BULL. AUSTRAL. MATH. SOC. MOS 2210

VOL. 1 (1969), 315-317.

Topologies on finite groups

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It has been shown by D. Stephen that the number N of open sets in a non-discrete topology on a finite set with n elements is not greater than $3 \times 2^{n-2}$. We show that for admissable topologies on a finite group $N \leq 2^{n/r}$, where r is the least order of its non-trivial normal subgroups. This is clearly a sharper bound.

Define semi-topological group and topological group as in [2]. (We do not assume that these topological spaces are Hausdorff.)

THEOREM 1. *Let X be a non-discrete finite semi-topological group of order n (> 1). Then the number N of open sets satisfies $N \leq 2^{n/r}$, where r is the least order of the non-trivial normal subgroups of X.*

Proof. Let 0_x be the intersection of all open sets containing $x \in X$. Because there are only a finite number of open sets 0_x is open. We show that 0_e (where e is the identity of X) is a normal subgroup.

Since, for each $x \in X$, $x0_e$ and $x^{-1}0_x$ are open (Theorem 1, Chapter II of [2]) and contain x and e respectively, $0_x \subseteq x\,0_e$ and $0_e \subseteq x^{-1}\,0_x$ which imply

$$0_x = x\,0_e . \tag{1}$$

Similarly

Received 28 April 1969. This paper was written while the authors held Research Assistantships at the University of Queensland. The authors greatly appreciate the suggestions made by Mr M.P. O'Donnell and Professor R. Výborný.

(2) $$0_x = 0_e x \ .$$

Clearly if $y \in 0_e$, $0_e \supseteq 0_y = y0_e$. (In fact since 0_e and $y0_e$ have the same finite number of elements $0_e = y0_e$.) Thus

(3) $$0_e^2 \subseteq 0_e \ .$$

Since X is a finite group and 0_e is non-empty, (1), (2) and (3) imply 0_e is a normal subgroup.

The 0_x are the cosets of 0_e . Indeed $\{0_e\}$ is an open basis at e for the topology. Thus the number of distinct 0_x is $\frac{n}{m}$, where m is the order of 0_e . (Since the topology was assumed to be non-discrete m is strictly greater than one.) The proof is completed by noting that the number of open sets is $2^{n/m} \leq 2^{n/r}$. (See [3]).

The following example shows that the bound in the above theorem is the best possible (under the conditions of the theorem).

EXAMPLE. Let X be the additive group of integers mod n and $0_e = \left\{0, \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}\right\}$, where p is the smallest prime dividing n . By Theorem 4.5 of [1], the topology on X which has $\{0_e\}$ as an open basis at 0 makes X a topological group. It is obvious that the number of open sets is $2^{n/p}$.

COROLLARY 2. *Every finite semi-topological group is a topological group.*

Proof. By Theorem 1 0_e is a normal subgroup and $\{0_e\}$ is an open basis for the topology. The result now follows from Theorem 4.5 of [1].

For completeness we include the following remark.

REMARK. It is easily shown that if there is a topology on a group X (not necessarily finite) such that the mapping $(x,y) \to x^{-1}y$ is continuous in the first variable then $(x,y) \to xy$ is continuous in each variable separately and $x \to x^{-1}$ is continuous. If X is also finite then by Corollary 2, X is a topological group.

https://doi.org/10.1017/S0004972700042180 Published online by Cambridge University Press

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https://doi.org/10.1017/S0004972700042180 Published online by Cambridge University Press