*Multi-scale (time and mass) dynamics of space objects*
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# Four- and five-body periodic Caledonian orbits

**Valerie Chopovda† and Winston L. Sweatman**

School of Mathematical and Computational Sciences, Massey University, Auckland, New Zealand. emails: valerie.chopovda@gmail.com, w.sweatman@massey.ac.nz

**Abstract.** We consider four- and five-body problems with symmetrical masses (Caledonian problems). Families of periodic orbits originate from the collinear Schubart orbits. We present and discuss some of these periodic orbits.

**Keywords.** Celestial mechanics; Stellar dynamics; Caledonian few-body problem.

## 1. Introduction

The Caledonian four-body and five-body problems are formed by point-mass systems with, respectively, four and five masses Roy & Steves (2000); Steves *et al.* (2020). Symmetry about the centre of mass makes these systems less complicated than the general N-body problem. Although simpler, Caledonian few-body problems can provide insight for the more general problem (Sweatman (2015)).

The present paper considers families of periodic orbits in the Caledonian few-body problem. Similar families for the three-body problem were found by Hénon (1976), who discovered a family of periodic orbits arising from Schubart's orbit (Schubart (1956)). We have explored various mass ratios and outline some of the results.

## 2. The Caledonian symmetrical four- and five-body problems

The Caledonian symmetrical four-body problem consists of four masses arranged in a system that has rotational symmetry about the centre of mass. The pair of masses $m_1$ and $m_3$ are sited opposite, across the centre of mass one another, with equal masses so that $m_3 = m_1$. Working within the rest frame of the centre of mass, their position vectors are equal and opposite, $\mathbf{r}_3 = -\mathbf{r}_1$, as are their velocities, $\mathbf{v}_3 = -\mathbf{v}_1$. Masses $m_2$ and $m_4$ are similarly related: $m_4 = m_2$, $\mathbf{r}_4 = -\mathbf{r}_2$ and $\mathbf{v}_4 = -\mathbf{v}_2$. If a system has such rotational symmetry at any time, then the rotational symmetry is preserved for all time.

The Caledonian five-body problem is created by including an additional mass $m_0$ at the centre of mass of a Caledonian four-body problem (see Figure 1). This fifth mass remains at rest for all time: $\mathbf{r}_0 = 0$, $\mathbf{v}_0 = 0$, $\forall t$. We can regard the Caledonian four-body problem as being the special case of the Caledonian five-body problem when this mass is zero: $m_0 = 0$.

To explore the Caledonian four- or five-body problem numerically, the equations of motion are regularised using an adaptation of the global regularisation method (Heggie (1974)). In the four-body case, all possible close two-body encounters are regularised by introducing Levi-Civita coordinates for each distinct inter-body distance and a corresponding time transformation (cf. Sivasankaran *et al.* (2010)). The same regularised

† Present address: Department of Engineering Science, University of Auckland, New Zealand.

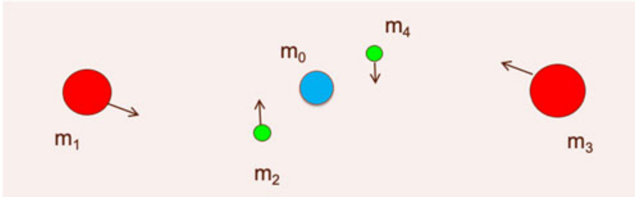


**Figure 1.** The Caledonian five-body problem: pairs $[m_1, m_3]$ and $[m_2, m_4]$ are symmetric about $m_0$. To obtain the Caledonian four-body problem $m_0$ is removed or, alternatively, set to zero ($m_0 = 0$).

coordinates also work for the five-body case, as the extra mass, $m_0$, will only come close to $m_1$ or $m_3$ when these masses approach one another. Similarly, $m_0$ is only close to $m_2$ or $m_4$ when these masses are themselves close. We cannot regularise close interactions involving all four or all five masses.

Families of periodic orbits can be found with the approach used previously by Hénon (1976) for three-body systems and by Chopovda & Sweatman (2018) for the equal-mass Caledonian four-body problem. Initial conditions from known periodic solutions are used to generate approximate initial conditions for a neighbouring orbit. These are integrated through a period and the difference between initial and final points is used to produce improved initial conditions by the process of differential correction (cf. Sweatman (2014) and papers cited therein).

## 3. Families of periodic orbits generated from the four-body Schubart orbits

The orbits within our families are considered periodic in the sense that the masses return to the same configuration at the end of a period as they had at the beginning. This is sometimes called a relatively periodic orbit. However, typically, the configuration is rotated by some angle during the orbital period. This angle of rotation and, also, the angular momentum vary continuously along each family of orbits.

The families of orbits are parameterised by the mass ratio $[m_1 : m_2]$. A range of mass ratios have been explored (cf. Chopovda (2019)) and from these some features can be identified that are common to all the families. For a particular mass ratio, the family begins and ends at the one-dimensional limit of motion, in which the four masses remain ordered and move along a fixed line. The initial periodic orbit of an individual family is a symmetric four-body Schubart orbit (Sweatman (2002, 2006); Sekiguchi & Tanikawa (2004)). In such orbits the masses perform an interplay motion in which close encounters alternate between the central and outer parts of the system. If we label the masses in their collinear order $m_1$, $m_2$, $m_4$, $m_3$, then, in the centre, the inner masses $m_2$ and $m_4$ have close interactions. On the outside $m_1$ interacts with $m_2$ whilst $m_3$ interacts with $m_4$. If a family begins at a particular Schubart orbit, then the final family member is also a Schubart orbit but with the masses reordered so that the inner and outer masses are exchanged, i.e., $m_4$, $m_1$, $m_3$, $m_2$ with the previous labelling. As the Schubart orbits are strictly one-dimensional, they have no angular momentum, and there is zero rotation during a period. Members of the families near either Schubart orbit similarly display an interplay-type motion.

Figure 2 presents three Schubart orbits. Each line on a chart represents the position of one of the masses, these values are plotted against time. The two orbits towards the top are for the case where the masses are unequal and one mass value is three times the size of the other. In the uppermost plot the smaller masses are on the inside and for the central plot they are on the outside. When the inner masses of the Schubart

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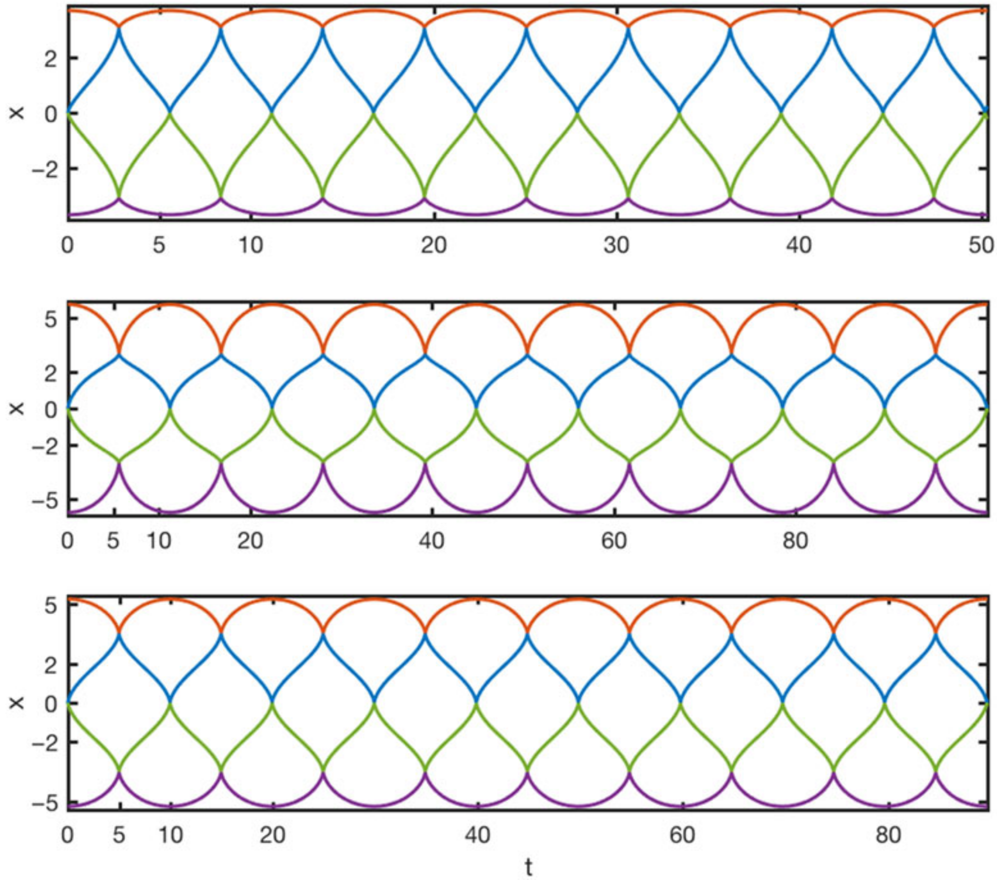


**Figure 2.** Four-body Schubart orbits for mass ratios $[m_1 : m_2] = [3 : 1]$ (top), $[m_1 : m_2] = [1 : 3]$ (middle) and $[m_1 : m_2] = [1 : 1]$ (bottom), where $m_1 = m_3$ is the mass of a body on the outside of the system and $m_2 = m_4$ is the mass of a body on the inside of the system.

orbit are relatively smaller, their motion is rapid compared with the outer masses so that sufficient momentum can be transferred to keep the outer masses apart (Figure 2 (top)). In contrast, when the inner masses are larger their relative motion approaches two-body motion, and the smaller outer masses have little effect on other masses unless passing very close (Figure 2 (middle)). The plot at the bottom of Figure 2 is for the equal masses case.

Another feature common to all the families of orbits is a double choreography orbit that occurs midway through the family. In the double choreography orbits each pair of symmetrical masses shares a common path and the masses return to their initial positions after two (relative) periods. Figure 3 shows double choreography orbits for three different mass ratios. In orbit D1, the larger mass is approximately seven times larger than the smaller. For orbit D2 the factor is close to three. The equal masses case, orbit D3, was described by Chen (2001) and has fourfold symmetry. This feature contrasts with what occurs for the three-body family, as shown by Hénon (1976), for which the size of the orbits approaches infinity (cf. Chopovda & Sweatman (2018)).

Figure 4 illustrates a specific family of orbits corresponding to the 3:1 mass ratio. Members of the family lie between the two Schubart orbits of this mass ratio (cf. Figure 2). Unlike the Schubart orbits, most of these orbits have a non-zero rotation. For example in

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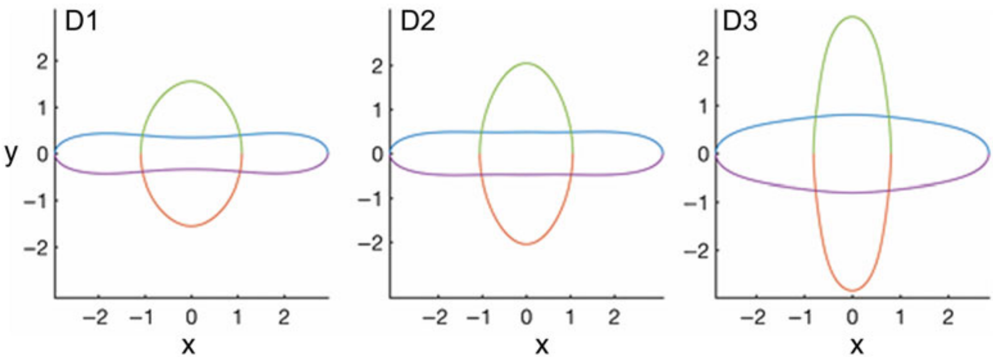


**Figure 3.** Double choreography orbits with mass ratios [254:1746], [496:1504] and [1:1]

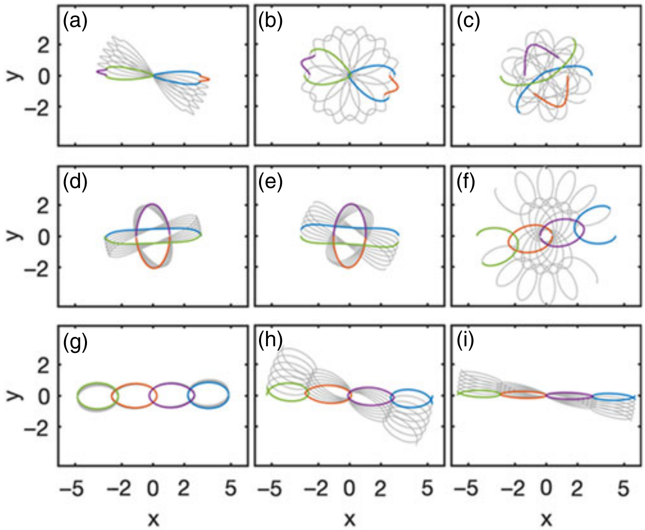


**Figure 4.** The family of orbits for the 3:1 mass ratio. The family begins from the Schubart orbit with the larger masses on the outside (Figure 2 (top)). It then progresses through the orbits shown, alphabetically, and finishes at the Schubart orbit with the smaller masses on the outside (Figure 2 (middle)).

orbit A, early in the family, a negative angle of rotation produces a clockwise precession of the orbit. Starting from the Schubart orbit with the larger masses on the outside (Figure 2 (top)), the angle of rotation decreases from zero into negative values. The angle of rotation continues to decrease through most of the family until near orbit H, at which stage it has changed by a magnitude greater than a complete rotation. From orbit H, the angle then increases until the end of the family at the final Schubart orbit (Figure 2 (middle)), at which stage the net change from the start is essentially a whole negative rotation which is equivalent to a zero rotation angle, modulo 360°. The interplay motion for family members close to a Schubart orbit is illustrated by orbits A and B near the start and orbits F, G, H and I towards the end. Orbit B completes about one twelfth of a rotation in each (relative) period and hence B is approximately absolutely periodic (i.e., returns to the initial conditions) over 12 (relative) periods. Such absolutely periodic orbits occur whenever the angle of rotation is a rational multiple of a degree. The double choreography orbit of the family will occur between orbits D and E. The interplay orbit

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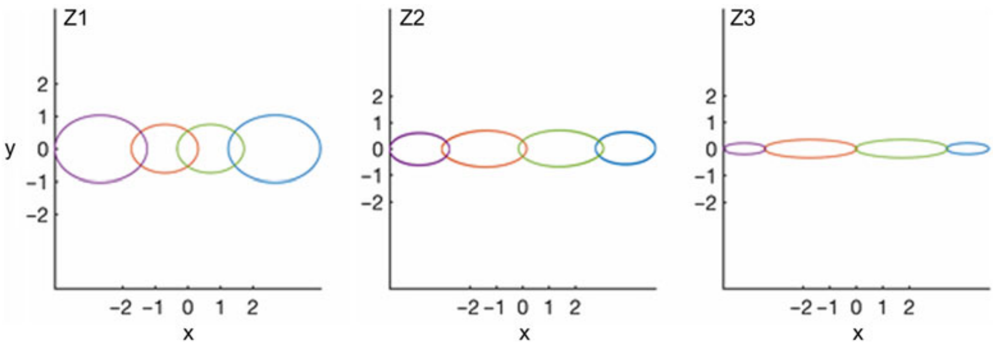


**Figure 5.** Absolutely periodic interplay orbits. The ratios of the outer masses to the inner masses are [3:197], [73:127] and [1056:944].

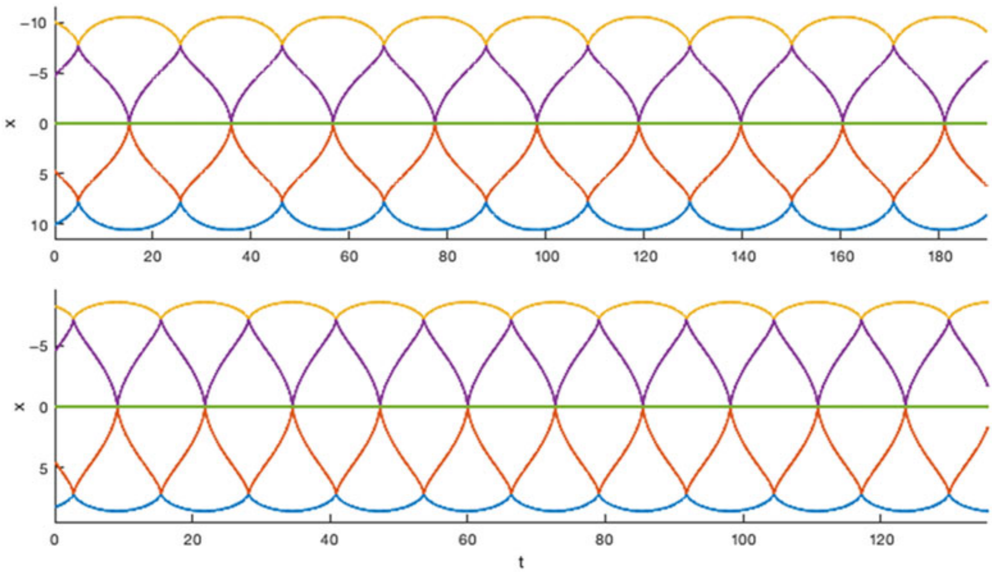


**Figure 6.** Five-body Schubart orbits with equal masses (top), and with larger central and outermost masses $[m_1 : m_2 : m_0] = [1 : 0.5 : 2]$ (bottom).

G approximates an absolutely periodic orbit that occurs when the angle of rotation is zero, modulo 360°.

For the case of equal masses, the family of orbits has been described by Chopovda & Sweatman (2018). This family contrasts with the unequal-mass family presented here in that the equal-mass family is symmetric with the same orbits occurring on either side of the double choreography orbit. The equal-mass family begins and ends at the same Schubart orbit: that shown in Figure 2 (bottom). The equal-mass family has an absolutely periodic orbit similar to orbit G, however, this occurs on both sides of the double choreography. The orbit presented by Sweatman (2014) is close in appearance to this orbit and was a starting point for a more exact calculation by Chopovda & Sweatman (2018). Families of orbits with near equal masses have two absolutely periodic interplay orbits, similar to orbit G. These occur on either side of the double choreography orbit as in the equal-mass case. However, for more disparate mass ratios there is only one such absolutely periodic interplay orbit, and for this the larger masses must be the inner masses of the system, as is the case for orbit G.

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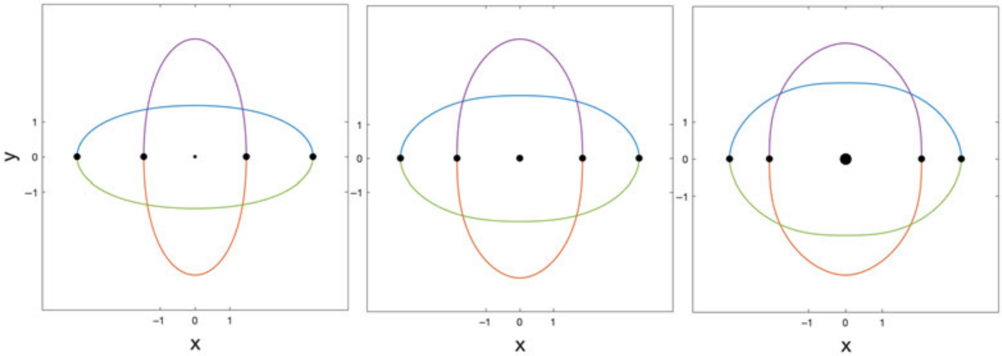


**Figure 7.** Five-body double choreographies. The outer masses are all equal and the outer to central $[m_1 : m_0]$ mass ratios are from left to right [35:20], [3:4] and [1:4], respectively.

Figure 5 shows three absolutely periodic interplay orbits with different mass ratios. For orbits Z1 and Z2 the outer masses are smaller than the inner ones. Orbit Z3 does have larger outer masses but they are of comparable size to the inner masses. For more extreme mass ratios, such as the 3:1 ratio above (Figure 4), there is no such absolutely periodic interplay orbit with the larger masses on the outside.

## 4. Caledonian five-body orbits

Similar families of orbits will occur in the Caledonian five-body problem when a central mass $m_0$ is added to the system. The limit $m_0 = 0$ gives the four-body families already explored. Figure 6 shows two five-body Schubart orbits. Above is the equal masses case. Below is a case with relatively larger central and outermost masses, $m_0$ and $m_1 = m_3$, respectively. The masses between, $m_2 = m_4$, are smaller. Figure 7 shows three double-choreography orbits including a central mass $m_0$ and four equal outer masses $m_1 = m_2 = m_3 = m_4$. The effect of adding the central mass is to make the periodic orbits rounder.

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