A GRAPHICAL EXPOSITION OF THE ISING PROBLEM [1]

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(Received 25 July 1969)

Communicated by C. A. Hurst

1. Historical introduction

Ising [1] proposed the problem which now bears his name and solved it for the one-dimensional case only, leaving the higher dimensional cases as unsolved problems. The first solution to the two dimensional Ising problem was obtained by Onsager [6]. Onsager's method was subsequently explained more clearly by Kaufman [3]. More recently, Kac and Ward [2] discovered a simpler procedure involving determinants which is not logically complete.

Their purpose was to indicate the ideas involved in a combinatorial development of a proof of the two-dimensional Ising problem, and they provided heuristic arguments only. Feynman's unpublished simplification of their treatment contains precisely the same logical gap, but clarifies the issue by means of an elegant conjecture. We are grateful to R. P. Feynman for his permission to present his work here and to M. Cohen for his kindness in communicating these details.

Our own interest in the Ising problem is enhanced by the fact that among its many equivalent formulations, it can be regarded as an enumeration problem for graphs, see [8]. An unpublished earlier draft of this article (1958) served as a catalytic agent: Sherman [7] succeeded in proving Feynman's conjecture by deriving an appropriate combinatorial theorem.

2. Statement of the two-dimensional Ising problem

Newell and Montroll [5] give a very clear exposition of the problem, but we will restate it here for the sake of completeness. Consider as in Figure 1 a large two-dimensional lattice L with N points. We may regard L as embedded on a torus (by identifying both pairs of opposite sides as usual), so that each point of L is incident with exactly 4 lines of L.

[1] The preparation of this article was supported in part by a grant from the U.S. Air Force of Scientific Research.

This paper in unpublished form played an important part in the construction of the proof of the equivalence of the algebraic and combinatorial solution of the two-dimensional Ising model. Although it is some years since it was written its publication will be welcomed by those interested in this important subject. Ed.

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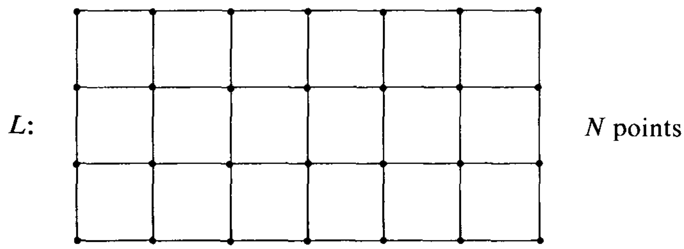

Figure 1

We need a few definitions in order to state the formula for the partition function Q belonging to L. Let k be Boltzmann's constant. The physical description of the problem is that we have a lattice graph in which each point has two possible states. These states may be regarded as (a) orientation: up or down, (b) two different kinds of atoms, (c) positive or negative charge, (d) a particle of a gas being present or absent at each point (as in Lee and Yang [4]), etc.

Two points of L are called *nearest neighbours* or *adjacent* if they are joined by a line of the lattice graph. If two points are not nearest neighbours then it is assumed that there is no interaction energy between them. For definiteness, it is stipulated that each point of the lattice is assigned an orientation which is called either 'up' or 'down', and is symbolized by writing at the i'th point the number σ_i which is either $+1$ or -1 respectively. Sometimes σ_i is called a *point variable* below.

A *configuration* σ or *state* of a lattice L is defined by the assignment of either $+1$ or -1 at each point. The *energy* of a configuration is given by:

(1) $$E = -\varepsilon \sum \sigma_i \sigma_j,$$

where the sum is taken over all nearest neighbours in the lattice. Thus each pair of nearest neighbours has an interaction energy of $-\varepsilon$ if their orientations are the same, and $+\varepsilon$ if their orientations are opposed as in Figure 2.

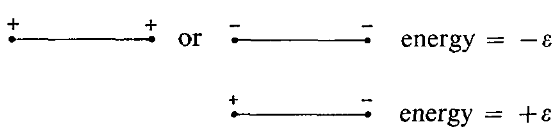

Figure 2

Let T be the absolute temperature of the system. The *partition function* Q of the lattice is then defined as:

(2) $$Q = \sum_{\sigma} \exp\left(-\frac{E}{kT}\right),$$

where the sum is taken over all possible configurations σ of the lattice system. Substituting (2) into (1) and letting $\alpha = \varepsilon/kT$ we get

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(3) $$Q = \sum_{\sigma} \exp(\alpha \sum_{\lambda} \sigma_i \sigma_j).$$

In equation (3), the first sum is taken over all configurations and the second sum is taken over all pairs λ of nearest neighbours. It is well known that all thermodynamic properties of a system are computable from its partition function Q. In particular, the specific heat C is obtained by successive differentiation of Q, namely

$$C = k\alpha^2 \frac{\partial^2 \ln Q}{\partial \alpha^2}.$$

If we divide both sides of this last equation by N, then the left side becomes the specific heat per particle and the right side depends on $\ln Q/N$.

An individual term of the right side of equation (3) is of the form $e^{\alpha\sigma_i\sigma_j}$. However, by a well known identity involving hyperbolic functions, we have, since $\sigma_i\sigma_j = \pm 1$, the equation

(4) $$e^{\alpha\sigma_i\sigma_j} = \cosh \alpha(1+\sigma_i\sigma_j \tanh \alpha).$$

It is convenient to make the substitution

(5) $$z = \tanh \alpha$$

On substituting equation (5) into (4) and then (4) into (3), we obtain:

(6) $$Q = \cosh^{2N} \alpha \sum_{\sigma} \prod_{\lambda} (1+\sigma_i\sigma_j z),$$

where the product is taken over all nearest neighbours, i.e. over all lines λ in the lattice graph. The reason that the exponent $2N$ occurs in equation (6) is that the number of lines in L is $2N$ since each point of the lattice L drawn on a torus is on 4 lines and each line joins 2 points.

A typical term of this product has a certain power of z, say z^b. The coefficient of z^b is a product of the variables σ_i of degree $2b$, and is therefore based on a choice of b lines. After summing, a term of this product is not zero if and only if each point-variable σ_i occurs an even number of times. Since there are two possible values $\sigma_i = \pm 1$ for each of the N point-variables, each term of the product contributes 2^N to the sum. Let A_b be the number of labelled subgraphs of the lattice in which each point has even degree and the number of lines is b, and define $A_0 = 1$. Then (6) may be written in the combinatorial form

(7) $$Q = (2^N \cosh^{2N} \alpha)(\sum_b A_b z^b).$$

Thus the two-dimensional Ising problem of finding the partition function Q has been reduced to the graphical enumeration problem of finding $\sum A_b z^b$.

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3. A system of weights

Several admissible subgraphs are illustrated in Figure 3.

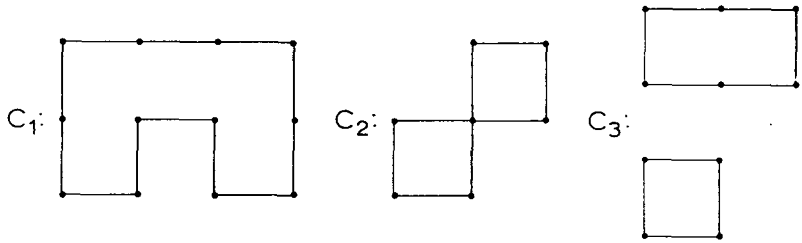

Figure 3

Each line in the graphs shown in Figure 3 is not directed. That is, each line of an admissible subgraph does not have a prescribed direction indicated by an arrow. However, it turns out to be convenient first to consider directed subgraphs on the lattice, and then express the number of ordinary undirected subgraphs in terms of the directed ones. A particular cyclic orientation of the first admissible subgraph of Figure 3 is indicated in Figure 4.

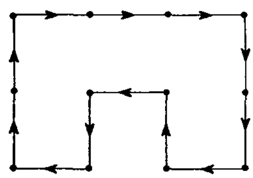

Figure 4

In the process of traversing closed directed walks G on the lattice, there are exactly four different possibilities for a directed line to take after the last directed line, namely, it can turn left, turn right, go straight ahead, or about-face. Let w_j be the weight assigned to the turn taken at the j'th point of G. Then in Figure 5 the weights are shown for the four possible kinds of turns described above.

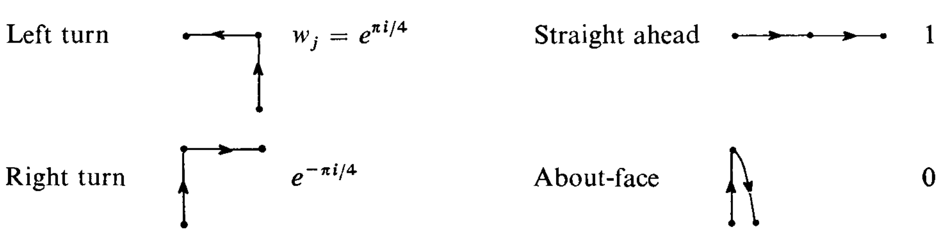

Figure 5

Thus in general the weight at a turn whose counterclockwise angle is θ is given by

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(8) $$w_j = e^{\theta i/2},$$

except that we use weight 0 for the about-face in order to exclude this inadmissible configuration. Let C be a directed cycle of G (not dependent on its starting point) and let W_C be the weight of C, defined by

(9) $$W_C = -\tfrac{1}{2}\prod_j w_j z,$$

where the product is taken over all the turns in C. Consider

(10) $$\sum_{C\subset G} W_C = -\tfrac{1}{2}\sum_{C\subset G}\Big(z^{b(C)}\prod_{j\in C} w_j\Big)$$

where C varies over all simple closed walks of an admissible subgraph G and $b(C)$ is the length (number of lines) of C. There are two possibilities. Either the closed directed admissible subgraph C consists of a collection of exact retracings of some closed walk or it does not. In any case let n_C be the number of exact retracings in C. For example in Figure 6, $n_C = 2$.

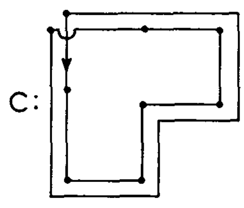

Figure 6

Consider the sum $\sum_C W_C/n_C$. We now modify the right side of equation (10). *Let s_b be the sum of all the weights of all closed walks of length b through a given point of L.* There are b different possible starting points of the labelled admissible graph C. Since W_C has a factor of z in it for each line of C, the power of z due to C is z^b. There are N different locations for the graph C. Since C winds around n_C times, there are only b/n_C distinguishable starting points for the closed walk C. Collecting these observations, we have

(11) $$\sum_{C\subset L} W_C/n_C = -\tfrac{1}{2}N\sum_{b=4}^{\infty}\frac{s_b}{b}z^b.$$

Let β denote the left side of equation (11).

It will be seen that s_b is computable (it is related to the probability of eventual return in b steps of a two-dimensional random walk). We illustrate in Figure 7 two closed directed walks of length 8.

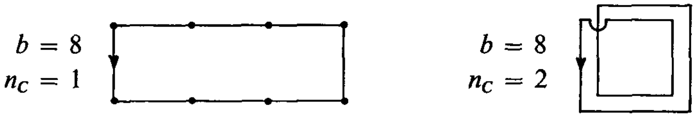

Figure 7

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Next consider an undirected subgraph C which is a simple closed curve with no crossings, such as C_1 in Figure 3. As a closed directed walk this can occur twice, once for each orientation of this cycle. For each of these closed directed walks the weight is $\frac{1}{2}$. For the total angle through which turns occur while traversing C in either direction is 2π. Then on adding $-\frac{1}{2}(-1) = \frac{1}{2}$ twice, we get 1. Therefore, on summing over both orientations of the cycle, we have $\sum W_C = z^{12}$, in which the coefficient of z^{12} is 1.

We next consider an admissible graph C consisting of two simple closed curves with no crossings or, in graphical language, of two disjoint cycles such as C_3 in Figure 3. Since by definition

$$\beta = \sum_C W_C/n_C,$$

we have

$$\beta^2 = \sum_{(C,C')} \frac{W_C}{n_C}\frac{W_{C'}}{n_{C'}}.$$

Now it is clear that $\beta^2/2!$ gives weight 1 for this situation; for on orienting two disjoint cycles C and C' there are two choices of orientation for each cycle and also a factor of two for the interchanging of the names of the cycles C and C'.

Continuing in this way we get

$$1+\beta+\frac{\beta^2}{2!}+\frac{\beta^3}{3!}+\cdots = e^{\beta}. \tag{12}$$

Kac and Ward tried to count oriented admissible graphs by making a $1-1$ correspondence between terms of a determinant and oriented graphs. For this purpose they require the following observation.

THEOREM. *Let G be an admissible or inadmissible subgraph which is not oriented. Then the sum of the terms in the determinant* (*see* [2] *for details*) *contributed by all the corresponding oriented graphs is* 1 *or* 0 *respectively.*

They do not prove this theorem but illustrate several special cases, as Feynman does. However, the theorem is known to be true, since the final result agrees with the formulation obtained by Onsager [6], and all the steps are reversible.

The insight of Kac and Ward [2] is that the sum $\sum_b A_b z^b$ is given by the value of a certain complicated determinant. On the other hand Feynman's insight is that this sum is given by e^{β}. A rather different approach which effectively contains Feynman's conjecture is presented in equation (31) of the paper by Hurst [10].

FEYNMAN'S CONJECTURE. *It can be proved directly that the following equation holds.*

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(13)
$$e^{\beta} = \sum_b A_b z^b$$

We have already seen from the consideration of admissible graphs, each connected component of which consists of a simple closed curve, that e^{β} is a reasonable candidate for the function which counts all the admissible graphs correctly. But now consider some important simple cases which are counted correctly even though they do not consist only of disjoint cycles.

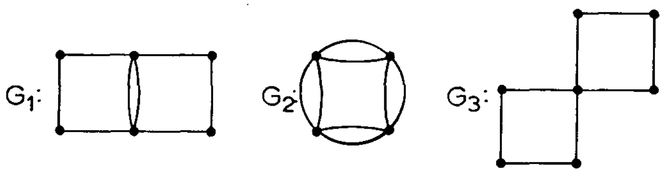

Figure 8

We illustrate in Figure 9 the two ways of traversing the lines of G_1 by closed directed walks.

2 of this type 4 of this type

Figure 9

We now compute the contributions to the weight of the unoriented graph G_1 which arise from the two ways of traversing G_1 (as shown in the first part of Figure 9) and also from the four ways of traversing G_1 (one of which is shown in the second part of Figure 9). In this calculation we omit the powers of z from the weights. If the formulation of Feynman is to be verified, we must have the sum of these six weights add up to 0 since the unoriented graph G_1 of Figure 8 is inadmissible (due to the fact that it contains more than one line joining the same two points).

The weight of each closed directed line sequence is given by $-\frac{1}{2}e^{i\theta/2}$, where θ is the total angle of turn. Since $\theta = 0$ in the orientation of the first part of Figure 9, this gives a contribution of $-\frac{1}{2}$ to the weight. The opposite orientation contributes another summand $-\frac{1}{2}$. Similarly the traversing of G by the two line-disjoint directed cycles of the second part of Figure 9 has a weight of $(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{4}$. But there are four representations of G_1 in this way: two for each of the two cycles. Thus the sum of the weights in all the possible orientations of G_1 is

$$2(-\tfrac{1}{2})+4(+\tfrac{1}{4}) = 0,$$

verifying that the inadmissible graph G_1 is not counted in e^{β}.

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We now consider the graph G_2 of Figure 8. We wish to verify that in the expansion of e^β, the sum of all terms corresponding to orientations of G_2 is 0. Here we must take into consideration that G_2 winds around the same simple cycle twice, so that $n_{G_2} = 2$. Thus we need to add the contribution from two terms in the expansion of e^β, namely β and $\beta^2/2$. We have here an analogous situation to that shown in Figure 9, for there are exactly two orientations of G_2 as a single closed directed line sequence (one is shown in the first part of Figure 10) and four possible orientations of G_2 regarded as two disjoint directed cycles (one is shown in the second part of Figure 10).

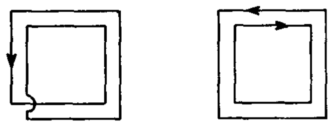

Figure 10

From the first part of Figure 10 we obtain a weight of

$$\frac{W_{G_2}}{n_{G_2}} = -\frac{\frac{1}{2}}{2} = -\frac{1}{4},$$

for here $\theta/2 = 2\pi$. Hence the contribution to the weight of the undirected graph G_2 from this orientation and its opposite is $-\frac{1}{2}$.

For each of the directed cycles shown in the second part of Figure 10, the weight is given by $-\frac{1}{2}e^{i2\pi/2} = +\frac{1}{2}$. Call these two directed cycles A and B. Then among other terms, β contains w_A+w_B; on substituting $w_A = w_B = \frac{1}{2}$ into $\beta^2/2$, we obtain $+\frac{1}{2}$. This completes the verification that the total contribution to e^β provided by orientations of G_2 is $-\frac{1}{2}+\frac{1}{2} = 0$.

Finally we consider three possible orientations of G_3 in Figure 11.

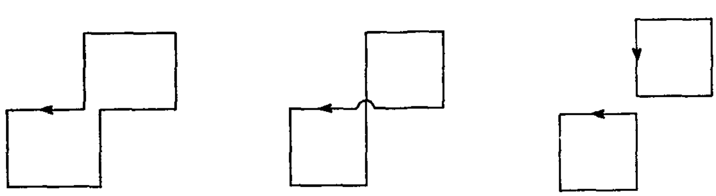

Figure 11

Without going into details we see that contributions to the weight of the undirected admissible graph G_3 due to these three orientations and four variations are

$$2(\tfrac{1}{2}) = +1, 2(-\tfrac{1}{2}) = -1, \text{ and } 4(\tfrac{1}{4}) = +1$$

respectively. Adding these, we get $1-1+1 = +1$, and G_3 is counted.

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Accepting Equation (13) and substituting it into Equation (7), we get

$$\frac{1}{N}\ln Q = \ln 2 + 2\ln\cosh\alpha + \frac{1}{N}\beta. \tag{14}$$

We require $\sum (s_b/b)z^b$ where s_b is the total weight of all closed paths of length b returning to the origin, that is, each s_b is the product of factors $e^{\theta i/2}$ where θ is the angle of turn. The following derivation will show that the numbers s_b are computable quantities.

Let (m, n) be the cartesian coordinates of a point of the lattice. As a directed path proceeds along the lattice, at each point it can go up, down, right, or left. We arbitrarily designate these four directions by $i = 1, 2, 3, 4$ as pictured in Figure 12.

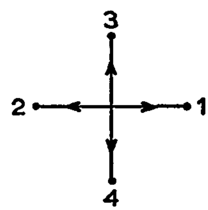

Figure 12

The following definition is crucial. Let $\beta_i(m, n; k)$ be the total weight of all directed walks, starting from the origin, whose k'th step terminates at the point (m, n) and which leave this point in direction i, for $i = 1, 2, 3, 4$. It is sometimes convenient to think of k as a time coordinate and (m, n) as space coordinates.

This last definition does not serve to define $\beta_i(m, n; 0)$. It is useful to define this quantity as a product of Kronecker deltas:

$$\beta_i(m, n; 0) = \delta_{i1}\,\delta_{m0}\,\delta_{n0} \tag{15}$$

As a special case of Equation (15) we have

$$\beta_i(0, 0; 0) = \delta_{i1}$$

This is equivalent to saying that all paths start from the origin at time 0 and direction 1. As an immediate consequence, we have

$$s_b = 4\beta_1(0, 0; b) \tag{16}$$

The step leading to any point (m, n) can come from any of the four points $(m-1, n)$, $(m+1, n)$, $(m, n-1)$, or $(m, n+1)$. We wish to establish recurrence relations for the quantities $\beta_i(m, n; k)$. For these relations it is convenient to use the abbreviations:

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$$\text{(17)} \quad \begin{bmatrix} \beta_1 = \beta_1(m-1, n; k) \\ \beta_2 = \beta_2(m+1, n; k) \\ \beta_3 = \beta_3(m, n-1; k) \\ \beta_4 = \beta_4(m, n+1; k) \end{bmatrix}$$

In the following equations (18) the factors of the form $e^{\pi i/4}$, etc. are the weights of the turns. There are just four ways of looking backward one step:

$$\text{(18)} \quad \begin{bmatrix} \beta_1(m, n; k+1) = \beta_1 & +0 & +\beta_3 e^{-\pi i/4} & +\beta_4 e^{\pi i/4} \\ \beta_2(m, n; k+1) = 0 & +\beta_2 & +\beta_3 e^{\pi i/4} & +\beta_4 e^{-\pi i/4} \\ \beta_3(m, n; k+1) = \beta_1 e^{\pi i/4} & +\beta_2 e^{-\pi i/4} & +\beta_3 & +0 \\ \beta_4(m, n; k+1) = \beta_1 e^{-\pi i/4} & +\beta_2 e^{\pi i/4} & +0 & +\beta_4 \end{bmatrix}$$

We discuss the derivation of the first equation (18). The quantity $\beta_1(m, n; k+1)$ involves the departure of the directed path from the point (m, n) along direction 1. The first term β_1 is multiplied by weight 1 since it is a continuation of the path along the same direction. The second term in the first equation of (18) is 0 since direction 2 followed by direction 1 constitutes an about-face. The remaining two terms of this first equation have weights obtained from a right turn and a left turn respectively.

We note that in equations (18) there are only $(k+1)$'s on the left side and there are only k's on the right. Therefore this process is Markovian! In fact, all these definitions have as their purpose the attainment of these Markovian quantities β_i. The problem remains to solve this set of difference equations. Feynman's solution is obtained by the ingenious use of Fourier analysis. Consider any function $f(n)$ defined only on integers n. Then its Fourier representation involves only frequencies from 0 to 2π.

$$\text{(19)} \qquad f(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{ihn}\varphi(h)dh.$$

Let $\vec{\beta}\,(m, n; b)$ be the vector whose four components are the quantities $\beta_i(m,n;b)$. Let $\vec{B}(u, v; b)$ be the Fourier transform of the $\vec{\beta}$ vector. Then we have

$$\text{(20)} \qquad \vec{\beta}(m, n; b) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} du\, dv\, e^{i(mu+nv)}\vec{B}(u, v; b)$$

Here, displacement by one step amounts to multiplication of each of the four components by e^{iu}, e^{-iu}, e^{iv}, e^{-iv} respectively. Therefore we obtain the following matrix transformation for proceeding from length b to length $b+1$ in the B vector.

$$\text{(21)} \qquad \vec{B}(u, v; b+1) = \begin{bmatrix} e^{-iu} & 0 & e^{-\pi i/4}e^{-iv} & e^{\pi i/4}e^{iv} \\ 0 & e^{iu} & e^{\pi i/4}e^{-iv} & e^{-\pi i/4}e^{iv} \\ e^{\pi i/4}e^{-iu} & e^{-\pi i/4}e^{iu} & e^{-iv} & 0 \\ e^{-\pi i/4}e^{-iu} & e^{\pi i/4}e^{iu} & 0 & e^{iv} \end{bmatrix} \vec{B}(u, v; b)$$

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This may be abbreviated:

$$\vec{B}(u, v; b+1) = M\vec{B}(u, v; b), \tag{21'}$$

by calling this matrix M. Then

$$\vec{B}(u, v; b) = M^b\vec{B}(u, v; 0). \tag{22}$$

Now recall that since the initial direction is stipulated as 1,

$$\vec{\beta}(m, n; 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \delta_{m0}\,\delta_{n0} \tag{23}$$

Therefore,

$$\vec{B}(u, v; 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{24}$$

This last equation can be verified by substituting this column vector into the above integral expression (20). In view of equation (15) it is sufficient to find the quantity $\beta_1(0, 0; b)$. To do this we first find $B_1(0, 0; b)$. Using both equations (20) and (22) together with (24) and (16) we obtain

$$s_b = 4\beta_1(0, 0; b) = \frac{4}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} du\,dv(1, 0, 0, 0)M^b \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

By symmetry considerations, there are three other equations which are entirely analogous to equation (25) for the other three directions. Using all four of these equations, we get the following very concise expression for s_b:

$$s_b = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} du\,dv\,\mathrm{Tr}\,(M^b) \tag{26}$$

in which Tr denotes the trace of a matrix. We note also that matrix M has u and v in it. Let $|M|$ be the determinant of M. Using (26) and the well known result that $\mathrm{Tr}(\ln M) = \ln|M|$, we obtain (since trace and summations are both linear):

$$\begin{aligned} \sum \frac{s_b z^b}{b} &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} du\,dv\,\mathrm{Tr}\left(\sum \frac{M^b}{b} z^b\right). \\ &= \frac{-1}{4\pi^2}\iint du\,dv\,\mathrm{Tr}\,(\ln(I - Mz)) \\ &= \frac{-1}{4\pi^2}\iint du\,dv\,\ln|I - Mz|. \end{aligned} \tag{27}$$

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Simplifying and recalling the various notations, we get

$$(28) \quad \frac{1}{N} \ln Q = \ln 2 + \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \ln (\cosh^2 2\alpha - \sinh 2\alpha(\cos u + \cos v)) du\, dv.$$

This is the partition function of the two-dimensional Ising problem and appears in Newell-Montroll [4]. If equation (28) is rewritten in terms of z, then it is of the form

$$\frac{1}{N} \ln Q = -\tfrac{1}{2} \sum \frac{s_b}{b} z^b + \ln 2 + 2 \ln \cosh \alpha.$$

The quantity A_b may be found from this last equation in principle only.

We conclude with a discussion of some unsolved graph-counting problems involving the Ising models. These are also listed in [9].

I. The Three-Dimensional Ising Problem

In graphical terms only, this celebrated unsolved problem may be stated in the following terms. Consider a labelled graph which is a three-dimensional lattice. A subgraph of this lattice is *admissible* if and only if each point is incident to an even number of lines. In other words, a labelled subgraph is admissible if and only if it can be written as the union of line-disjoint cycles. Let A_b be the number of different labelled admissible subgraphs with b lines. Find a generating function for the quantity A_b.

II. The Two-Dimensional Ising Problem with Magnetic Field

By the *area* of an admissible labelled subgraph of a two-dimensional lattice we mean the minimum area enclosed by disjoint cycles constituting this subgraph. Thus the area of an admissible subgraph is unique. Let $A_{b,c}$ be the number of admissible labelled subgraphs with b lines and area c. Find a generating function for the quantities $A_{b,c}$.

III. The Two-Dimensional Ising Problem with Interactions Occurring Not Only Between Nearest Neighbours

This problem involves the formulation of the partition function for the situation in which we not only have energy interaction between nearest neighbours, but also between pairs of points at distance 2 from each other in the lattice graph.

IV. The Paving Problem

Let us start with a two-dimensional lattice with N squares. Consider n_1 squares and n_2 double squares (like dominoes) such that $n_1 + 2n_2 = N$. In how many ways can the lattice be 'paved' by the n_1 'single titles' and he n_2 'double tiles'?

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V. The Cell-Growth Problem

Consider a one celled animal which has a square shape and can grow in the plane by adding a cell to any of its four sides. How many different connected animals with area c are there?

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https://doi.org/10.1017/S1446788700009836 Published online by Cambridge University Press