9
Bosonized gauge theories

Bosonization, the equivalence map between two-dimensional fermionic and bosonic operators, was developed in Chapter 6. In fact several such maps have been described. The simplest one has been the *abelian bosonization* that maps the free theory of a Dirac fermion into that of a single real scalar field. The map includes in particular an explicit bosonic expression for the left and right chiral fermions (6.19), the vector and axial abelian currents (6.3) and for a mass term (6.22). Using these transformations it is straightforward to write the bosonized Lagrangian or Hamiltonian that corresponds to two-dimensional QED and QCD. By its nature the abelian bosonization is more adequate to the abelian theory of QED. The bosonized version of QED will be discussed in the next section. We then apply this bosonization to QCD_2. Though it is possible to write QCD_2 in an abelian bosonization formulation, it will turn out not to be very useful. Instead, we will use the non-abelian bosonization discussed in Section 6.3. For that purpose we will need to gauge the WZW action. Once this is done the bosonized version of massless flavored QCD_2 follows easily. The massive case requires more care, as was explained in Section 6.3.3. Using the results of that section the full bosonized theory that corresponds to massive flavored QCD_2 will be written down.

References on bosonization were given in Chapter 6.

9.1 QED_2 – The massive Schwinger model

Recall that the fermionic Lagrangian of this model is given by,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\partial\!\!\!/ - e A\!\!\!/ - m)\Psi. \tag{9.1}$$

The Hamiltonian density of the system in the $A_1 = 0$ gauge takes the form,

$$\mathcal{H} = \bar{\Psi}(i\gamma_1\partial_1 + m)\Psi + \frac{1}{2}(F_{01})^2.$$

In bosonic variables, using (8, 10), the Hamiltonian becomes[1]

$$\mathcal{H} =: \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 - \frac{cm^2}{\pi}\cos(2\sqrt{\pi}\phi) + \frac{e^2}{2\pi}\left(\frac{1}{2}\frac{\theta}{\sqrt{\pi}} - \phi\right)^2\right] :_m,$$

[1] The treatment of the bosonized Schwinger model was done in [68] and [64].

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where c is the constant of bosonization and the normal ordering is with respect to the mass m, as was explained in Section 6.1.1.

After a shift in the definition of ϕ,

$$\phi \to \phi + \frac{1}{2}\frac{\theta}{\sqrt{\pi}},$$

and normal ordering with respect to $\mu = e/\sqrt{\pi}$, one finds,

$$\mathcal{H} =: \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\partial_1\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{cm\mu}{\pi}\cos(\theta + 2\sqrt{\pi}\phi)\right) : . \tag{9.2}$$

From this expression the periodicity in θ is manifest. The angle θ is the conjugate to the winding number, appearing in two dimensions for the abelian case, since $\Pi_1[U(1)] = \mathcal{Z}$ (looking at a circle of large radius in the two-dimensional plane). Physics is invariant under $\theta \to \theta + 2\pi$. From (8.10) it is clear that $\frac{e\theta}{2\pi}$ corresponds to a background electric field. The periodicity is due to the ability to produce electron-positron pairs in the vacuum when $|\frac{e\theta}{2\pi}| > \frac{1}{2}e$, and these pairs create their own electric field which reduces the original one.

When we set $m = 0$ we discover that the massless Schwinger model is in fact a theory of one free bosonic field with a mass equal to μ.

In the strong coupling limit, the bosonized form of the Hamiltonian is very useful. The theory contains a meson of a mass that is approximately μ, and the number of bound states depends on the value of θ. It can be shown that there are no bound states for $|\theta| > \pi/2$. For $0 < |\theta| \leq \pi/2$ there is a stable two-body bound state, while for $\theta = 0$ there is also a three-body bound state.

Note that even though the Hamiltonian density (9.2) resembles that of a sine-Gordon model, it does not admit soliton solutions due to the mass term $\frac{1}{2}\mu^2\phi^2$. We will come back later to analyze this bosonized Hamiltonian, in the context of the question whether the system admits screening or confinement in Chapter 14.

Finally, let us show how the anomaly arises in the bosonized version. The equation of motion for the electromagnetic field is,

$$\partial^\mu F_{\mu\nu} = eJ_\nu. \tag{9.3}$$

The vector fermion current, in bosonic version, is (6.9),

$$J_\nu = \frac{1}{\sqrt{\pi}}\epsilon_{\nu\alpha}\partial^\alpha\phi. \tag{9.4}$$

Taking the time component, we get,

$$\partial^1\left(F_{10} - \frac{e}{\sqrt{\pi}}\phi\right) = 0. \tag{9.5}$$

From here, with the vanishing conditions at space infinity,

$$F_{10} = \frac{e}{\sqrt{\pi}}\phi. \tag{9.6}$$

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Now, the axial current, in bosonic version, is (6.12),

$$J_5^\mu = \frac{1}{\sqrt{\pi}} \partial^\mu \phi. \tag{9.7}$$

For the case of a massless fermion, the scalar field is a free field of mass $\frac{e}{\sqrt{\pi}}$, and so,

$$\partial_\mu J_5^\mu = -\frac{1}{\sqrt{\pi}} \frac{e^2}{\pi} \phi = \frac{e}{\pi} F_{01}, \tag{9.8}$$

which is the anomaly equation.

Note that in the bosonic version, the anomaly is a result of the equations of motion, while in the fermionic one it is the result of one loop.

9.2 Abelian bosonization of flavored QCD_2

Let us now apply the prescription of flavored Dirac fermions for the analysis of QCD_2. It is convenient to start with the Hamiltonian of the theory in its fermionic formulation which we derive from (8.16),

$$H = (e_c)^2 \sum_{a,b=1}^{N_C} \left(E_b^a\right)^2 + \sum_{a,b=1}^{N_C} \sum_{i=1}^{N_F} \bar{\Psi}^{ai} \gamma_1 \left(i\delta_a^b \partial_1 - A_a^b\right) \Psi_{bi} + m \sum_{a=1}^{N_C} \sum_{i=1}^{N_F} \bar{\Psi}^{ai} \Psi_{ai}, \tag{9.9}$$

in the gauge,

$$A_0 = 0; \quad A_b^a = 0 \ for\ a = b; \quad E_b^a = 0 \ \ for\ a \neq b. \tag{9.10}$$

The Gauss law of the system is given by,

$$\partial_1 E_b^a = i[A, E]_b^a + \frac{1}{2} \sum_{i=1}^{N_F} \Psi^{\dagger ai} \Psi_{bi} - \frac{\delta_b^a}{2N_C} \sum_{i=1}^{N_F} \sum_{d=1}^{N_C} \Psi^{\dagger di} \Psi_{di} \tag{9.11}$$

Bosonizing now the various parts of the Hamiltonian one then gets,[2]

$$\begin{aligned}
H &= H_\Psi^0 + H_E - H^I \\
H_\Psi^0 &= \Sigma_{ai} \left[\frac{1}{2} [\pi_{ai}^2 + (\partial_1 \phi_{ai})^2] + \frac{cm\mu}{\pi} : (1 - \cos(2\sqrt{\pi}\phi_{ai})) : \right] \\
H_E &= \frac{e_c^2}{8\pi N_c} \sum_{ab} \left[\sum_i (\phi_{ai} - \phi_{bi}) \right]^2 \\
H^I &= \frac{2c^2\mu^2}{\pi^{\frac{3}{2}}} \Sigma_{a\neq b} \Sigma_{ij} K_{ij,ab} N_\mu \left[\cos\sqrt{\pi} \int_{-\infty}^{x} (\pi_{ai} - \pi_{aj} + \pi_{bj} - \pi_{bi})(\xi) d\xi \right] \\
&\quad \left[\sin(\sqrt{\pi}(\phi_{ai} + \phi_{aj} - \phi_{bj} - \phi_{bi})(\xi)) \right] \left[\sum_{ab} (\phi_{ak} - \phi_{bk}) \right]^{-1},
\end{aligned} \tag{9.12}$$

[2] Abelian bosonization of two-dimensional QCD was discussed in [24] and [201] and was further elaborated in [62].

H^0_Ψ is the free "fermionic" part, after bosonization, thus in terms of bosonic variables; H_E is the first term of the Hamiltonian (9.9) rewritten in terms of the boson variables corresponding to the fermions, by eliminating the electric fields through the Gauss law. Thus although originally coming from the kinetic part of the gauge potentials, it actually involves the interactions. This is a result of the fact that there are no transverse vectors in $1+1$ dimensions. K^{ab}_{ij} is a properly generalized ordering operator.[3]

In the case of one flavor, $i=j=1$, H^I does not involve the π variables.

The interaction involves non-local terms which relate to color non-singlets. For static and $e_c \to \infty$ approximations one finds that for $N_F = 2$ the interaction is field independent. For $N_F \geq 3$, on the other hand, the limit is singular. This singularity should not be there in the predictions of physical quantities, but it renders any further treatment very complicated.

It is thus clear that a different method of bosonization is required for the treatment of flavored QCD_2. In the following it will be shown that the "non-abelian bosonization", based on the WZW model discussed in Section 6.3, is an adequate tool for this purpose.

Before proceeding to non-abelian bosonization and in particular to gauge the color symmetry group of the colored-flavored WZW model, we describe briefly another approach, in which the flavor sector appears in the form of a WZW model, but for the color degrees of freedom the gauged abelian bosonization is invoked. As we have seen above one can use the Gauss law to express the gauge fields in terms of the appropriate fermionic bilinear, which translate into bosonic group elements as,

$$2\partial_1 e_a = \sum_i \Psi^{\dagger ia}\Psi_{ia} = \frac{i}{\pi}\partial_1 \mathrm{Tr}_F(\log g_a), \tag{9.13}$$

where $g_a \in U(N_F)$ is one out of N_C such matrices, and $e_a = 2\sqrt{\pi}E^a_a$, the diagonal element. One can also express A^b_a for $a \neq b$ in terms of fermion densities. Inserting these into the QCD_2 Hamiltonian one gets,

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}^0 + \mathcal{H}^I \\
\mathcal{H}^I &= -\sum_{a,b}\frac{(e_c)^2}{32\pi^2 N_C}\left[\mathrm{Tr}\log\left(g_a g_b^{-1}\right)\right]^2 - \sum_{a,b}\pi\mu^2\frac{\mathrm{Tr}(g_a g_b^{-1})}{\mathrm{Tr}\log(g_a g_b^{-1})} \\
&\quad + \sum_a mc\mu\sqrt{N_\mathrm{F}}\,\mathrm{Tr}(g_a),
\end{aligned} \tag{9.14}$$

$\mathcal{H}^0$ includes the fermion kinetic term. For $N_\mathrm{F} = 2$ the potential is free from singularities, for $N_\mathrm{F} \geq 3$ it is not. In the case of $N_\mathrm{F} = 2$ the low lying baryonic spectrum can be extracted. Here we will not follow this approach further and instead will move on to the fully non-abelian bosonization.

[3] See Cohen *et al.* [62].

9.3 Non-abelian bosonization of QCD_2

Whereas abelian bosonization has been very useful to address various abelian systems, we have seen in the last section that the implementation of this approach to QCD_2 is quite limited. Instead the natural approach is to make use non-abelian bosonization, namely, the WZW action.[4] Recall from Section 6.3 that the bosonized action of massless free colored flavored fermions can be expressed either using an $SU(N_\mathrm{C}) \times SU(N_\mathrm{F}) \times U(1)$ scheme where it reads,

$$S = N_\mathrm{C} S[g] + N_\mathrm{F} S[h] + \frac{1}{2}\int \mathrm{d}^2 x \partial_\mu \phi \partial^\mu \phi, \tag{9.15}$$

or a $U(N_\mathrm{F} \times N_\mathrm{C})$ where the action takes the form,

$$S[u] = N_\mathrm{C} S[g] + N_\mathrm{F} S[h] + \frac{1}{2}\int \mathrm{d}^2 x (\partial_\mu \phi \partial^\mu \phi + S[l]). \tag{9.16}$$

Note that l is still an $SU(N_\mathrm{C} N_\mathrm{F})$ matrix while g and h are expressed now as $SU(N_\mathrm{F})$ and $SU(N_\mathrm{C})$ matrices, respectively, but the matrix l involves only products of color and flavor matrices (not any of them separately). For massive Dirac fermions we can use only the latter frame in which the mass term action reads,

$$S_m[u] = m'^2 N_{\tilde{m}} \int \mathrm{d}^2 x Tr(u + u^\dagger). \tag{9.17}$$

To determine the bosonized action of two-dimensional QCD_2 one needs to couple the colored degrees of freedom to the gauge fields. Thus, we first have to gauge the WZW model.

9.3.1 Gauging the WZW action

Since there are two possible bosonization schemes (for the massless case) we need to invoke a gauging procedure for both of them. We start first by gauging an $SU(N_\mathrm{C})$ WZW model which is what is needed in the product scheme, we later adopt it also to the $U(N_\mathrm{F} \times N_\mathrm{C})$. Gauging the colored WZW is achieved by gauging the vector subgroup $SU_V(N_\mathrm{C})$ of $SU_L(N_\mathrm{C}) \times SU_R(N_\mathrm{C})$. There are various methods to gauge the model. Here we present two of them. One is a trial and error method, and the other is by gauging via covariantizing the current. Those methods are applicable also in the $U(N_\mathrm{F} N_\mathrm{C})$ bosonization scheme.

The gauging of the WZW model and the full non-abelian bosonization of QCD in two dimensions was analyzed in [75] and [99]. Bosonization of QCD in two dimensions was reviewed in [101].

Trial and error Noether method

The WZW action on the $SU(N_\mathrm{C})$ group manifold is, as stated above, invariant under the global vector transformation $h \to UhU^{-1}$, where $U \subset SU(N_\mathrm{C})$. Now

[4] The hybrid of abelian and non-abelian bosonizations was implemented in [107].

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we want to vary h with respect to the associated local infinitesimal transformation $U = 1 + i\epsilon(x) = 1 + iT^A\epsilon^A(x)$,

$$\delta_\epsilon h = i[\epsilon, h], \quad \delta_\epsilon h^{-1} = i[\epsilon, h^{-1}]. \tag{9.18}$$

The variation of the action $S^{(0)}[h] \equiv S[h]$ under such a transformation is,

$$\delta_\epsilon S^{(0)}[h] = -\int \mathrm{d}^2x \mathrm{Tr}(\partial_\mu \epsilon J^\mu), \tag{9.19}$$

where the Noether vector current is given by,

$$J_\mu = \frac{i}{4\pi}\{[h^\dagger \partial_\mu h + h\partial_\mu h^\dagger] - \varepsilon_{\mu\nu}[h^\dagger\partial^\nu h - h\partial^\nu h^\dagger]\}. \tag{9.20}$$

We introduce now the first correction term $S^{(1)}$ given by,

$$S^{(1)} = \int \mathrm{d}^2x \mathrm{Tr}(A_\mu J^\mu) \quad \delta_\epsilon S^{(1)}[h] = -\int \mathrm{d}^2x \mathrm{Tr}[\partial_\mu \epsilon(J^\mu + J^{'\mu})]. \tag{9.21}$$

The variation of $S^{(1)}$ is derived using the infinitesimal variation of the gauge field $\delta A_\mu = -D_\mu\epsilon = -(\partial_\mu\epsilon + i[A_\mu, \epsilon])$. $J^{'\mu}$ is found to be,

$$J^{'}_\mu = \frac{-1}{4\pi}\{[h^\dagger A_\mu h + hA_\mu h^\dagger - 2A_\mu] - \varepsilon_{\mu\nu}[h^\dagger A^\nu h - hA^\nu h^\dagger]\}. \tag{9.22}$$

The second iteration will be given by adding $S^{(2)}$, where now $J^{'\mu}$ is replacing J^μ,

$$S^{(2)} = \int \mathrm{d}^2x \mathrm{Tr}(A_\mu J^{'\mu}), \quad \delta_\epsilon S^{(2)}[h] = -2\int \mathrm{d}^2x \mathrm{Tr}(\partial_\mu\epsilon J^{'\mu}). \tag{9.23}$$

It is therefore obvious that,

$$\delta_\epsilon\left[S^{(0)} + S^{(1)} - \frac{1}{2}S^{(2)}\right] = 0. \tag{9.24}$$

Hence the action we are looking for is $S[h, A_\mu] \equiv [S^{(0)} + S^{(1)} - \frac{1}{2}S^{(2)}]$, given by,

$$\begin{aligned} S[h, A_\mu] = &\frac{1}{8\pi}\int \mathrm{d}^2x \mathrm{Tr}(D_\mu h D^\mu h^\dagger) \\ &+\frac{1}{12\pi}\int_B \mathrm{d}^3y \varepsilon^{ijk}\mathrm{Tr}(h^\dagger\partial_i h)(h^\dagger\partial_j h)(h^\dagger\partial_k h) \\ &-\frac{1}{4\pi}\int \mathrm{d}^2x\varepsilon_{\mu\nu}\mathrm{Tr}[iA^\mu(h^\dagger\partial^\nu h - h\partial^\nu h^\dagger + ih^\dagger A^\nu h)], \end{aligned} \tag{9.25}$$

which can also be written in light-cone coordinates,

$$\begin{aligned} S[h, A_+, A_-] = S[h] &+ \frac{i}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}(A_+ h\partial_- h^\dagger + A_- h^\dagger\partial_+ h) \\ &-\frac{1}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}(A_+ hA_- h^\dagger - A_- A_+). \end{aligned} \tag{9.26}$$

Gauging via covariantization of the Noether current

In four space-time dimensions the current, in terms of bosonic matrices, involves up to third power gauge potentials. In D space-time dimensions the bare current

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will contain (D-1) derivatives, and is gauged by replacing the ordinary derivatives with covariant derivatives and by adding terms which contain products of $F_{\mu\nu}$ with powers of h and $h^\dagger$ and also covariant derivatives $D_\mu h$ and $D_\mu h^\dagger$. In two dimensions, however, there is no room for such terms in the gauge covariant current, as these involve $\epsilon_{\mu_1\ldots\mu_D}$ in D dimensions, with one free index and the others contracted with $F_{\mu\nu}$s and D_μs, and in two dimensions they cannot be constructed. Therefore the covariantized current is given by,

$$J_\mu(h, A_\mu) = \frac{i}{4\pi}\{[h^\dagger D_\mu h + hD_\mu h^\dagger] - \varepsilon_{\mu\nu}[h^\dagger D^\nu h - hD^\nu h^\dagger]\}. \tag{9.27}$$

Knowing the current we deduce the action via, $J_\mu = \frac{\delta S}{\delta A_\mu}$, getting (9.26) directly.

Finally, we combine the gauged WZW action of the color group manifold, the WZW of the flavor group manifold and the action term for the gauge fields, to get the bosonic form of the action of massless QCD_2. The well known fermionic form of the action is (a mass term will be added later),

$$S_F[\Psi, A_\mu] = \int \mathrm{d}^2x\Big\{-\frac{1}{2e_c^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) - \bar\Psi^{ai}[(i\partial\!\!\!/ + A\!\!\!/\,)\Psi_i]_a\Big\}, \tag{9.28}$$

where e_c is the coupling constant to the color potentials (note it has mass dimensions in 1+1 space-time), and,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]. \tag{9.29}$$

The bosonized action is,

$$\begin{aligned} S[g, h, A_+, A_-] = N_{\mathrm{C}}S[g] + N_{\mathrm{F}}S[h] + \frac{1}{2}\int \mathrm{d}^2 \times \partial_\mu\phi\partial^\mu\phi \\ + \frac{N_{\mathrm{F}}}{2\pi}\int \mathrm{d}^2x\mathrm{Tr}[i(A_+h\partial_-h^\dagger + A_-h^\dagger\partial_+h) \\ - (A_+hA_-h^\dagger - A_-A_+)] \\ - \frac{1}{2e_c^2}\int \mathrm{d}^2x\mathrm{Tr}F_{\mu\nu}F^{\mu\nu}. \end{aligned} \tag{9.30}$$

9.3.2 Multiflavor QCD_2 using the $U(N_F \times N_C)$ scheme

Let us now repeat the gauging of the $SU_V(N_{\mathrm{C}})$ subgroup in the framework of the $U(N_{\mathrm{F}} \times N_{\mathrm{C}})$ bosonization procedure.

Using the gauging prescription discussed in Section 9.3.1 we first get the action in which the whole $SU(N_{\mathrm{C}}N_{\mathrm{F}})$ is gauged, namely,

$$\begin{aligned} S[u, A_+, A_-] = S[u] + \frac{i}{2\pi}\int \mathrm{d}^2x\mathrm{Tr}(A_+u\partial_-u^\dagger + A_-u^\dagger\partial_+u) \\ - \frac{1}{2\pi}\int \mathrm{d}^2x\mathrm{Tr}(A_+uA_-u^\dagger - A_-A_+) \\ + m'^2N_{\tilde m}\int \mathrm{d}^2x\mathrm{Tr}(u + u^\dagger), \end{aligned} \tag{9.31}$$

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where we have also added a mass term with $m^{'2} = m_q c\tilde{m}$. Now since we are interested in gauging only the $SU(N_\mathrm{C})$ subgroup of $U(N_\mathrm{F}N_\mathrm{C})$, we take A_μ to be spanned by the generator $T^D \subset SU(N_\mathrm{C})$ via $A_\mu = e_c A_\mu^D T^D$. We then add to this action the kinetic term for the gauge fields $-\frac{1}{2e_c^{'2}}\int \mathrm{d}^2x \mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$. The coupling $e_c^{'}$ is related to the color gauge coupling e_c by $e_c^{'} = \sqrt{N_\mathrm{F}}e_c$, so that after taking the trace over flavor we get the expected kinetic term with coupling e_c. The resulting action is invariant under local color and global flavor,

$$u \to V(x)uV^{-1}(x), \quad A_\mu \to V(x)(A_\mu - i\partial_\mu)V^{-1}(x); \quad V(x) \subset SU_V(N_\mathrm{C})$$

$$u \to WuW^{-1}; \quad W \subset U(N_\mathrm{F}).$$

The symmetry group is now $SU_V(N_\mathrm{C}) \times U(N_\mathrm{F})$, just as for the gauged fermionic theory. We choose the gauge $A_- = 0$, so now the action takes the form,

$$\begin{aligned} S[u, A_+] = S[u] + \tfrac{1}{e_c^{'2}}\int \mathrm{d}^2x \mathrm{Tr}(\partial_- A_+)^2 + \tfrac{i}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}(A_+ u\partial_- u^\dagger) \\ + m^{'2}N_{\tilde{m}}\int \mathrm{d}^2x \mathrm{Tr}(u + u^\dagger). \end{aligned} \tag{9.32}$$

Upon the decomposition $u = \tilde{g}\tilde{h}le^{-i\sqrt{\frac{4\pi}{N_\mathrm{C}N_\mathrm{F}}}\phi}$, we see that the current that couples to A_+ is $\tilde{h}\partial_-\tilde{h}^\dagger$. In terms of u it is the color projection $(u\partial_- u^\dagger)_C = \frac{1}{N_\mathrm{F}}\mathrm{Tr}_F[u\partial_- u^\dagger - \frac{1}{N_\mathrm{C}}\mathrm{Tr}_C u\partial_- u^\dagger]$. Thus the coupling of the current to the gauge field $\frac{i}{2\pi}\int \mathrm{d}^2x \mathrm{Tr}(A_+\tilde{h}\partial_-\tilde{h}^\dagger)$.

We can further manipulate the action to a form which will be convenient for taking the strong coupling limit (see Chapter 13). We define $\tilde{H}(x)$ by $\partial_-\tilde{H} = i\tilde{h}\partial_-\tilde{h}^\dagger$. We take the boundary conditions to be $\tilde{H}(-\infty, x_-) = 0$ and then integrate out A_+ obtaining,

$$\begin{aligned} \tilde{S}[u] = S[u] - (\tfrac{e_c}{4\pi})^2 N_\mathrm{F}\int \mathrm{d}^2x \mathrm{Tr}(\tilde{H}^2) \\ + m^{'2}N_{\tilde{m}}\int \mathrm{d}^2x \mathrm{Tr}(u + u^\dagger). \end{aligned} \tag{9.33}$$

In Chapter 13 this form of action will constitute the starting point of determining the baryonic spectrum of QCD_2 in the strong coupling limit. In Chapter (14) we will use this action to analyze the string tension and the confining behavior of massive QCD_2.