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CAUCHY'S PROBLEM FOR HARMONIC FUNCTIONS WITH ENTIRE DATA ON A SPHERE

DMITRY KHAVINSON

ABSTRACT. We give an elementary potential-theoretic proof of a theorem of G. Johnsson: all solutions of Cauchy's problems for the Laplace equations with an entire data on a sphere extend harmonically to the whole space R^N except, perhaps, for the center of the sphere.

1. **Introduction.** G. Johnsson has given in his thesis [J] a complete solution of the following problem.

Let Γ be a quadratic surface in $\mathbf{R}^N$, and consider the following Cauchy problem:

$$\begin{cases} \Delta u := \sum_1^N \frac{\partial^2 u}{\partial x_j^2} = 0 & \text{near } \Gamma; \\ (\frac{\partial}{\partial x_j})^k (u - f) = 0 & \text{on } \Gamma; \qquad j = 1, \ldots, N;\ k = 0, 1; \end{cases} \tag{1.1}$$

where the "data" function f is an entire function of N variables. Find the maximal domain Ω in $\mathbf{R}^N$ (or, $\mathbf{C}^N$) to which all solutions of (1.1) extend as real-analytic (or, holomorphic) functions.

In fact, Johnsson has even solved the problem for all second-order operators that have the Laplacian as their principal part. Johnsson's work is rather deep, and based on so-called "globalizing family" arguments stemming out from the work of Bony and Schapira [BS] and Zerner [Z], blended with local uniformization of solutions of Cauchy's problems pioneered by Leray [L].

Similar and even somewhat more general results based on a set of interesting topological ideas—R. Thom's theorem—have been independently obtained by B. Sternin and V. Shatalov and their school (*cf.* [SS] and references therein). One of the remarkable corollaries of those investigations is the following

THEOREM 1. *Let $\Gamma = \{x \in \mathbf{R}^N : |x| = 1\}$ be the unit sphere. The solution u of the Cauchy problem (1.1) with an entire data f on Γ extends harmonically to the whole space $\mathbf{R}^N \setminus \{0\}$.*

Note that a (simple) partial case of this theorem when f is a polynomial has been established earlier by the author and H. S. Shapiro in [KS1]. On the other hand, in [KS2] we have proven the following

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THEOREM 2. *Let Γ be an ellipsoidal surface in $\mathbf{R}^N$ and Ω denote its interior. The solution of the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \Gamma \end{cases} \tag{1.2}$$

extends as an entire harmonic function to $\mathbf{R}^N$.

The purpose of this note is twofold. First, it is to give a simple proof of Johnsson's Theorem 1 (but not his other results!) based on elementary potential theory. Second, and this is to some extent surprising, we show that the estimates needed to establish Theorem 1 are essentially those used in the proof of Theorem 2 in case of the sphere, perhaps with slight modifications (Lemmas 5, 6).

Throughout the paper we use standard multivariate notations. $\mathcal{P}_m = \mathcal{P}_{m,N}$ denotes the space of polynomials in N variables of degree at most m, and $\mathcal{H}_k = \mathcal{H}_{k,N}$ is the subspace of homogeneous polynomials of degree k in $\mathcal{P}_m$. If the functions f, g coincide up to their first derivatives on a surface Γ (*i.e.*, $(\frac{\partial}{\partial x_i})^k(f-g)|_\Gamma = 0, j = 1, \ldots, N; k = 0, 1)$, we write $f|_\Gamma \equiv g|_\Gamma$. ∇f denotes the gradient of a function f. A_N, B_N, C_N, *etc.*, denote constants that only depend on the dimension of the space.

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2. **Auxiliary lemmas.** Let $\Gamma = \{x \in \mathbf{R}^N : |x| = 1\}$ be the unit sphere and f is an entire function. As in [KS2], write the Taylor expansion of f as $f = \sum_0^\infty f_m$, where $f_m \in \mathcal{H}_m$. The following lemma is well-known (*cf.* [KS1] and references therein).

LEMMA 1. *$f_m \equiv u_m + (|x|^2 - 1)v_m$ on Γ where $u_m \in \mathcal{P}_m$, $v_m \in \mathcal{P}_{m-2}$ are harmonic polynomials.*

PROOF. It is well-known (*cf.*, *e.g.*, [ABR, p. 76]) that $f_m \in \mathcal{H}_m$ can be written as a finite sum $f_m = h_m + |x|^2 h_{m-2} + |x|^4 h_{m-4} + \cdots$, where $h_j \in \mathcal{H}_j$ and h_j are harmonic. Hence, on Γ we have: $f_m = u_m := h_m + h_{m-2} + h_{m-4} + \cdots$, $\operatorname{grad} f_m = \operatorname{grad} u_m + \operatorname{const}(f_m - h_m)x = \operatorname{grad}\{u_m + \operatorname{const}(h_{m-2} + h_{m-4} + \cdots)(|x|^2 - 1)\}$ and the lemma follows.

Let

$$u_m = u_{m,0} + \cdots + u_{m,m}, \quad v_m = v_{m,0} + \cdots + v_{m,m-2} \tag{2.1}$$

denote the decomposition of u_m and v_m into homogeneous polynomials; thus $u_{m,j}$, $v_{m,j}$ are in $\mathcal{H}_j$ and harmonic.

LEMMA 2. *The solution U_m of the Cauchy Problem*

$$\begin{cases} \Delta U_m = 0 & \text{near } \Gamma; \\ U_m \equiv f_m & \text{on } \Gamma \end{cases}$$

is given by

$$U_m = \sum_{k=0}^{m} u_{m,k} + \sum_{k=0}^{m-2} \frac{2}{2-N-2k}\left(\frac{v_{m,k}}{|x|^{N-2+2k}} - v_{m,k}\right), \tag{2.2}$$

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where $u_{m,k}$ *and* $v_{m,k}$ *are the same as in (2.1) (and Lemma 1). (In trivial cases,* $m = 0, 1$, $v_{m,k} = 0$.

PROOF (*cf.* [KS1]). First, note (*cf.*, *e.g.*, [ABR, p. 184]) that if $h \in H_k$ and is harmonic, then $\frac{h}{|x|^{N-2+2k}}$ is a homogeneous harmonic function of degree $2-N-k$ in $\mathbf{R}^N \setminus \{0\}$. Thus, the function in the right-hand side of (2.2) is indeed harmonic. Also,

$$\frac{\partial}{\partial x_j}\left(\frac{v_{m,k}}{|x|^{N-2+2k}}\right)\Big|_\Gamma = \frac{\partial v_{m,k}}{\partial x_j} + (2-N-2k)x_j v_{m,k}|_\Gamma$$

and hence,

$$\sum_{k=0}^{m-2} \frac{2}{2-N-2k}\left(\frac{v_{m,k}}{|x|^{N-2+2k}} - v_{m,k}\right)\Big|_\Gamma \equiv (|x|^2-1)v_m|_\Gamma. \tag{2.3}$$

(2.3) and Lemma 1 complete the proof of Lemma 2.

Now the strategy to prove Theorem 1 is rather straightforward: we shall show that the series $\sum_{m=0}^{\infty} |U_m(x)|$ converges for all $x \in \mathbf{R}^N \setminus \{0\}$. The following series of lemmas provide the needed estimates.

LEMMA 3 (*cf.* [KS2]). *Let* $F_m := \max\{|f_m(x)| : x \in \Gamma\}$, $G_m := \max\{\|\nabla f_m(x)\| : x \in \Gamma\}$. *Then* $(F_m)^{\frac{1}{m}} \longrightarrow 0$ *and* $(G_m)^{\frac{1}{m}} \longrightarrow 0$.

PROOF. Since the proofs of both statements are essentially the same, let us show that $(G_m)^{\frac{1}{m}} \longrightarrow 0$. (The other statement is also proved in [KS2].) Fix $1 \le j \le N$. For $t \in \mathbf{C}$, $x \in \Gamma$ we have

$$f(tx) = \sum_0^\infty t^m f_m(x),$$

so,

$$\frac{\partial f}{\partial x_j}(tx) = \sum_0^\infty t^m \frac{\partial f_m}{\partial x_j}(x),$$

i.e., $\frac{\partial f_m(x)}{\partial x_j}$ are the Taylor coefficients of the entire function $t \longmapsto \frac{\partial f}{\partial x_j}(tx)$ on $\mathbf{C}$. The Cauchy-Hadamard estimate then implies

$$\left|\frac{\partial f_m}{\partial x_j}(x)\right| \le \frac{\max\{|\frac{\partial f}{\partial x_j}(tx)| : |t| \le T\}}{T^m}$$

for all $T > 0$. Hence,

$$\max_{x\in\Gamma}\left|\frac{\partial f_m}{\partial x_j}(x)\right| \le \frac{\max\{|\frac{\partial f}{\partial x_j}(z)| : |z| \le T\}}{T^m}$$

and

$$\max_{x\in\Gamma} \|\nabla f_m(x)\| \le \frac{\left(\sum_{j=1}^N \left(\max\{|\frac{\partial f}{\partial x_j}(z)| : |z| \le T\}\right)^2\right)^{\frac{1}{2}}}{T^m}.$$

Taking the m-th root and letting $m \longrightarrow \infty$ gives

$$\lim_{m\to\infty} (G_m)^{\frac{1}{m}} \le \frac{1}{T}$$

for arbitrary T, implying the assertion.

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LEMMA 4. *Let $h \in P_m$ be any harmonic polynomial, and $h = h_0 + h_1 + \cdots + h_m$ its decomposition into homogeneous polynomials. Then,*

$$\max_{x\in\Gamma} |h_k(x)| \le C_N k^{\frac{N}{2}} \max_{x\in\Gamma} |h(x)|, \quad 1 \le k \le m.$$

Also,

$$|h_0(x)| = |h_0(0)| \le \max\{|h(x)| : x \in \Gamma\}.$$

This lemma is from [KS2]. For the reader's convenience we include a proof.

PROOF. The statement concerning h_0 is obvious, so suppose $k \ge 1$. Without loss of generality, suppose $\max\{|h(x)| : x \in \Gamma\} = 1$. If $d\sigma$ denotes surface measure on Γ, we have, since $\{h_k\}$ are orthogonal in $L^2(\Gamma, d\sigma)$,

$$\int_\Gamma |h_k|^2\, d\sigma \le \int_\Gamma |h|^2\, d\sigma \le |\Gamma|,$$

where $|\Gamma|$ is the $(N-1)$-dimensional measure of Γ. It follows easily that, if dx denotes Lebesgue measure in $\mathbf{R}^n$,

$$\int_B |h_k|^2\, dx \le A_N, \tag{2.4}$$

where A_N is a constant depending only on N, $B := \{x : |x| < 1\}$ is the unit ball. Fix $y \in \Gamma$. Then, for $0 < r < 1$, $|h_k|^2(ry)$, does not exceed the mean value of $|h_k|^2$ over the ball B' centered at ry, with radius $1 - r$, giving the estimate

$$|h_k(ry)|^2 \le \frac{1}{|B'|} \int_{|B'|} |h_k|^2\, dx. \tag{2.5}$$

Since the volume of $|B'| = A'_N(1-r)^N$, we obtain from (2.4), (2.5) and homogeneity of h_k:

$$|h_k(y)| \le \left[\frac{A''_N}{r^{2k}(1-r)^N}\right]^{\frac{1}{2}}$$

for all $0 < r < 1$. The choice of $r = 1 - (2k)^{-1}$ gives the desired estimate.

LEMMA 5. *Let $f_m \equiv u_m + (|x|^2 - 1)v_m$ on Γ be as in Lemma 1. Then,*

$$V_m := \max\{|v_m(x)| : x \in \Gamma\} \le C_N(G_m + m^{2N} F_m),$$

where $F_m := \max\{|f_m(x)| : x \in \Gamma\}$, $G_m := \max\{\|\nabla f_m(x)\| : x \in \Gamma\}$ are the same as in Lemma 2. Thus, in particular,

$$\lim_{m\to\infty} (V_m)^{\frac{1}{m}} = 0.$$

PROOF. By our hypothesis, for $1 \le j \le N$, we have on Γ

$$\frac{\partial f_m}{\partial x_j} = \frac{\partial u_m}{\partial x_j} + 2x_j v_m.$$

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So,

$$4\sum_1^N x_j^2|v_m|^2 = 4|v_m|^2 \le 2(\|\nabla f_m\|^2 + \|\nabla u_m\|^2)$$

on Γ, *i.e.*, for $x \in \Gamma$

$$|v_m(x)| \le C\big(G_m + \|\nabla u_m(x)\|\big). \tag{2.6}$$

To estimate $\|\nabla u_m\|$ on Γ, recall that $u_m = \sum_{k=0}^m u_{m,k}$, where $u_{m,k}$ are homogeneous harmonic polynomials of degree k.

The following assertion is perhaps of independent interest.

LEMMA 6. *Let $h \in H_k$ be a homogeneous polynomial of degree k. Then*

$$\max\{\|\nabla h(x)\| : x \in \Gamma\} \le k\sqrt{2}\max\{|h(x)| : x \in \Gamma\}.$$

PROOF OF LEMMA 6. Fix $x \in \Gamma$. First note that by Euler's equation the normal derivative of h at x equals

$$\frac{\partial h}{\partial n}(x) = \sum_1^N x_j \frac{\partial h}{\partial x_j}(x) = kh(x),$$

and hence,

$$\max\left\{\left|\frac{\partial h}{\partial n}(x)\right| : x \in \Gamma\right\} = k\max\{|h(x)| : x \in \Gamma\}. \tag{2.7}$$

Now, let $y : \|y\| = 1$ be any vector orthogonal to $x \in \Gamma$, *i.e.*, tangent to Γ at x. The two-dimensional plane $\langle x, y\rangle$ spanned by x, y "cuts" Γ along a unit circle T. If (ξ, η) stand for coordinates on $\langle x, y\rangle$, the restriction of $h|_{\langle x,y\rangle}$ is a (homogeneous) polynomial of degree k in two variables (ξ, η), and hence, according to Lemma 1 (for $N = 2$), it coincides on T with a harmonic polynomial $H_0(\xi, \eta)$, $\deg H_0 \le k$. In particular, on T $H_0 := \sum_0^k (a_j \cos j\theta + b_j \sin j\theta)$ becomes a trigonometric polynomial of order $\le k$, where θ is the polar angle in the plane $\langle x, y\rangle$. Then, invoking classical Chebyshev's inequality, we obtain

$$\begin{aligned} |D_{\bar{y}} h(x)| &= \left|\frac{dH_0}{d\theta}(x)\right| \le k\max\{|H_0(z)| : z \in T\} \\ &\le k\max\{|h(z)| : z \in \Gamma\}, \end{aligned} \tag{2.8}$$

for an arbitrary vector y at x tangent to Γ. From (2.7), (2.8), the lemma follows.

REMARK. In view of (2.7), the constant $\sqrt{2}$ may not be sharp: the maximum of normal and tangential derivatives cannot be attained at the same point. In particular, it would be interesting to know whether $\sqrt{2}$ can be replaced by 1. This is true, *e.g.*, when h is real-valued (*cf.* [S, Equation (12) ff.]).

PROOF OF LEMMA 5, CONT'D. From Lemmas 4, 6, and the fact that $u_m = f_m$ on Γ, we obtain for $x \in \Gamma$

$$\left|\frac{\partial}{\partial x_j} u_m(x)\right| \le \sum_{k=0}^m C_N k^{\frac{N}{2}+1} F_m$$

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and, finally,

$$\|\nabla u_m(x)\| \leq C_N m^{\frac{N+4}{2}} F_m \leq C_N m^{2N} F_m. \tag{2.9}$$

Now, (2.6) and (2.9) imply the lemma.

3. **Proof of Theorem 1.** Fix $R > 0$. To show that for any $x : \frac{1}{R} < |x| < R$, the series $\sum_{m=0}^{\infty} |U_m(x)| < A(R) < +\infty$, where U_m is defined by (2.2), it suffices to show that the series

(I) $\sum_{m=0}^{\infty} \sum_{k=0}^{m} |u_{m,k}(x)|$,

(II) $\sum_{m=2}^{\infty} \sum_{k=0}^{m-2} \frac{2}{|2-N-2k|} \frac{|v_{m,k}(x)|}{|x|^{N-2+2k}}$, and

(III) $\sum_{m=2}^{\infty} \sum_{k=0}^{m-2} \frac{2}{|2-N-2k|} |v_{m,k}(x)|$

all converge. Set $F_m = \varepsilon_m^m$, where, according to Lemma 3, $\varepsilon_m \longrightarrow 0$ when $m \longrightarrow \infty$. For series (I) we have, in view of Lemma 4,

$$|u_{m,k}(x)| \leq C_N k^{\frac{N}{2}} \varepsilon_m^m |x|^k \leq C'_N A_N^k \varepsilon_m^m R^k, \tag{3.1}$$

where $A_N > 1$ is a constant that depends on N. Thus,

$$\sum_{k=0}^{m} |u_{m,k}(x)| \leq C'_N \varepsilon_m^m \sum_{k=0}^{m} (A_N R)^k \leq C''_N \varepsilon_m^m (A_N R)^{m+1}, \tag{3.2}$$

and hence,

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} |u_{m,k}(x)| \leq C''_N A_N R \sum_{m=0}^{\infty} (\varepsilon_m A_N R)^m < A(R) < +\infty,$$

because $\varepsilon_m \longrightarrow 0$ when $m \longrightarrow +\infty$.

It is worth pausing here to observe the following Corollary (*cf.* Theorem 2, [KS2]).

COROLLARY 1. *The solution $u_0 := \sum_{m=0}^{\infty} \sum_{k=0}^{m} u_{m,k}(x)$ of the Dirichlet problem*

$$\begin{cases} \Delta u_0 = 0 & \textit{in } B; \\ u_0 = f & \textit{on } \Gamma \end{cases}$$

with entire data f extends to $\mathbf{R}^N$ (and hence to $\mathbf{C}^N$) as an entire harmonic function.

Note that the above argument immediately implies the convergence of series (III), as well. Indeed, let $V_m := \max\{|v_m(x)| : x \in \Gamma\} = \delta_m^m$. Then, Lemma 5 implies that $\delta_m \longrightarrow 0$ when $m \longrightarrow \infty$, while Lemma 4 provides the estimate

$$|v_{m,k}(x)| \leq C_N k^{\frac{N}{2}} \delta_m^m |x|^k, \tag{3.3}$$

which is identical with (3.1).

Finally, to establish the convergence of (II), we fix $x : 0 < \frac{1}{R} < |x| < R < \infty$. Without loss of generality, we can assume $|x| < 1$, since for $|x| > 1$ convergence of series (II) is implied by that of (III). Then, (3.3) yields (*cf.* (3.2)):

$$\sum_{k=0}^{m-2} \frac{|v_{m,k}(x)|}{|x|^{N-2+2k}} \leq C_N \delta_m^m \sum_{k=0}^{m-2} k^{\frac{N}{2}} R^{N-2+k} \leq C'_N \delta_m^m R^{N-2} (A_N R)^{m-1}.$$

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Therefore, as above,

$$\sum_{m=2}^{\infty}\sum_{k=0}^{m-2}\frac{|v_{m,k}(x)|}{|x|^{N-2+2k}} \le A(R) < +\infty,$$

and hence, series (II) converges as well.

From the estimates we have given it follows at once that the series $u = \sum_{m=0}^{\infty} U_m(x)$, giving the solution of the Cauchy problem (1.1) on the sphere converges absolutely everywhere in $\mathbf{C}^N \setminus \{z : \sum_0^N z_j^2 = 0\}$. Thus, we obtain the following corollary also due to G. Johnsson [J].

COROLLARY 2. *The solution of the Cauchy problem for the Laplace equation with an entire data on the sphere extends as an analytic (multi-valued for odd N) function to the whole complement in* $\mathbf{C}^N$ *of the isotropic cone* $\hat{\Gamma}_0 := \{z \in \mathbf{C}^N : \sum_1^N z_j^2 = 0\}$.

REMARK. It is plausible that this way of reasoning can be somewhat modified to give a proof of Johnsson's theorem for general ellipsoids $\Gamma := \{x \in \mathbf{R}^N : \sum_1^N a_j^{-2} x_j^2 = 1\}$. The singularity set then is the "caustic" $\tilde{\Gamma} := \{x \in \mathbf{R}^N, x_N = 0, \sum_1^{N-1} \frac{x_j}{a_j^2 - a_N^2} = 1\}$ (we assume that $a_1 > a_2 > \cdots > a_N$). All the estimates related to the Dirichlet problem extend to that case mutatis mutandis (*cf.* [KS2]). The difficulty lies in extending Lemma 2. Although, in the earlier unpublished joint work with H. S. Shapiro, we have been able to show explicitly that the analogues of functions U_m (*i.e.*, solutions of the Cauchy problem (1.1) with polynomial data on an ellipsoidal surface) do extend to the complement of the caustic $\tilde{\Gamma}$ in $\mathbf{R}^N$, the formulae for those solutions obtained in terms of ellipsoidal harmonics seem too complicated to allow establishing a readable analogue of Lemma 5.

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Department of Mathematics
University of Arkansas
Fayetteville, Arkansas 72701
U.S.A.
email: dmitry@comp.uark.edu