The Journal of Symbolic Logic
Volume 87, Number 3, September 2022

ALMOST DISJOINT AND MAD FAMILIES IN VECTOR SPACES AND CHOICE PRINCIPLES

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Abstract. In set theory without the Axiom of Choice (AC), we investigate the open problem of the deductive strength of statements which concern the existence of almost disjoint and maximal almost disjoint (MAD) families of infinite-dimensional subspaces of a given infinite-dimensional vector space, as well as the extension of almost disjoint families in infinite-dimensional vector spaces to MAD families.

§1. Introduction. In Tachtsis [17, 18], the research was centered around open problems which concerned the set-theoretic strength of statements on the existence of almost disjoint and maximal almost disjoint (MAD) families on any infinite set, such as "Every infinite set has an uncountable almost disjoint family"; "Every almost disjoint family in an infinite set X can be extended to a MAD family in X"; "No infinite MAD family in an infinite set has cardinality $\aleph_0$" in mild extensions of ZF (Zermelo–Fraenkel set theory without the AC) and of ZFA (ZF with the Axiom of Extensionality modified in order to allow atoms), that is, in ZF+ Weak Choice and in ZFA+ Weak Choice.

In the current paper, we investigate the open problem of the deductive strength of analogous statements in the realm of infinite-dimensional vector spaces and we determine their placement in the hierarchy of weak choice principles. Among other results, we will prove the following (complete definitions of notions and terms appearing in the following list, will be given in Section 2):

(1) $\mathsf{MC}^{\aleph_0}$ is equivalent to "For every field F and every infinite-dimensional vector space V over F, no MAD2 family in V has cardinality $\aleph_0$." (Theorem 4.1.)
(2) The statement "For every well orderable field F and every infinite-dimensional vector space V over F with a well orderable basis, no MAD2 family in V has cardinality $\aleph_0$" is provable in ZF. (Theorem 4.2.)
(3) $\mathsf{AC}^{\aleph_0}$ implies "For every field F and every infinite-dimensional vector space V over F, there is an AD2 family in V of cardinality $2^{\aleph_0}$." (Theorem 4.6.)

Received February 15, 2020.
2020 *Mathematics Subject Classification*. 03E25, 03E35, 15A03.
Key words and phrases. axiom of choice, weak axioms of choice, vector space, finite-dimensional vector space, infinite-dimensional vector space, almost disjoint family in a vector space, MAD family in a vector space, permutation model of ZFA set theory, Jech–Sochor first embedding theorem, Pincus' transfer theorems.

0022-4812/22/8703-0010
DOI:10.1017/jsl.2021.91

(4) BPI does not imply the statement "For every field F and every infinite-dimensional vector space V over F, there is an infinite AD1 family in V" in ZF. (Theorem 4.9.)
(5) In ZFA, MC implies "In every infinite-dimensional vector space V over any field, every AD1 family of infinite-dimensional subspaces of V can be extended to a MAD1 family in V". Hence, the latter statement does not imply AC in ZFA. (Theorem 4.10.)
(6) $\mathsf{AC}^{\mathsf{LO}}$ does not imply "In every infinite-dimensional vector space V, every AD1 family in V can be extended to a MAD1 family" in ZFA. Hence, neither LW nor $\mathsf{AC}^{\mathsf{WO}}$ imply the above proposition in ZFA. Furthermore, the above proposition is not provable in ZF. (Theorem 4.11.)
(7) For every uncountable regular cardinal $\aleph_\alpha$, there exists a Fraenkel–Mostowski model $\mathcal{N}_{\aleph_\alpha}$ such that for every infinite cardinal $\lambda < \aleph_\alpha$, DC_λ is true in $\mathcal{N}_{\aleph_\alpha}$, but there exists an infinite-dimensional vector space over some field which has an AD1 family in $\mathcal{N}_{\aleph_\alpha}$ that cannot be extended to a MAD1 family in $\mathcal{N}_{\aleph_\alpha}$. The result is transferable to ZF. (Theorem 4.13.)

The only sources of relevant information in this area of research which are known to us are the papers of Kolman [8] and Smythe [14]. However, their perspective is fairly different from the one here. In particular, the above papers deal with the possible cardinality of infinite almost disjoint and MAD families in countably infinite-dimensional vector spaces over a countable (possibly finite) field and the definability of MAD families in such spaces in certain models of ZF + ¬AC.

§2. Notation, terminology, and known results.

NOTATION 2.1. *Let X be a set and also let F be a field.*

If $f \in F^X$, where F^X is the set of all functions from X to F, then $\operatorname{supp}(f)$ *denotes the support of f, i.e.,* $\operatorname{supp}(f) = \{x \in X : f(x) \neq 0_F\}$, *where 0_F is the additive identity of F.*

If $A \subseteq X$, then the element χ_A of F^X denotes the characteristic function of A, i.e., $\chi_A(x) = 1_F$ if $x \in A$ and $\chi_A(x) = 0_F$ if $x \in X \setminus A$, where 1_F is the multiplicative identity of F.

$[X]^{<\omega}$ *denotes the set of finite subsets of X and, for $n \in \omega$, $[X]^n$ denotes the set of n-element subsets of X.*

DEFINITION 2.2. Let $(V, +, \cdot)$ be a vector space over a field F.

If $X \subseteq V$, then $\langle X \rangle$ denotes the *linear span* of X.

V is called *finite-dimensional* if V is finitely generated, i.e., if there exists a finite set $X \subseteq V$ such that $V = \langle X \rangle$. Otherwise, V is called *infinite-dimensional*.

If $W_1, \dots, W_n$ (where n is some positive integer) are subspaces of V, then the *sum of the subspaces W_i* of V is the subspace of V, $\sum_{i=1}^{n} W_i = \left\{ \sum_{i=1}^{n} x_i : \forall i \in \{1, \dots, n\}(x_i \in W_i) \right\}$.

As usual, '$W \leq V$' denotes that W is a vector subspace of V.

It is part of the folklore that, in ZF, every finite-dimensional vector space has a basis. However, it is a celebrated result of Blass [1] that, in ZF, AC is equivalent to "For every field F, every vector space over F has a basis". In particular, Blass

https://doi.org/10.1017/jsl.2021.91 Published online by Cambridge University Press

showed that the latter algebraic statement implies the Axiom of Multiple Choice (MC), which is equivalent to AC in ZF, but it is not equivalent to AC in ZFA (see [3] [7, Theorems 9.1 and 9.2]).

DEFINITION 2.3. Let X be an infinite set. A family $\mathcal{A}$ of infinite subsets of X is called *almost disjoint* in X if for all $A, B \in \mathcal{A}$ with $A \neq B$, $A \cap B$ is finite. An almost disjoint family $\mathcal{A}$ in X is called MAD in X if for every almost disjoint family $\mathcal{B}$ in X with $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} = \mathcal{B}$.

DEFINITION 2.4. Let V be an infinite-dimensional vector space over a field F and let $\mathcal{A}$ be a family of infinite-dimensional vector subspaces of V.

(1) $\mathcal{A}$ is called *almost disjoint*-1 in V (abbreviated by AD1) if for all $U, W \in \mathcal{A}$ with $U \neq W$, $U \cap W$ is finite-dimensional.

(2) $\mathcal{A}$ is called *almost disjoint*-2 in V (abbreviated by AD2) if for every (non-empty) finite subset $\mathcal{B} \subset \mathcal{A}$, $B \cap (\sum_{X \in \mathcal{B} \setminus \{B\}} X)$ is finite-dimensional for all $B \in \mathcal{B}$.

(3) Let $i \in \{1, 2\}$. If $\mathcal{A}$ is an ADi family in V, then $\mathcal{A}$ is called *maximal almost disjoint MAD- i* in V (abbreviated by MADi) if for every ADi family $\mathcal{B}$ in V with $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{A} = \mathcal{B}$.

REMARK 2.5. It is clear that AD2 $\Rightarrow$ AD1. However, the converse may fail to be true. Indeed, let V be the infinite-dimensional vector space $\mathbb{R}^\omega$ over $\mathbb{R}$ (where addition and scalar multiplication are the usual coordinatewise operations). Consider the following infinite-dimensional subspaces of V: $W_1 = \langle \{\chi_{\{2n\}} : n \in \omega\} \rangle$, $W_2 = \langle \{\chi_{\{2n+1\}} : n \in \omega\} \rangle$, and $W_3 = \langle \{\chi_{\{2n,2n+1\}} : n \in \omega\} \rangle$. Then the family $\mathcal{A} = \{W_1, W_2, W_3\}$ is AD1 but not AD2.

DEFINITION 2.6. Let X be a set.

X is called *denumerable* if $|X| = \aleph_0$, i.e., if there is a bijection $f : \omega \to X$.

X is called *countable* if X is either finite or denumerable.

X is called *Dedekind-finite* if $\aleph_0 \not\leq |X|$, i.e., if there is no one-to-one function $f : \omega \to X$. Otherwise, X is called *Dedekind-infinite*.

If X is infinite, then X is called *amorphous* if it cannot be written as a disjoint union of two infinite subsets.[1]

Next, we list the set-theoretic principles that will be used in this paper.

DEFINITION 2.7.

(1) The *Axiom of Multiple Choice* MC (Form 67 in [3]): For every set X of non-empty sets there is a function F with domain X such that for all $x \in X$, $F(x)$ is a non-empty finite subset of x. The function F is called a *multiple choice function* for X.
(2) The *Axiom of Countable Multiple Choice* $\mathsf{MC}^{\aleph_0}$ (Form 126 in [3]): Every denumerable family of non-empty sets has a multiple choice function.
(3) $\mathsf{AC}^{\mathsf{LO}}$ (Form 202 in [3]): Every linearly ordered family of non-empty sets has a choice function.
(4) $\mathsf{AC}^{\mathsf{WO}}$ (Form 40 in [3]): Every well-ordered family of non-empty sets has a choice function.

[1]Clearly every amorphous set is Dedekind-finite, but the converse may fail; the set A of the denumerably many added Cohen reals in the Basic Cohen Model, is Dedekind-finite but not amorphous—see [3, Model $\mathcal{M}1$].

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(5) The *Axiom of Countable Choice* $\mathsf{AC}^{\aleph_0}$ (Form 8 in [3]): Every denumerable family of non-empty sets has a choice function.
(6) LW (Form 90 in [3]): Every linearly ordered set can be well ordered.
(7) Let κ be an infinite well-ordered cardinal number. DC_κ (Form 87(κ) in [3]): Let S be a non-empty set and let R be a binary relation such that for every $\alpha < \kappa$ and every α-sequence $s = (s_\xi)_{\xi<\alpha}$ of elements of S there exists $y \in S$ such that $s\ R\ y$. Then there is a function $f : \kappa \to S$ such that for every $\alpha < \kappa$, $(f \restriction \alpha)\ R\ f(\alpha)$.

(Note that $\mathsf{DC}_{\aleph_0}$ is a reformulation of the Principle of Dependent Choices (DC) (Form 43 in [3]).)
(8) The *Boolean Prime Ideal Theorem* BPI (Form 14 in [3]): Every Boolean algebra has a prime ideal.
(9) Let κ be an infinite well-ordered cardinal number.

$\mathsf{MA}(\kappa)$ is the principle: if $(P, \leq)$ is a non-empty c.c.c. partial order[2] and if $\mathcal{D}$ is a family of $\leq \kappa$ dense sets in P, then there is a filter F of P such that $F \cap D \neq \emptyset$ for all $D \in \mathcal{D}$. (Such a filter F of P is called a $\mathcal{D}$-*generic* filter of P.)

For $\kappa = \aleph_0$, $\mathsf{MA}(\aleph_0)$ is Form 339 in [3].

Martin's Axiom MA: $\forall\kappa < 2^{\aleph_0}(\mathsf{MA}(\kappa))$, where the parameter κ runs through the well-ordered cardinal numbers.

REMARK 2.8. $\mathsf{MC}^{\aleph_0}$ is equivalent to its partial version $\mathsf{PMC}^{\aleph_0}$, i.e. the statement "Every denumerable family $\mathcal{A}$ of non-empty sets has an infinite subfamily $\mathcal{B}$ with a multiple choice function" (a multiple choice function for an infinite subfamily of $\mathcal{A}$ is called a *partial multiple choice function* for $\mathcal{A}$); see [3].

Both $\mathsf{AC}^{\mathsf{LO}}$ and LW are equivalent to AC in ZF, but none of them are equivalent to AC in ZFA (see [3]). Furthermore, both LW and $\mathsf{AC}^{\mathsf{WO}}$ are strictly weaker than $\mathsf{AC}^{\mathsf{LO}}$ in ZFA, and $\mathsf{AC}^{\mathsf{WO}}$ is strictly weaker than AC in ZF (see [3]).

For any infinite well-ordered cardinal number κ, DC_κ implies "Every family which has cardinality κ and comprises non-empty sets, has a choice function." Furthermore, $\forall\kappa(\mathsf{DC}_\kappa)$ (where the parameter κ runs through the infinite well-ordered cardinals) is equivalent to AC in ZFA (see [7, Theorem 8.1((b),(c))]).

We also recall the following renowned results about almost disjoint and MAD families; for their proof, see (for example) Kunen [9, Chapter II].

THEOREM 2.9. *The following hold*:

(i) (ZF) *There is an almost disjoint family in* ω *of cardinality* $2^{\aleph_0}$.

(ii) *Assume* $\mathsf{MA}(\kappa)$ *for some well-ordered cardinal* κ *with* $\aleph_0 \leq \kappa < 2^{\aleph_0}$. *If* $\mathcal{A} \subseteq \mathcal{P}(\omega)$ *is an almost disjoint family with cardinality* κ, *then* $\mathcal{A}$ *is not MAD*. (*The case where* $\kappa = \aleph_0$ *is provable in* ZF.)

(iii) (ZFC) *For every infinite set* X, *every almost disjoint family in* X *can be extended to a MAD family in* X.

§3. Diagram of results. In Figure 1, we summarize main results of our paper. Some explanations about Figure 1 are in order:

[2] A partially ordered set $(P, \leq)$ is called c.c.c. if every antichain in P (i.e. every subset of P comprising pairwise incompatible elements) is countable. (Where, for a partially ordered set $(P, \leq)$, two elements p and q of P are called *compatible* if there exists $r \in P$ such that $r \leq p$ and $r \leq q$.)

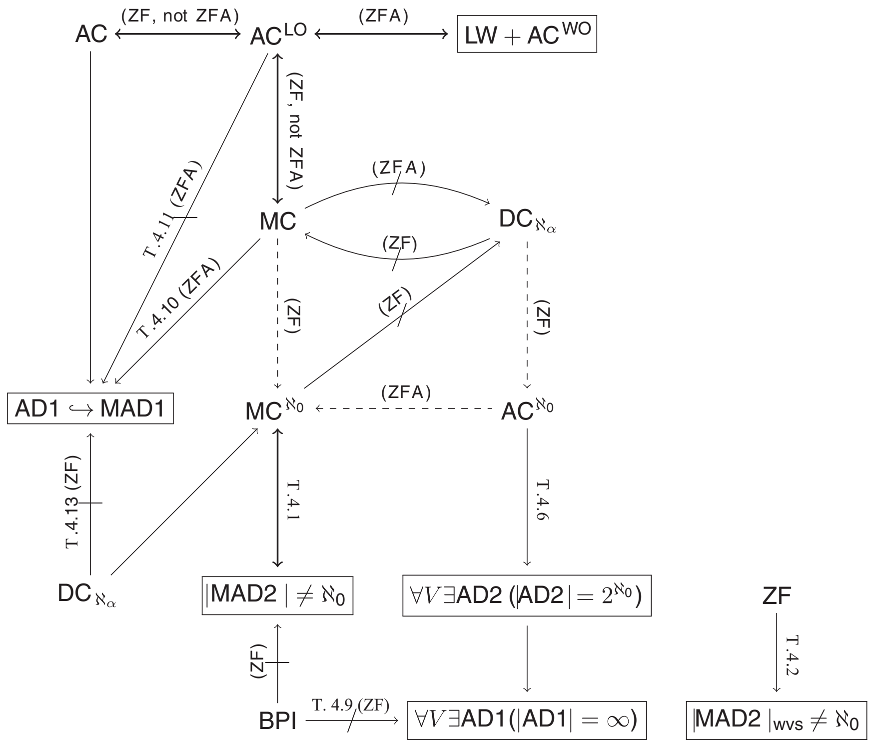

FIGURE 1. Main results of the paper.

(1) For use in Figure 1, we insert some notation here:
- $\mathsf{AD1} \hookrightarrow \mathsf{MAD1}$: For every field F and every infinite-dimensional vector space V over F, every AD1 family of infinite-dimensional subspaces of V can be extended to a MAD1 family in V.
- $|\mathsf{MAD2}| \neq \aleph_0$: For every field F and every infinite-dimensional vector space V over F, no MAD2 family in V has cardinality $\aleph_0$.
- $|\mathsf{MAD2}|_{\mathsf{wvs}} \neq \aleph_0$: For every well orderable field F and every infinite-dimensional vector space V over F with a well orderable basis, no MAD2 family in V has cardinality $\aleph_0$.
- $\forall V \exists \mathsf{AD2}(|\mathsf{AD2}| = 2^{\aleph_0})$: For every field F and every infinite-dimensional vector space V over F, there is an AD2 family in V of cardinality $2^{\aleph_0}$.
- $\forall V \exists \mathsf{AD1}(|\mathsf{AD1}| = \infty)$: For every field F and every infinite-dimensional vector space V over F, there is an infinite AD1 family in V.

(2) Arrows or negated arrows without a label that refer to some result in the paper, represent implications or non-implications, respectively, that are "known" or "straightforward." The reader is referred to Howard and Rubin [3] for known results.

(3) A dashed arrow from A to B which is labeled with '(ZF)' (respectively, with '(ZFA)') means that A implies B, but the implication is not reversible in ZF (respectively, in ZFA).

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(4) If proposition A is equivalent to proposition B, then we use a thick left-right arrow between A and B.

§4. Main results. Our first result, Theorem 4.1, shows that the statement "For every field F and for every infinite-dimensional vector space V over F, no MAD2 family in V has cardinality $\aleph_0$" is equivalent to a well-known weak choice principle, namely $\mathsf{MC}^{\aleph_0}$.

THEOREM 4.1. *The following statements are equivalent*:
(i) $\mathsf{MC}^{\aleph_0}$.
(ii) *For every field F and every infinite-dimensional vector space V over F, no MAD*2 *family in V has cardinality* $\aleph_0$.

PROOF. (i) $\Rightarrow$ (ii) Let F be any field, and also let V be an infinite-dimensional vector space over F. Let $\mathcal{A} = \{W_n : n \in \omega\}$ be an AD2 family of infinite-dimensional subspaces of V with cardinality $\aleph_0$ (the mapping $n \mapsto W_n$ is a bijection). Let $U_0 = W_0$ and for $n \in \omega \setminus \{0\}$, let

$$U_n = W_n \setminus (W_0 + \cdots + W_{n-1}).$$

Since $\mathcal{A}$ is AD2, it follows that for all $n \in \omega$, $U_n \neq \emptyset$; in particular, $\langle U_n \rangle$ is an infinite-dimensional subspace of V for all $n \in \omega$. By $\mathsf{MC}^{\aleph_0}$, let f be a multiple choice function for the denumerable, disjoint family $\mathcal{B} = \{U_n : n \in \omega\}$. Then

$$Z = \left\langle \bigcup \{f(U_n) : n \in \omega\} \right\rangle$$

is an infinite-dimensional subspace of V since, for every $n \in \omega$, any choice set for $\{U_i : i < n+1\}$ is a linearly independent subset of V with cardinality $n+1$ (and hence Z has arbitrarily large finite linearly independent subsets).

Furthermore, $Z \notin \mathcal{A}$ and $\mathcal{A} \cup \{Z\}$ is AD2. Indeed, if $Z \in \mathcal{A}$, then $Z = W_n$ for some $n \in \omega$. But then, for any $u \in f(U_{n+1}) \subseteq Z$, we have $u \notin W_n = Z$, which is a contradiction.

For the second assertion, let $\{W_{n_1}, W_{n_2}, \dots, W_{n_k}\} \subset \mathcal{A}$, where $n_1 < n_2 < \cdots < n_k$. Fix $j \in \{1, \dots, k\}$. Since $f(U_n) \subset W_n \setminus (W_0 + \cdots + W_{n-1})$ for all $n \in \omega \setminus \{0\}$, it follows that

$$W_{n_j} \cap \left(\left(\sum_{n_i \neq n_j} W_{n_i} \right) + Z \right) = W_{n_j} \cap \left(\left(\sum_{n_i \neq n_j} W_{n_i} \right) + \left\langle \bigcup \{f(U_i) : i \leq n_k\} \right\rangle \right). \tag{1}$$

Let

$$W_{n_j} \cap \left(\sum \{W_i : i \in (n_k+1) \setminus \{n_j\}\} \right) = \langle u_1, u_2, \dots, u_m \rangle,$$

(recall that $\mathcal{A}$ is AD2). Then it is not hard to verify that

$$W_{n_j} \cap \left(\left(\sum_{n_i \neq n_j} W_{n_i} \right) + \left\langle \bigcup \{f(U_i) : i \leq n_k\} \right\rangle \right) \subseteq \left\langle \{u_1, u_2, \dots, u_m\} \cup \left(\bigcup \{f(U_i) : i \leq n_k\} \right) \right\rangle,$$

and thus, by Equation (1), we have

$$W_{n_j} \cap \left(\left(\sum_{n_i \neq n_j} W_{n_i} \right) + Z \right) \subseteq \left\langle \{u_1, u_2, \dots, u_m\} \cup \left(\bigcup \{ f(U_i) : i \leq n_k \} \right) \right\rangle.$$

Therefore, $W_{n_j} \cap ((\sum_{n_i \neq n_j} W_{n_i}) + Z)$ is finite-dimensional.

On the other hand, it is easy to see that

$$Z \cap \left(\sum_{j=1}^{k} W_{n_j} \right) \subseteq \left\langle \bigcup \{ f(U_j) : j \leq n_k \} \right\rangle,$$

and hence $Z \cap (\sum_{j=1}^{k} W_{n_j})$ is finite-dimensional. We leave the details of the above observations to the interested reader. Thus $\mathcal{A} \cup \{Z\}$ is an AD2 family in V which properly contains $\mathcal{A}$. Hence, $\mathcal{A}$ is not MAD2.

(ii) $\Rightarrow$ (i) Assume the hypothesis. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a denumerable family of non-empty sets. Without loss of generality, we assume that $\mathcal{A}$ is disjoint and that every member of $\mathcal{A}$ is infinite.

Let $A = \bigcup \mathcal{A}$ and also let

$$V = \{ f \in \mathbb{Z}_2^A : |\operatorname{supp}(f)| < \aleph_0 \},$$

i.e., $V = \{ f \in \mathbb{Z}_2^A : |f^{-1}(\{1\})| < \aleph_0 \}$. Then V with pointwise operations is an infinite-dimensional vector space over $\mathbb{Z}_2$ (and note that $\{\chi_{\{a\}} : a \in A\}$ is a basis for V).

For each $i \in \omega$ we let

$$V_i = \{ f \in V : \forall x \in A \setminus A_i (f(x) = 0) \},$$

(so an element f of V is in V_i if and only if $\operatorname{supp}(f) \subset A_i$). It is clear that for every $i \in \omega$, V_i is an infinite-dimensional vector subspace of V (and furthermore, note that $\{\chi_{\{a\}} : a \in A_i\}$ is a basis for V_i). We let

$$\mathcal{V} = \{ V_i : i \in \omega \}.$$

$\mathcal{V}$ is an AD2 family in V since, if J is a finite subset of ω (with at least two elements) and $i \in J$, then $V_i \cap (\sum_{j \in J \setminus \{i\}} V_j) = \{0\}$. Furthermore, $|\mathcal{V}| = \aleph_0$, and thus (by our hypothesis) $\mathcal{V}$ is not MAD2 in V. So there exists an infinite-dimensional subspace W of V such that $W \notin \mathcal{V}$ and $\mathcal{W} = \mathcal{V} \cup \{W\}$ is AD2 in V.

Since the field of scalars (i.e., $\mathbb{Z}_2$) is finite and $\mathcal{W}$ is AD2, we have that

$$\forall J \in [\omega]^{<\omega} \left(|W \cap \left(\sum_{j \in J} V_j \right)| < \aleph_0 \right). \tag{2}$$

By (2) and the fact that W is infinite-dimensional (and thus W is infinite), it follows that for every $i \in \omega$, $W \nsubseteq \sum_{j \leq i} V_j$, and thus there exists a strictly increasing sequence $(n_i)_{i \in \omega}$ of natural numbers and a sequence $(F_{n_i})_{i \in \omega}$ of finite sets which have the following two properties:

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(a) for every $i \in \omega$, $F_{n_i} \subseteq W \cap (\sum_{j \leq n_i} V_j)$ and
(b) for every $i \in \omega$ and every $f \in F_{n_i}$, $\mathrm{supp}(f \restriction A_{n_i}) \neq \emptyset$.
We define

$$h = \left\{ \left(A_{n_i}, \bigcup \{\mathrm{supp}(f \restriction A_{n_i}) : f \in F_{n_i}\} \right) : i \in \omega \right\}.$$

Then h is a multiple choice function for the infinite subfamily $\{A_{n_i} : i \in \omega\}$ of $\mathcal{A}$ (and hence h is a partial multiple choice function for $\mathcal{A}$).

The conclusion now follows from the fact that $\mathsf{MC}^{\aleph_0}$ is equivalent to $\mathsf{PMC}^{\aleph_0}$ (see Remark 2.8). This completes the proof of "(ii) $\Rightarrow$ (i)" and of the theorem. ⊣

Next, following the ideas of the proof of "(i) $\Rightarrow$ (ii)" of Theorem 4.1, we establish the algebraic statement of Theorem 4.2 within ZF.

THEOREM 4.2. *The statement "For every well orderable field F and every infinite-dimensional vector space V over F with a well orderable basis, no MAD2 family in V has cardinality $\aleph_0$" is provable in* ZF.

PROOF. Let V be an infinite-dimensional vector space over a well-ordered field $F = \{r_\beta : \beta < \lambda\}$, having a well-ordered basis $B = \{b_\alpha : \alpha < \kappa\}$, where λ is a well-ordered cardinal number (possibly finite) and κ is an infinite well-ordered cardinal number. Then $[F]^{<\omega}$ and $[B]^{<\omega}$ are well orderable (recall that, in ZF, $|[\mu]^{<\omega}| = \mu$ for any infinite well-ordered cardinal μ—see [11, Proposition 4.21(ii)]).

Let $\mathcal{A} = \{W_n : n \in \omega\}$ be an AD2 family of infinite-dimensional subspaces of V with cardinality $\aleph_0$, and also let $U_0 = W_0$, and for $n \in \omega \setminus \{0\}$, $U_n = W_n \setminus (W_0 + \cdots + W_{n-1})$.

For every $n \in \omega$, let

$$J_n = \left\{ \{b_{\alpha_1}, \dots, b_{\alpha_k}\} \in [B]^{<\omega} : \exists \{r_{\beta_1}, \dots, r_{\beta_k}\} \in [F]^{<\omega} \left(\sum_{j=1}^{k} r_{\beta_j} b_{\alpha_j} \in U_n \right) \right\}.$$

Since J_n is non-empty (for $U_n \neq \emptyset$ for all $n \in \omega$), and $[B]^{<\omega}$ and $[F]^{<\omega}$ are well-ordered, we may pick the smallest $\{b_{\alpha_1}, \dots, b_{\alpha_k}\} \in J_n$, and for this element of J_n, the smallest $\{r_{\beta_1}, \dots, r_{\beta_k}\} \in [F]^{<\omega}$ (with respect to some prescribed well orderings of $[B]^{<\omega}$ and $[F]^{<\omega}$) such that the vector

$$u_n = \sum_{j=1}^{k} r_{\beta_j} b_{\alpha_j}$$

is an element of U_n. We may now follow the proof of "(i) $\Rightarrow$ (ii)" of Theorem 4.1 in order to establish that

$$Z = \langle \{u_n : n \in \omega\} \rangle,$$

is an infinite-dimensional subspace of V such that $Z \notin \mathcal{A}$ and $\mathcal{A} \cup \{Z\}$ is AD2. ⊣

REMARK 4.3. As the referee pointed out to us, Theorem 4.2 can be also proved by using an absoluteness argument. Indeed, we can encode the well-orderable field F, the well-orderable basis B, and the denumerable AD2 family $\mathcal{A}$ as a set of ordinals, E say. Then in $L[E]$ (i.e., the class of all sets constructible from E), which is a model of ZFC (see Jech [6, Relative Constructibility and Theorem 13.22, p. 192]), we have that $\mathcal{A}$ is not maximal, and therefore not maximal in V either (and we recall here that if A is an arbitrary set and if M is an inner model of ZF such that $A \cap M \in M$, then $L[A] \subset M$, see [6, Theorem 13.22]).

While the algebraic statement of Theorem 4.2 requires no choice principles for its proof, substantial difficulty arises when one tries to prove the following related statements: "For every countable field F and every vector space V over F with a denumerable basis, no MADi family in V has cardinality κ for any well-ordered cardinal number κ with $\aleph_0 \leq \kappa < 2^{\aleph_0}$" ($i = 1, 2$). Kolman [8] and Smythe [14], using a slight modification of Solovay's partial order (see Kunen [9, Definition 2.7, p. 55] for this partial order), established that Martin's Axiom MA implies the above statements for $i = 2$ and $i = 1$, respectively. Furthermore, by employing AC, they obtained that for every vector space V with a denumerable basis, every MADi ($i = 1, 2$) family in V has power $2^{\aleph_0}$.[3] We label the above two results as Theorem 4.4.

Theorem 4.4. [8, 14] *Let κ be a well-ordered cardinal number with $\aleph_0 \leq \kappa < 2^{\aleph_0}$. Then, in* ZF, $\mathsf{MA}(\kappa)$ *implies "For every countable field F and every vector space V over F with a denumerable basis, no MADi family* ($i = 1, 2$) *in V has cardinality κ."*

We would like to note here that it is an open problem whether or not the instance $\mathsf{MA}(\aleph_0)$ of MA implies either of $\mathsf{AC}^{\aleph_0}$ and $\mathsf{MC}^{\aleph_0}$ (and hence, by Theorem 4.1, it is unknown whether $\mathsf{MA}(\aleph_0)$ implies "For every field F and every infinite-dimensional vector space V over F, no MAD2 family in V has cardinality $\aleph_0$").

It is also an open problem whether or not $\mathsf{AC}^{\aleph_0}$ implies $\mathsf{MA}(\aleph_0)$, whereas it has been established recently by Tachtsis [15, Theorem 2.11] that $\mathsf{ZFA} + \mathsf{MC}$ (and hence $\mathsf{ZFA} + \mathsf{MC}^{\aleph_0}$) cannot prove $\mathsf{MA}(\aleph_0)$ restricted to complete Boolean algebras. (Further study on the relative strength of $\mathsf{MA}(\aleph_0)$ restricted to complete Boolean algebras has been conducted in [19].)

By Theorem 4.1 and [15, Theorem 2.11], we obtain the following corollary.

Corollary 4.5. *The statement "For every field F and every infinite-dimensional vector space V over F, no MAD2 family in V has cardinality $\aleph_0$" does not imply* $\mathsf{MA}(\aleph_0)$ *restricted to complete Boolean algebras in* ZFA.

Theorem 4.6. $\mathsf{AC}^{\aleph_0}$ *implies "For every field F and every infinite-dimensional vector space V over F, there is an AD2 family in V of cardinality $2^{\aleph_0}$."*

Proof. Assume the hypothesis. We will use the following lemma, which has been established by Howard and Tachtsis [5]. For the reader's convenience, and in the interest of making our paper self-contained, we include its proof here. ⊣

Lemma 4.7. $\mathsf{AC}^{\aleph_0}$ *implies "For every field F, every infinite-dimensional vector space V over F has a denumerable linearly independent subset."*

Proof. Let F be any field and let V be an infinite-dimensional vector space over F. For each $n \in \omega \setminus \{0\}$, let

$$A_n = \{(v_0, v_1, \dots, v_n) \in V^{n+1} : v_0 \neq 0, \text{ and for all } 1 \leq i \leq n, v_i \notin \langle v_0, v_1, \dots, v_{i-1}\rangle\}.$$

[3] In [8], the definition of an almost disjoint family in an infinite-dimensional vector space is the same as the one given for an AD1 family here. However, Kolman mostly uses the AD2 definition in his proofs.

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Since V is infinite-dimensional, it follows that $A_n \neq \emptyset$ for all $n \in \omega \setminus \{0\}$. Let $\mathcal{A} = \{A_n : n \in \omega \setminus \{0\}\}$ and let, by $\mathsf{AC}^{\aleph_0}$, $f = \{(n, (v_0^{(n)}, v_1^{(n)}, \dots, v_n^{(n)})) : n \in \omega \setminus \{0\}\}$ be a choice function for $\mathcal{A}$. Note that by definition of A_n, $\operatorname{ran}(f(n))$ is an $(n+1)$-sized set of linearly independent vectors of V.

Let $A = \bigcup\{\operatorname{ran}(f(n)) : n \in \omega \setminus \{0\}\}$. It is clear that A is denumerable. Furthermore, since A has finite sequences of linearly independent vectors of arbitrary finite length, we may construct via mathematical induction a denumerable linearly independent subset of V. Indeed, let $w_0 = v_0^{(1)}$. Then w_0 is linearly independent, since $w_0 \neq 0$ (see the definition of A_n). Assume that for some $n \in \omega \setminus \{0\}$ we have chosen linearly independent vectors $w_0, w_1, \dots, w_n \in A$. Since $\dim(\langle w_0, w_1, \dots, w_n\rangle) = n+1$ and $\operatorname{ran}(f(n+1))$ consists of $n+2$ linearly independent vectors, there exists $v \in \operatorname{ran}(f(n+1)) \setminus \langle w_0, w_1, \dots, w_n\rangle$. Let $j_{n+1} = \min\{j : j < n+2$ and $v_j^{(n+1)} \in \operatorname{ran}(f(n+1)) \setminus \langle w_0, w_1, \dots, w_n\rangle\}$. Put $w_{n+1} = v_{j_{n+1}}^{(n+1)}$. This completes the inductive step.

From the above construction, we conclude that $\{w_n : n \in \omega\}$ is a denumerable linearly independent subset of V. ⊣

Now we turn to the proof of the theorem. Fix a field F and an infinite-dimensional vector space V over F. By Lemma 4.7, let $I = \{v_n : n \in \omega\}$ be a denumerable linearly independent subset of V (the mapping $n \mapsto v_n$ is a bijection). We also let $W = \langle I \rangle$. Then W is an infinite-dimensional subspace of V (and I is a denumerable basis for W). By Theorem 2.9(i), let $\mathcal{A}$ be an almost disjoint family in ω of cardinality $2^{\aleph_0}$. We let

$$W_X = \langle \{v_n : n \in X\}\rangle \; (X \in \mathcal{A}).$$

Then $\mathcal{W} = \{W_X : X \in \mathcal{A}\}$ is an AD2 family in V with cardinality $2^{\aleph_0}$ (since $|\mathcal{W}| = |\mathcal{A}| = 2^{\aleph_0}$). To see that $\mathcal{W}$ is AD2, note that if $\mathcal{C}$ is a finite subset of $\mathcal{W}$, then for every $C \in \mathcal{C}$ we have

$$C \cap \left(\sum_{D \in \mathcal{C}\setminus\{C\}} D\right) \subseteq \left\langle \bigcup\{\{v_n : n \in X \cap Y\} : X, Y \in \mathcal{A},\ X \neq Y, \text{and } W_X, W_Y \in \mathcal{C}\}\right\rangle.$$

Since the set $\bigcup\{\{v_n : n \in X \cap Y\} : X, Y \in \mathcal{A}, X \neq Y$, and $W_X, W_Y \in \mathcal{C}\}$ is finite (for $\mathcal{C}$ is finite and $\mathcal{A}$ is almost disjoint in ω), we conclude that $C \cap (\sum_{D \in \mathcal{C}\setminus\{C\}} D)$ is finite-dimensional. ⊣

Tachtsis [16] established that "For every field F, every infinite-dimensional vector space V over F has an infinite linearly independent subset" (which is formally weaker than "For every field F, every infinite-dimensional vector space V over F has a denumerable linearly independent subset") implies $\mathsf{MC}^{\aleph_0}$. This, together with Theorem 4.1, gives us the following result.

THEOREM 4.8. *"For every field F, every infinite-dimensional vector space V over F has an infinite linearly independent subset" implies "For every field F and for every infinite-dimensional vector space V over F, no $MAD2$ family in V has cardinality $\aleph_0$."*

Läuchli [10] proved that it is relatively consistent with ZFA that there is an infinite-dimensional vector space V over $\mathbb{Q}$ (the set of rational numbers) which has no basis, but every proper subspace of V is finite-dimensional. The latter result can be transferred to ZF via the Jech–Sochor First Embedding Theorem (see [7, Theorem 6.1 and Problem 1 (p. 94)]). Therefore, the statement "Every infinite-dimensional vector space has an infinite AD1 family" is not provable in ZF, and thus neither is "Every infinite-dimensional vector space has an infinite AD2 family" provable in ZF (recall that AD2 $\Rightarrow$ AD1).

In the next theorem, we provide a substantial strengthening of the latter result by showing that ZF + BPI cannot prove "Every infinite-dimensional vector space has an infinite AD1 family" (and thus ZF + BPI can neither prove "Every infinite-dimensional vector space has an infinite AD2 family").

THEOREM 4.9. *The statement "Every infinite-dimensional vector space has an infinite AD*1 *family" is not provable in* ZF + BPI.

PROOF. We will first establish the independence of the above statement from BPI in ZFA and then we will transfer the result to ZF by using a suitable transfer theorem of Pincus [13].

We consider the Mostowski Linearly Ordered Model of ZFA, which is labeled as 'Model $\mathcal{N}3$' in Howard–Rubin [3]: The set A of atoms is denumerable and is equipped with an ordering $\leq$ chosen so that $(A, \leq)$ is order-isomorphic to the set $\mathbb{Q}$ of rational numbers with the usual ordering; G is the group of all order automorphisms of $(A, \leq)$; and $\mathcal{F}$ is the finite support (normal) filter on G, i.e., $\mathcal{F}$ is the filter on G which is generated by the subgroups $\mathrm{fix}_G(E) = \{\phi \in G : \forall e \in E(\phi(e) = e)\}$, $E \in [A]^{<\omega}$. $\mathcal{N}3$ is the permutation model determined by A, G and $\mathcal{F}$.

Let us recall that every $x \in \mathcal{N}3$ has a least (finite) support, which we shall denote by E_x (see Jech [7, Lemma 4.5]). Furthermore, it is a renowned result of Halpern [2] that BPI is true in the model $\mathcal{N}3$. Thus we only need to show that there exists an infinite-dimensional vector space in $\mathcal{N}3$ which has no infinite AD1 family in $\mathcal{N}3$.

To this end, we consider the set

$$V = \{f \in \mathbb{Z}_2^A : |\mathrm{supp}(f)| < \aleph_0\},$$

equipped with pointwise operations of addition and multiplication with scalars from $\mathbb{Z}_2$. Then $(V, +, \cdot)$ is an infinite-dimensional vector space over $\mathbb{Z}_2$, and $(V, +, \cdot) \in \mathcal{N}3$ since, for every $\phi \in G$, $\phi((V, +, \cdot)) = (V, +, \cdot)$.

We assert that V has no infinite AD1 family in $\mathcal{N}3$. Assume not, and let $\mathcal{W} \in \mathcal{N}3$ be an infinite AD1 family of infinite-dimensional subspaces of V. Let $E = \{e_1, e_2, \dots, e_n\} \subset A$, where $e_1 < e_2 < \cdots < e_n$, be a finite support of $\mathcal{W}$. Then E determines $n + 1$ pairwise disjoint, open intervals in the ordering of A, namely $(-\infty, e_1)$, (e_i, e_{i+1}) $(1 \leq i < n)$, $(e_n, +\infty)$. Let

$$\mathcal{Z} = \{(-\infty, e_1)\} \cup \{(e_i, e_{i+1}) : 1 \leq i < n\} \cup \{(e_n, +\infty)\}.$$

A couple of observations are in order:

(a) Since $\mathcal{W}$ is AD1 and the field of scalars (i.e. $\mathbb{Z}_2$) is finite, $\mathcal{W}$ is almost disjoint in V in the sense of Definition 2.3, i.e., for every $W, W' \in \mathcal{W}$, if $W \neq W'$ then $W \cap W'$ is finite.

(b) Since $\mathcal{W}$ is infinite and $\mathcal{Z} \cup \{E\}$ is a finite partition of A, a straightforward pigeonhole-type argument yields that there exist distinct $W, W' \in \mathcal{W}$ and an

$I \in \mathcal{Z}$ such that:

$$\forall X \in \{W, W'\}(\{I \cap \mathrm{supp}(f) : f \in X\} \text{ is infinite}). \tag{3}$$

Fix $W, W' \in \mathcal{W}$ and $I \in \mathcal{Z}$ as in (b). Since (by (b)) $W \neq W'$, (a) yields that $W \cap W'$ is finite. This, together with (3) and the fact that $E_W \cup E_{W'}$ is finite, readily implies that there exist distinct $f_W \in W$, $f_{W'} \in W'$ such that $(I \cap \mathrm{supp}(f_W)) \setminus E_W \neq \emptyset$ and $(I \cap \mathrm{supp}(f_{W'})) \setminus E_{W'} \neq \emptyset$.

Let $z = \min((I \cap \mathrm{supp}(f_W)) \setminus E_W)$ and also let $z' \in I \setminus E_W$ such that $z' < z$ and $(z', z) \cap E_W = \emptyset$. Construct a $\phi \in \mathrm{fix}_G(E \cup E_W)$ so that $\phi(z) = z'$ and $f_W, \phi(f_W)$ agree on $A \setminus \{z', z\}$. Since E_W is a support of W and $\phi \in \mathrm{fix}_G(E \cup E_W)$, we have $\phi(W) = W$. Thus $\phi(f_W) \in \phi(W) = W$, and since W is a subspace of V, we have $f_W + \phi(f_W) \in W$. Furthermore, it is clear that $\mathrm{supp}(f_W + \phi(f_W)) = \{z', z\} \subset I \setminus E_W$.

Using $f_W + \phi(f_W)$ and suitable elements of $\mathrm{fix}_G(E \cup E_W)$, it is not hard to verify that there exist infinitely many $g_W \in W$ such that $\mathrm{supp}(g_W) \in [I \setminus E_W]^2$ and $\max(\mathrm{supp}(g_W)) < z'$, and infinitely many $h_W \in W$ such that $\mathrm{supp}(h_W) \in [I \setminus E_W]^2$ and $z < \min(\mathrm{supp}(h_W))$. In particular, for every $a, b \in I \setminus E_W$ such that $a < b < z'$ and $(a, z') \cap E_W = \emptyset$, we may consider a $\pi \in \mathrm{fix}_G(E \cup E_W)$ such that $\mathrm{supp}(\pi(f_W + \phi(f_W))) = \{a, b\}$. Letting $g_W = \pi(f_W + \phi(f_W))$, we have $g_W \in W$ (for $f_W + \phi(f_W) \in W$, $\pi \in \mathrm{fix}_G(E \cup E_W)$ and E_W is a support of W), $\mathrm{supp}(g_W) = \{a, b\} \in [I \setminus E_W]^2$ and $\max(\mathrm{supp}(g_W)) = b < z'$. We may work in a similar manner for the h_W's, and thus we leave this to the interested reader. By the above considerations, we have:

$$\forall g_W \forall h_W(\max(\mathrm{supp}(g_W)) < z' < z < \min(\mathrm{supp}(h_W))). \tag{4}$$

As with f_W, we may let $t = \min((I \cap \mathrm{supp}(f_{W'})) \setminus E_{W'})$, $t' \in I \setminus E_{W'}$ such that $t' < t$ and $(t', t) \cap E_{W'} = \emptyset$, and a $\psi \in \mathrm{fix}_G(E \cup E_{W'})$ such that $f_{W'} + \psi(f_{W'}) \in W'$ and $\mathrm{supp}(f_{W'} + \psi(f_{W'})) = \{t', t\} \subset I \setminus E_{W'}$. Furthermore, similarly to the arguments of the previous paragraph, we may conclude that there exist infinitely many $g_{W'} \in W'$ such that $\mathrm{supp}(g_{W'}) \in [I \setminus E_{W'}]^2$ and $\max(\mathrm{supp}(g_{W'})) < t'$, and infinitely many $h_{W'} \in W'$ such that $\mathrm{supp}(h_{W'}) \in [I \setminus E_{W'}]^2$ and $t < \min(\mathrm{supp}(h_{W'}))$. Hence, we have:

$$\forall g_{W'} \forall h_{W'}(\max(\mathrm{supp}(g_{W'})) < t' < t < \min(\mathrm{supp}(h_{W'}))). \tag{5}$$

Since $W \cap W'$ is finite and the sets of functions h_W and of $h_{W'}$ are infinite subsets of W and W', respectively, we may pick an $h_W \in W \setminus W'$ and an $h_{W'} \in W' \setminus W$ (and thus $h_W \neq h_{W'}$).

There are the following cases:

(i) $\min(\mathrm{supp}(h_W)) \leq \min(\mathrm{supp}(h_{W'}))$.[4] In view of (4), we may construct an $\eta \in \mathrm{fix}_G(E)$ such that $\eta(h_W) = h_{W'}$ and η fixes each of the elements g_W of W, which are infinitely many. Since $h_{W'} = \eta(h_W) \in \eta(W)$ and $h_{W'} \notin W$, we obtain that $\eta(W) \neq W$. Furthermore, as $W \in \mathcal{W}$, E is a support of $\mathcal{W}$ and $\eta \in \mathrm{fix}_G(E)$, we have $\eta(W) \in \eta(\mathcal{W}) = \mathcal{W}$. Since η fixes an infinite subset of

[4]Note that if $\min(\mathrm{supp}(h_W)) = \min(\mathrm{supp}(h_{W'}))$, then one may alternatively use the argument in case (ii).

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W pointwise, namely the set of the functions g_W, we deduce that $\eta(W) \cap W$ is infinite. But this contradicts the fact that $\mathcal{W}$ is almost disjoint in V.

(ii) $\min(\mathrm{supp}(h_{W'})) < \min(\mathrm{supp}(h_W))$. In view of (5), we may construct a $\sigma \in \mathrm{fix}_G(E)$ such that $\sigma(h_{W'}) = h_W$ and σ fixes each of the elements $g_{W'}$ of W', which are infinitely many. Working similarly to (i), we infer that $\sigma(W') \neq W'$, $\sigma(W') \in \mathcal{W}$, and $\sigma(W') \cap W'$ is infinite; thus contradicting $\mathcal{W}$'s being almost disjoint in V.

By the above arguments, we conclude that the vector space V has no infinite AD1 family in the model $\mathcal{N}3$, as asserted.

Now, we refer the reader to Pincus [12] or [13] (or to Howard–Rubin [3, Note 103, pp. 284–285]) for the definition of the terms "boundable statement" and "injectively boundable statement." To transfer the above result to ZF, we will apply the following transfer theorem of Pincus (see [13, Theorem 4 and note added in proof]): "If a conjunction of injectively boundable statements and BPI has a Fraenkel–Mostowski model, then it also has a ZF-model."

It is easy to see that Ψ = "There exists an infinite-dimensional vector space which has no infinite AD1 family" is a boundable statement, and thus Ψ is injectively boundable (boundable statements are (up to equivalence) injectively boundable, see Pincus [12, p. 722]). Letting $\Phi = \mathsf{BPI} \wedge \Psi$, we have that Φ satisfies the hypotheses of [13, Theorem 4] stated above, and thus Φ has a ZF-model. This completes the proof of the theorem. ⊣

THEOREM 4.10. (ZFA) MC *implies "For every field F and for every infinite-dimensional vector space V over F, every $AD1$ family of infinite-dimensional subspaces of V can be extended to a $MAD1$ family in V." Hence, the latter statement does not imply* AC *in* ZFA.

PROOF. Assume the hypothesis. Let V be an infinite-dimensional vector space over a field F, and let $\mathcal{A}$ be an AD1 family in V. Assume that $\mathcal{A}$ is not MAD1 (otherwise, there is nothing to show). By MC, let f be a multiple choice function for $\wp(\wp(V)) \setminus \{\emptyset\}$.

By transfinite recursion, we will construct a MAD1 family in V which contains $\mathcal{A}$. Let

$$\mathcal{R}_0 = \{Z : Z \leq V, Z \text{ is infinite-dimensional}, Z \notin \mathcal{A}, \text{ and } \mathcal{A} \cup \{Z\} \text{ is } AD1\}.$$

Since $\mathcal{A}$ is not maximal, $\mathcal{R}_0 \neq \emptyset$. We let

$$\mathcal{S}_0 = \{\mathcal{U} : \mathcal{U} \text{ is a } \subsetneq\text{-maximal subset of } f(\mathcal{R}_0) \text{ such that } \bigcap \mathcal{U} \text{ is infinite-dimensional}\},$$

$$\mathcal{T}_0 = \left\{\bigcap \mathcal{U} : \mathcal{U} \in \mathcal{S}_0\right\},$$

and

$$\mathcal{A}_0 = \mathcal{A} \cup \mathcal{T}_0.$$

The family $\mathcal{A}_0$ has the following properties:

(a) $\mathcal{A} \subsetneq \mathcal{A}_0$. Indeed, pick any element T of $\mathcal{T}_0$. Then $T = \bigcap \mathcal{U}$ for some $\mathcal{U} \in \mathcal{S}_0$. We assert that $T \notin \mathcal{A}$. If not (i.e., if $T \in \mathcal{A}$), then we choose any element Z of $\mathcal{U}$. Since $Z \in \mathcal{R}_0$ (for $Z \in f(\mathcal{R}_0) \subseteq \mathcal{R}_0$), we have that $\mathcal{A} \cup \{Z\}$ is AD1. However, $T \cap Z = (\bigcap \mathcal{U}) \cap Z = \bigcap \mathcal{U}$ (for $Z \in \mathcal{U}$). Since $\bigcap \mathcal{U}$ is infinite-dimensional, we have

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obtained a contradiction to the fact that $\mathcal{A} \cup \{Z\}$ is AD1. Therefore, $T \in \mathcal{A}_0 \setminus \mathcal{A}$, and thus $\mathcal{A} \subsetneq \mathcal{A}_0$.

(b) $\mathcal{A}_0$ is AD1. Let $U, W \in \mathcal{A}_0$ with $U \neq W$. There are the following cases:

(b1) $U, W \in \mathcal{A}$. Then $U \cap W$ is finite-dimensional since $\mathcal{A}$ is AD1.

(b2) $U, W \in \mathcal{T}_0$. Then $U = \bigcap \mathcal{U}$ and $W = \bigcap \mathcal{U}'$ for some $\mathcal{U}, \mathcal{U}' \in \mathcal{S}_0$. Since $U \neq W$, we have $\mathcal{U} \neq \mathcal{U}'$ so $\mathcal{U} \nsubseteq \mathcal{U}'$ and $\mathcal{U}' \nsubseteq \mathcal{U}$, for $\mathcal{U}$ and $\mathcal{U}'$ are maximal subsets of $f(\mathcal{R}_0)$ such that $\bigcap \mathcal{U}$ and $\bigcap \mathcal{U}'$ are infinite-dimensional. It follows that $\mathcal{U}, \mathcal{U}' \subsetneq \mathcal{U} \cup \mathcal{U}'$, so $\bigcap(\mathcal{U} \cup \mathcal{U}') = (\bigcap \mathcal{U}) \cap (\bigcap \mathcal{U}') = U \cap W$ is finite-dimensional.

(b3) $U \in \mathcal{A}$, $W \in \mathcal{T}_0$. There exists $\mathcal{U} \in \mathcal{S}_0$ such that $W = \bigcap \mathcal{U}$. Pick any element Z of $\mathcal{U}$. Then $U \cap Z$ is finite-dimensional, for $U \in \mathcal{A}$ and $\mathcal{A} \cup \{Z\}$ is AD1. Since $U \cap W \subseteq U \cap Z$, we have that $U \cap W$ is finite-dimensional.

(b4) $U \in \mathcal{T}_0$, $W \in \mathcal{A}$. Similarly to (b3), we conclude that $U \cap W$ is finite-dimensional.

Hence, $\mathcal{A}_0$ is AD1.

We assume that for some ordinal number α we have constructed an $\subseteq$-increasing sequence $(\mathcal{A}_\beta)_{\beta<\alpha}$ of AD1 families in V such that $\mathcal{A} \subsetneq \mathcal{A}_0$. Let

$$\mathcal{B}_\alpha = \bigcup \{\mathcal{A}_\beta : \beta < \alpha\},$$

and also let

$$\mathcal{R}_\alpha = \{Z : Z \leq V, Z \text{ is infinite-dimensional}, Z \notin \mathcal{B}_\alpha, \text{ and } \mathcal{B}_\alpha \cup \{Z\} \text{ is AD1}\}.$$

Since $(\mathcal{A}_\beta)_{\beta<\alpha}$ is a chain of AD1 families, $\mathcal{B}_\alpha$ is AD1. If $\mathcal{R}_\alpha = \emptyset$, then $\mathcal{B}_\alpha$ is a MAD1 family in V which contains $\mathcal{A}$, and we are done. Otherwise, (that is, if $\mathcal{R}_\alpha \neq \emptyset$), we define

$$\mathcal{A}_\alpha = \mathcal{B}_\alpha \cup \Big\{ \bigcap \mathcal{U} : \mathcal{U} \text{ is a } \subseteq\text{-maximal subset of } f(\mathcal{R}_\alpha) \text{ such that } \bigcap \mathcal{U} \text{ is infinite-dimensional} \Big\}.$$

Since Ord (i.e., the class of all ordinal numbers) is a proper class, the recursion must terminate at some ordinal stage. By the above construction, the ending of the recursion yields a MAD1 family in V which contains $\mathcal{A}$.

The second assertion of the theorem follows from the first and the fact that MC does not imply AC in ZFA (see [7, Theorem 9.2(i)]). This completes the proof of the theorem. ⊣

In the next theorem, we show that $\mathsf{AC}^{\mathsf{LO}}$ does not imply "In every infinite-dimensional vector space V, every AD1 family in V can be extended to a MAD1 family" in ZFA. Hence, neither LW nor $\mathsf{AC}^{\mathsf{WO}}$ imply the above algebraic proposition in ZFA (see also Remark 2.8). We also recall here that $\mathsf{AC}^{\mathsf{LO}}$ (and hence LW) is equivalent to AC in ZF.

Thus Theorem 4.11, on the one hand, justifies that the above proposition cannot be proved from the ZF axioms alone, and on the other hand, it indicates that this proposition is actually a *strong axiom*. The latter fact is also suggested by the forthcoming Theorem 4.13.

THEOREM 4.11. $\mathsf{AC}^{\mathsf{LO}}$ *does not imply "In every infinite-dimensional vector space V, every AD1 family in V can be extended to a MAD1 family" in* ZFA. *Hence, neither*

LW *nor* $\mathsf{AC}^{\mathsf{WO}}$ *imply the above proposition in* ZFA. *Furthermore, the above proposition is not provable in* ZF.

PROOF. We will use a permutation model which was constructed by Howard and Tachtsis [4], and whose description is as follows: We start with a model M of ZFA + AC with an $\aleph_1$-sized set A of atoms, which is a disjoint, denumerable union of $\aleph_1$-sized sets so that

$$A = \bigcup\{A_i : i \in \omega\},\ |A_i| = \aleph_1.$$

Let G be the group of all permutations of A, which fix A_i for all $i \in \omega$. Note that any element of G also fixes the bijection $i \mapsto A_i$, and hence fixes the ordered partition $(A_i)_{i\in\omega}$ of A. (And also note that for every $i \in \omega$ and every permutation ϕ of A (not necessarily in G), $\phi(i) = i$ since the natural numbers are pure sets (i.e., their transitive closure contains no atoms) and pure sets are fixed by any permutation of A.) Let $\mathcal{F}$ be the (normal) filter of subgroups of G generated by the subgroups $\mathrm{fix}_G(E)$, where $E \subset A$ and $|E| < \aleph_1$. Let $\mathcal{N}$ be the Fraenkel–Mostowski model determined by M, G and $\mathcal{F}$.

In [4, proof B of Theorem 2], it was shown that each of LW and $\mathsf{AC}^{\mathsf{LO}}$ (and hence $\mathsf{AC}^{\mathsf{WO}}$) are true in $\mathcal{N}$. Therefore, it suffices to show that in $\mathcal{N}$, there exists an infinite-dimensional vector space over some field which has an AD1 family in $\mathcal{N}$ that cannot be extended to a MAD1 family in $\mathcal{N}$.

To this end, we consider the following infinite-dimensional vector space V over $\mathbb{Z}_2$,

$$V = \{f \in \mathbb{Z}_2^A : |\operatorname{supp}(f)| < \aleph_0\},$$

where V is equipped with pointwise operations. Then $(V, +, \cdot) \in \mathcal{N}$ (for $(V, +, \cdot)$ is fixed by every permutation of A in G). For each $n \in \omega$, consider the infinite-dimensional vector subspace of V,

$$V_n = \{f \in V : \operatorname{supp}(f) \subset A_n\}.$$

Let

$$\mathcal{V} = \{V_n : n \in \omega\}.$$

$\mathcal{V}$ is an AD1 family in V which belongs to the model $\mathcal{N}$ (any element of G fixes $\mathcal{V}$ pointwise, so $\mathcal{V}$ is also denumerable in $\mathcal{N}$).

$\mathcal{V}$ is not MAD1 in $\mathcal{N}$. Indeed, let H be a choice function for $\mathcal{A} = \{A_n : n \in \omega\}$. (Note that by definition of the filter $\mathcal{F}$, $\operatorname{ran}(H)$ is a support of (every element of) H, and hence $H \in \mathcal{N}$.) Let $W = \langle\{F_n : n \in \omega\}\rangle$, where for $n \in \omega$, $F_n \restriction A_m = 0$ (the zero function) for $m \in \omega \setminus \{n\}$, and $F_n \restriction A_n = \chi_{\{H(n)\}}$. Then W is an infinite-dimensional subspace of V such that $W \notin \mathcal{V}$ and $\mathcal{V} \cup \{W\}$ is AD1 in $\mathcal{N}$. ⊣

CLAIM 4.12. *There is no MAD*1 *family in* $\mathcal{N}$ *containing* $\mathcal{V}$.

PROOF. We prove the claim by contradiction. Let $\mathcal{M}$ be a MAD1 family in $\mathcal{N}$ which contains $\mathcal{V}$ (and thus, by the observations in the previous paragraph, $\mathcal{V} \subsetneq \mathcal{M}$), and also let $E \subset A$ be a support of $\mathcal{M}$. We assert the following:

$$(\forall Z \in \mathcal{M} \setminus \mathcal{V})(\forall f \in Z)(\forall n \in \omega)(\operatorname{supp}(f \restriction A_n) \subseteq E \cap A_n). \tag{6}$$

If not, then there exist $Z \in \mathcal{M} \setminus \mathcal{V}$, $f \in Z$, $n \in \omega$, and $a \in \operatorname{supp}(f \restriction A_n) \setminus (E \cap A_n)$.

Since $Z \cap V_n$ is finite-dimensional (actually, $Z \cap V_n$ is finite since it is a finite-dimensional space over $\mathbb{Z}_2$) and any support of Z meets A_n in a countable set, there exists $b \in A_n \setminus \{a\}$ such that for every $g \in Z$, $b \notin \operatorname{supp}(g \restriction A_n)$. If not, i.e., if for every $b \in A_n \setminus \{a\}$ there exists $g \in Z$ with $b \in \operatorname{supp}(g \restriction A_n)$, then let E_Z be a support of Z. (So $E_Z \cap A_n$ is countable.) Fix $c \in R$, where $R = A_n \setminus [(\operatorname{supp}(f \restriction A_n) \cup E \cup E_Z]$, and let $g \in Z$ such that $c \in \operatorname{supp}(g \restriction A_n)$. For every $r \in R \setminus (\{c\} \cup \operatorname{supp}(g)\})$, let $\phi_r = (c, r)$ (i.e., ϕ_r interchanges the atoms c and r and leaves all the other atoms of A fixed). Then $S = \{\phi_r(g) : r \in R \setminus (\{c\} \cup \operatorname{supp}(g)\})\}$ is an infinite, linearly independent subset of Z (since $\phi_r \in \operatorname{fix}_G(E_Z)$ and $g \in Z$), and thus $\{\sum_{x \in F} x : F \in [S]^{<\omega}\}$ is an infinite, linearly independent subset of $Z \cap V_n$. But this contradicts $Z \cap V_n$'s being finite-dimensional.

Let $\pi = (a, b)$, where (by the above argument) $b \in A_n \setminus (\{a\} \cup \operatorname{supp}(g \restriction A_n))$ for all $g \in Z$. Then $\pi \in \operatorname{fix}_G(E)$, and hence $\pi(\mathcal{M}) = \mathcal{M}$. Thus $\pi(Z) \in \mathcal{M}$ (and also $\pi(Z) \notin \mathcal{V}$), and furthermore, $\pi(Z) \neq Z$ (for $\pi(f) \in \pi(Z) \setminus Z$, since $f \in Z$ and $b \in \operatorname{supp}(\pi(f) \restriction A_n))$.

Since Z is not included in any finite sum of the V_i's, and $Z \cap V_i$ is finite-dimensional for all $i \in \omega$, it is not hard to verify that $\pi(Z) \cap Z$ contains an infinite linearly independent subset. But this contradicts the fact that $\mathcal{M}$ is AD1. Thus (6) is true as asserted.

On the basis of (6), and working similarly to the argument that $\mathcal{V}$ is not MAD1 in $\mathcal{N}$, we may show that neither $\mathcal{M}$ is MAD1 in $\mathcal{N}$. But this contradicts our assumption on $\mathcal{M}$.

Thus the AD1 family $\mathcal{V}$ cannot be extended to a MAD1 family in the model $\mathcal{N}$, finishing the proof of the claim. ⊣

For the last assertion of the theorem (i.e., nonprovability of the algebraic statement in ZF), note that since the negation of "In every infinite-dimensional vector space V, every AD1 family in V can be extended to a MAD1 family" is a boundable statement, and has a ZFA-model, it follows from the Jech–Sochor First Embedding Theorem (see [7, Theorem 6.1 and Problem 1 (p. 94)]) that it has a (symmetric) ZF-model. This completes the proof of the theorem. ⊣

We note that the model $\mathcal{N}$ of the proof of Theorem 4.11 is actually an element of a class of permutation models. Indeed, for any uncountable regular cardinal $\aleph_\alpha$, we may construct a Fraenkel–Mostowski model $\mathcal{N}_{\aleph_\alpha}$ by taking: (a) an $\aleph_\alpha$-sized set A of atoms, which is a disjoint, denumerable union of $\aleph_\alpha$-sized sets so that $A = \bigcup\{A_i : i \in \omega\}$, $|A_i| = \aleph_\alpha$; (b) G to be the group of all permutations of A, which fix A_i for all $i \in \omega$; and (c) $\mathcal{F}$ to be the (normal) filter of subgroups of G generated by the subgroups $\operatorname{fix}_G(E)$, where $E \subset A$ and $|E| < \aleph_\alpha$.

Now, in each of the models $\mathcal{N}_{\aleph_\alpha}$, DC_λ is true for every infinite cardinal $\lambda < \aleph_\alpha$ (see also [7, Lemma 8.4, p. 123]), and in much the same way as the proof of Theorem 4.11, the statement "In every infinite-dimensional vector space V, every AD1 family in V can be extended to a MAD1 family" is false in $\mathcal{N}_{\aleph_\alpha}$. Furthermore, by applying the following transfer theorem of Pincus (see [13, Theorem 4]): "For every ordinal η, if Φ is a conjunction of injectively boundable statements and "$\forall \xi < \eta(\mathsf{DC}_\xi)$" and Φ has a

https://doi.org/10.1017/jsl.2021.91 Published online by Cambridge University Press

Fraenkel–Mostowski model, then Φ has a ZF-model," the above ZFA-independence result can be transferred to ZF. Hence, we obtain the following theorem.

THEOREM 4.13. *For every uncountable regular cardinal $\aleph_\alpha$, there exists a Fraenkel–Mostowski model $\mathcal{N}_{\aleph_\alpha}$ such that for every infinite cardinal $\lambda < \aleph_\alpha$, DC_λ is true in $\mathcal{N}_{\aleph_\alpha}$, but there exists an infinite-dimensional vector space over some field which has an AD1 family in $\mathcal{N}_{\aleph_\alpha}$ that cannot be extended to a MAD1 family in $\mathcal{N}_{\aleph_\alpha}$. The result is transferable to* ZF.

§5. Concluding remarks and open questions. As the referee pointed out, the vector space $V = \{f \in \mathbb{Z}_2^A : |\operatorname{supp}(f)| < \aleph_0\}$ (for some infinite set A), which is used in the majority of the proofs, is essentially the same as $[A]^{<\omega}$ equipped with the symmetric difference as addition. (Note also that $F : V \to [A]^{<\omega}$ defined by $F(f) = \operatorname{supp}(f)$ is a bijection.) So, an almost disjoint (either AD1 or AD2) family in V can be viewed as an almost disjoint family in $[A]^{<\omega}$ (in the sense of Definition 2.3), and thus one may also adopt this approach in order to carry out the arguments in the proofs of certain results, for example, of Theorem 4.9.

It is thus natural and interesting to investigate the possibility of extracting some combinatorial property that bridges the linear-algebraic approach with the set-approach in the realm of almost disjointness. To provide further incentive towards this direction, let us recall in view of the above discussion and the characterization of $\mathsf{MC}^{\aleph_0}$ given by Theorem 4.1, our result of [17, Theorem 7] that $\mathsf{MC}^{\aleph_0}$ is also equivalent to the statement "For every infinite set X, no infinite MAD family in X has cardinality $\aleph_0$"—see also open question (6) in the subsequent list and [18].

(1) Is $\mathsf{MC}^{\aleph_0}$ equivalent to "For every infinite-dimensional vector space V over any field, no MAD1 family in V has cardinality $\aleph_0$"?
(2) What is the status of "For every infinite-dimensional vector space V with a well orderable basis, no MADi family in V has cardinality $\aleph_0$" $(i = 1, 2)$?
(3) What weak choice principles are implied by "For every countable field F and every vector space V over F with a denumerable basis, no MADi $(i = 1, 2)$ family in V has cardinality κ, for any well-ordered cardinal number κ with $\aleph_0 \leq \kappa < 2^{\aleph_0}$"?
(4) Does MC imply "AD2 families in infinite-dimensional vector spaces can be extended to MAD2 families"?
(5) Is there a model of ZFA, or of ZF, in which "AD1 families in infinite-dimensional vector spaces can be extended to MAD1 families" is true, but MC is false?
(6) For $i \in \{1, 2\}$, does BPI imply "Every ADi family in an infinite-dimensional vector space V can be extended to a MADi family in V"?
(7) Is the statement "Every infinite-dimensional vector space has an infinite AD1 family" false in the Basic Cohen Model of ZF + BPI? (For the latter model, see Howard and Rubin [3, Model $\mathcal{M}1$, p. 146] or Jech [7, Section 5.3].)

Acknowledgements. We are grateful to the anonymous referee for carefully reading our paper and for several comments and suggestions which improved the quality and the exposition of the paper. We are especially thankful to the referee for giving us further motivation that resulted in establishing Theorem 4.9, which answers his/her

corresponding (open until now) question posed to us and which is a considerable strengthening of a former version of the theorem. The open questions (6) and (7) in the above list were also suggested to us by the referee.

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