphism $\phi : G \to \mathrm{Aut}(\mathcal{A})$, whose restriction to the substructure of time translations is also a homomorphism. The *instantaneous time reversal transformation* T is the image of time reversal in this representation, $T := \phi(\tau)$.

I have intentionally tried to formulate this definition with a high degree of generality. It is indeed in the same spirit as a recent approach to algebraic quantum field theory, viewed as an embedding of a category of relativistic spacetimes into a category of algebras.[33] More generally, the theorist might take time translations to be associated with a discrete group, or even a semigroup (without inverses). Or, in the context of general relativistic spacetimes, we might restrict time translations to local regions by requiring G to be a *local Lie group*; this restricts the definition of group operations to local regions (Olver 1993, p.18). On the Representation View, this perspective can be applied whenever our spacetime structure and our state space structure are defined by a functor between categories.

However, the application of Definition 2.3 in the simplest cases of dynamical systems is straightforward: suppose time translations are given by the Lie group $(\mathbb{R}, +)$ and that we construct the group with time reversal $G = (\mathbb{R}, +) \rtimes \{\iota, \tau\}$ as above, which satisfies $\tau t \tau^{-1} = -t$. Then we can take the symmetries of any state space, like a Hilbert space or classical configuration space, and find a representation of G given by a group homomorphism $\phi(G)$ amongst those symmetries, and where the restriction of ϕ to the time translations $(\mathbb{R}, +)$ is a Lie group homomorphism. We can write time translations on state space as ϕ_t for each $t \in \mathbb{R}$, commonly called a 'phase flow'. Time reversal on state space is then the element $T := \phi_\tau$. Since a representation is a homomorphism, the fact that $\tau t \tau^{-1} = -t$ carries over to state space too, as the statement that for T reverses the direction of each time translation:[34]

$$T\phi_t T^{-1} = \phi_{\tau t \tau^{-1}} = \phi_{-t}. \tag{2.7}$$

[33] Cf. Brunetti, Fredenhagen, and Verch (2003) and Rédei (2014).
[34] Experts may recognise this statement as indicative of time reversal invariance, and indeed it is. But this does not mean that time reversal invariance is assured: as we will see in Chapter 4, a time reversal violating system is one in which a representation of the group of time translations with time reversal does not exist.

Formally speaking, this transformation T is just the standard instantaneous time reversal operator; it is 'instantaneous' only in the sense that it is represented by a transformation on (instantaneous) state space. But, this by itself is no more paradoxical than the fact that a time translation $\phi_t : \mathcal{S} \to \mathcal{S}$ by t is a transformation on instantaneous state space, too. There is no special philosophical problem with either one: both represent structural facts about time and do not need to be interpreted as 'turning around' a spatial slice at an instant.

The objections of Albert and Callender about what it means to 'reverse the arrow of time at an instant' are thus dissolved. And, when Earman (2002b) does propose that time reversal must "turn around" the phase relations of a wave packet at an instant, we can always view this as shorthand for certain facts about the representative of the time reversal group element. But, the ultimate interpretation of T is as we have said above: it is the representative of the time reversal group element, which reverses the direction of time translations as in Eq. (2.7). We will see more about how such 'shorthand' facts arise in Chapter 3.

2.8 Summary

Time reversal faces a problem of definability: to justify its meaning in a way that makes it relevant to the direction of time. This gives rise to two camps of interpretation. The Instantaneous Camp introduces a richly structured time reversal operator T on state space, while the Time Reflection Camp favours a simpler picture in which time reversal just 'reverses time', $\tau : t \mapsto -t$.

I have argued that these statements are not incompatible. To clarify how this is the case, we must separate the concept of time into 'spacetime' and 'state space' components, where the latter gets its meaning from the fact that it is a structure-preserving copy of the former. This is the Representation View, a powerful approach to interpreting spacetime symmetries that we will use throughout this book. As we have seen above, the meaning of time reversal is essentially encoded in the structure of time, and in particular the time translations. The richly structured time reversal operator T is not a mystery in this view: it is just the representative of time reversal τ on a state space representation.

Thus, Albert and Callender are right that time reversal is a simple concept at its foundation: it simply reverses time translations. But, just as Earman has proposed, an instantaneous time reversal transformation is also a natural part of state space: it arises whenever the dynamics of a theory is associated with a representation of time translations. Of course, much more would need to be said about what exactly the time reversal operator T winds up looking like, theory by theory. That is the subject of Chapter 3.

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