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A NOTE ON KARAMATA'S GENERALISED REGULAR VARIATION

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Abstract

In this note, a one sided aspect of regular variation is considered, and some different results which can be obtained by bounding the ratio $f(x\lambda)/f(x)$ above or below are given.

In this note we consider some aspects of a one-sided type of regular variation. We let f be a positive measurable function on $[0,\infty)$, and consider the condition

$$(1) \qquad \limsup_{x\to+\infty} f(x\lambda)/f(x) < +\infty \quad \text{for } \lambda \text{ in a set of positive measure contained in } [1,+\infty).$$

Later, we show how this condition is related to some used by Karamata and other authors. We make the further assumptions throughout that f is integrable on all intervals $[0,x]$, $x>0$, and that $\int_0^1 u^k f(u)du$ converges for any k. These are merely for convenience in the notation, since we are only interested in the asymptotic behaviour of f.

One of the important uses of the theory of regular variation is to relate the asymptotic behaviour of the integral $\int_0^x u^k f(u)du$ to that of $f(x)$ and, in general, we would like to know when

$$(2) \qquad 0 < \liminf_{x\to+\infty} \frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du}, \quad \text{and} \quad \limsup_{x\to+\infty} \frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du} < +\infty.$$

In this note we find conditions under which the two inequalities in (2) hold for any k, and hence show that there is an essential dis-symmetry between them. In Lemma 1 below, we see that the right hand inequality in (2) holds whenever (1) holds; but that, for the left hand inequality in (2), we require in addition that $\limsup_{\lambda\to+\infty}\lambda^{k+1}\liminf_{x\to+\infty} f(x\lambda)/f(x) = +\infty$.

Our main result is:

THEOREM 1. *Suppose* (1) *holds, and* $\int_0^\infty u^k f(u)du = +\infty$. *Then*

$$\liminf_{x\to+\infty}\frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du} > 0$$

holds if

$$\limsup_{\lambda\to+\infty}\lambda^{k+1}\liminf_{x\to+\infty} f(x\lambda)/f(x) = +\infty,$$

while

$$\liminf_{x\to+\infty}\frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du} > 0$$

implies

$$\lim_{\lambda\to+\infty}\lambda^{k+1}\liminf_{x\to+\infty} f(x\lambda)/f(x) = +\infty.$$

The proof of Theorem 1 is accomplished through a series of lemmas, some of which are of independent interest.

LEMMA 1. *If* (1) *holds, then for any* k,

$$\limsup_{x\to+\infty}\frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du} = c_k < +\infty. \tag{3}$$

PROOF OF LEMMA 1. Suppose not; then for some k there is a sequence $x_n \to +\infty$ for which

$$0 = \liminf_{n\to+\infty}\int_0^{x_n} u^k f(u)du/x_n^{k+1}f(x_n) = \liminf_{n\to+\infty}\int_0^1 u^k f(ux_n)du/f(x_n)$$

$$\geqq \int_0^1 u^k \liminf_{n\to+\infty} f(ux_n)/f(x_n)du,$$

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by Fatou's Lemma. This means that $\liminf_{n\to+\infty} f(\lambda x_n)/f(x_n) = 0$ a.e. $\lambda \in [0,1)$, $\liminf_{x\to+\infty} f(\lambda x)/f(x) = 0$ a.e. $\lambda \in [0,1)$, and $\limsup_{x\to+\infty} f(x\lambda)/f(x) = +\infty$ a.e. $\lambda > 1$, contradicting (1).

LEMMA 2. *If* (1) *holds, there are constants* N, x_0 *and* $\lambda_0 \geqq 1$ *for which* $f(x\lambda)/f(x) \leqq \lambda^N$, *whenever* $\lambda \geqq \lambda_0$ *and* $x \geqq x_0$.

PROOF OF LEMMA 2. *Let* $\phi(x) = \log f(e^x)$, let $\psi(\lambda) = \limsup_{x\to+\infty}[\phi(x+\lambda) - \phi(x)]$, and for integers $n \geqq 1$ define the sets

$$S_n = \{\lambda \geqq 0 : \psi(\lambda) \leqq n\lambda\}$$

so that

$$\bigcup_{n\geqq 1} S_n = \{\lambda \geqq 0 : \psi(\lambda) < +\infty\},$$

and hence by (1), S_{n_0} has positive measure for some $n_0 \geqq 1$. Clearly, if $\lambda_1, \lambda_2 \in S_{n_0}$,

$$\psi(\lambda_1 + \lambda_2) \leqq \psi(\lambda_1) + \psi(\lambda_2) \leqq n_0(\lambda_1 + \lambda_2).$$

So $\lambda_1 + \lambda_2 \in S_{n_0}$, and by a well known result of Steinhaus, these properties mean that S_{n_0} contains an interval $[A, \infty)$. Recalling the definition of ϕ, this means that

$$\limsup_{x\to+\infty} f(x\lambda)/f(x) \leqq \lambda^{n_0}, \quad \text{for} \quad \lambda \geqq e^A.$$

This inequality, apparently stronger than (1), was imposed in place of (1) in the original version of this paper; a referee showed me the above argument, for which I am grateful. Now to derive the uniform bound of the lemma, we trivially modify Letac (1970). Define ϕ as above, and for $\varepsilon > 0$ let

$$T_n = \{\lambda \geqq A : \phi(x+\lambda) - \phi(x) \leqq (n_0 + \varepsilon)\lambda, \text{ for } x \geqq n\}.$$

Then $\bigcup_{n\geqq 1} T_n = [A, \infty)$ and so for some n_1, T_{n_1} has positive measure, and is easily seen, like S_{n_0}, to be a semigroup. Hence T_{n_1} contains an interval $[B, \infty)$. This means that $f(x\lambda)/f(x) \leqq \lambda^{n_0+\varepsilon} \equiv \lambda^N$ for $\lambda \geqq \lambda_0 = e^B$ and $x \geqq x_0 = e^{n_1}$. This completes the proof of Lemma 2.

LEMMA 3. *If* (1) *holds then*

$$\limsup_{x\to+\infty} \int_{x\lambda_0}^{x\lambda} u^k f(u)\,du / x^{k+1} f(x) = f_k(\lambda) < +\infty, \quad \text{for} \quad \lambda \geqq \lambda_0,$$

where λ_0 *is defined in Lemma* 2.

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PROOF OF LEMMA 3. For $\lambda \geqq \lambda_0$,

$$\limsup_{x\to+\infty} \int_{x\lambda_0}^{x\lambda} u^k f(u)du / x^{k+1} f(x) = \limsup_{x\to+\infty} \int_{\lambda_0}^{\lambda} u^k f(ux)du / f(x)$$

$$\leqq \limsup_{x\to+\infty} \sup_{\lambda_0 \leqq u \leqq \lambda} [f(ux)/f(x)] \int_{\lambda_0}^{\lambda} u^k du$$

$$\leqq \sup_{\lambda_0 \leqq u \leqq \lambda} u^N \int_{\lambda_0}^{\lambda} u^k du < +\infty,$$

by using the result of Lemma 2.

LEMMA 4. *If* (3) *holds and* $\liminf_{x\to+\infty} (x^{k+1} f(x) / \int_0^x u^k f(u)du) = b > 0$ *then*

$$\liminf_{x\to+\infty} f(x\lambda)/f(x) \geqq bc_k^{-1} \lambda^{b-k-1}, \quad \textit{for } \lambda \geqq 1.$$

PROOF OF LEMMA 4. For $\lambda > 1$, $\varepsilon > 0$ and $x \geqq x_0(\varepsilon)$,

$$\frac{\int_0^{x\lambda} u^k f(u)du}{\int_0^x u^k f(u)du} = \exp \int_x^{x\lambda} \frac{u^{k+1} f(u) u^{-1} du}{\int_0^u y^k f(y)dy}$$

$$\geqq \exp\left[(b-\varepsilon)\int_x^{x\lambda} u^{-1} du\right]$$

$$= \lambda^{b-\varepsilon},$$

(c.f. Matuszewska (1962) page 336). Hence, by what we have just proved and the definitions of b and c_k,

$$\lambda^{b-\varepsilon} \int_0^x u^k f(u)du \leqq \int_0^{x\lambda} u^k f(u)du,$$

$$(b-\varepsilon) \int_0^{x\lambda} u^k f(u)du \leqq \lambda^{k+1} x^{k+1} f(x\lambda),$$

$$x^{k+1} f(x) \leqq (c_k + \varepsilon) \int_0^x u^k f(u)du,$$

for $x \geqq x_1(\varepsilon, \lambda)$. This means that

$$\lambda^{b-\varepsilon} x^{k+1} f(x) \leqq (c_k + \varepsilon)(b-\varepsilon)^{-1} \lambda^{k+1} x^{k+1} f(x\lambda),$$

leading to the desired result.

LEMMA 5. *Let* (1) *hold, suppose* $\int_0^\infty u^k f(u)du = +\infty$, *and let* $h(\lambda) = \liminf_{x\to+\infty} f(x\lambda)/f(x)$, *for* $\lambda > 1$. *Then*

$$\liminf_{x\to+\infty} \frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du} \geqq \frac{\lambda^{k+1}h(\lambda) - \lambda_0^{c_k}}{f_k(\lambda)} \quad \text{for} \quad \lambda \geqq \lambda_0,$$

where c_k *is defined in* (3) *and* λ_0 *and* $f_k(\lambda)$ *in Lemmas* 2 *and* 3.

PROOF. Take $\lambda > \lambda_0$. If $h(\lambda) = 0$ there is nothing to prove and so take $h(\lambda) > 0$. Given $\varepsilon > 0$ with $h(\lambda) - \varepsilon > 0$ take $x_0 = x_0(\varepsilon, \lambda)$ so that $f(x\lambda)/f(x) \geqq h(\lambda) - \varepsilon$, and $\int_{x\lambda_0}^{x\lambda} u^k f(u)du / x^{k+1}f(x) \leqq f_k(\lambda) + \varepsilon$, for $x \geqq x_0$; the latter can be accomplished by Lemma 3. Now

$$\begin{aligned}\int_0^x u^k f(u)du &= \int_0^{x_0} u^k f(u)du + \int_{x_0}^x u^k f(u)du \\ &\leqq o\left(\int_0^x u^k f(u)du\right) + \int_{x_0}^x u^k f(u\lambda)du/[h(\lambda)-\varepsilon] \\ &\leqq o\left(\int_0^x u^k f(u)du\right) + \lambda^{-k-1}\int_0^{x\lambda} u^k f(u)du/[h(\lambda)-\varepsilon] \\ &\leqq o\left(\int_0^x u^k f(u)du\right) + \lambda^{-k-1}\int_0^{x\lambda_0} u^k f(u)du/[h(\lambda)-\varepsilon] \\ &\quad + \lambda^{-k-1}\int_{x\lambda_0}^{x\lambda} u^k f(u)du/[h(\lambda)-\varepsilon] \\ &\leqq o\left(\int_0^x u^k f(u)du\right) + \lambda^{-k-1}(\lambda_0^{c_k}+\varepsilon)\int_0^x u^k f(u)du/[h(\lambda)-\varepsilon] \\ &\quad + \lambda^{-k-1}[f_k(\lambda)+\varepsilon]x^{k+1}f(x)/[h(\lambda)-\varepsilon].\end{aligned}$$

where we have used also the fact that

$$\limsup_{x\to+\infty} \frac{\int_0^{x\lambda} u^k f(u)du}{\int_0^x u^k f(u)du} \leqq \lambda^{c_k},$$

for $\lambda \geqq 1$, which we will prove later. Continuing, we get

$$\lambda^{k+1}[h(\lambda)-\varepsilon] \leqq o(1) + \lambda_0^{c_k} + \varepsilon + [f_k(\lambda)+\varepsilon]\frac{x^{k+1}f(x)}{\int_0^x u^k f(u)du},$$

which leads to the required result. To complete the proof, note that for $\lambda > 1$, $\varepsilon > 0$ and $x \geqq x_0(\varepsilon, \lambda)$, by Lemma 1,

$$\frac{\int_0^{x\lambda} u^k f(u)du}{\int_0^x u^k f(u)du} = \exp \int_x^{x\lambda} \frac{u^{k+1} f(u) u^{-1} du}{\int_0^u y^k f(y) dy}$$
$$\leqq \exp\left[(c_k + \varepsilon)\int_x^{x\lambda} u^{-1} du\right]$$
$$= \lambda^{c_k+\varepsilon}$$

which is what we needed.

PROOF OF THEOREM 1. If $\liminf_{x\to+\infty}(x^{k+1}f(x)/\int_0^x u^k f(u)du) = b > 0$, then by Lemma 4, for $\lambda \geqq 1$,

$$\lambda^{k+1}\liminf_{x\to+\infty} f(x\lambda)/f(x) \geqq bc_k^{-1}\lambda^b \to +\infty \quad \text{as} \quad \lambda \to +\infty.$$

Conversely, if $\limsup_{\lambda\to+\infty}\lambda^{k+1}\liminf_{x\to+\infty} f(x\lambda)/f(x) = +\infty$, there is a $\lambda_1 > \lambda_0$ for which $\lambda_1^{k+1}\liminf_{x\to+\infty} f(x\lambda_1)/f(x) > 2\lambda_0^{c_k}$, where c_k is defined in Lemma 1 and λ_0 in Lemma 2. From Lemma 5, taking $\lambda = \lambda_1$, we see that $\liminf_{x\to+\infty}(x^{k+1}f(x)/\int_0^x u^k f(u)du) > \lambda_0^{c_k}/f_k(\lambda_1) > 0$.

REMARKS. (i) When $\int_0^\infty u^k f(u)du < +\infty$, similar methods can be used to consider $\int_x^\infty u^k f(u)du$, instead of $\int_0^x u^k f(u)du$.

(ii) When f is nonincreasing, (1) holds trivially, and Lemmas 1, 2 and 3 are obvious. Lemma 4 was proved, in essence, by Feller (1969) under this restriction.

(iii) A function f, positive and measurable on $[A,\infty)$ for some $A > 0$, was said by Karamata to be R-0 *varying* if for some $m \leqq 1$, $M \geqq 1$, $a > 1$,

$$m \leqq f(x\lambda)/f(x) \leqq M, \quad \text{for } 1 \leqq \lambda \leqq a \quad \text{and} \quad x \geqq A. \tag{4}$$

An R-0 varying function f is S-0 *varying* if for some $c \in (0,1]$ and $C \geqq 1$,

$$c \leqq \liminf_{x\to+\infty} f(x\lambda)/f(x) \leqq \limsup_{x\to+\infty} f(x\lambda)/f(x) \leqq C, \quad \text{for } \lambda \geqq 1. \tag{5}$$

The concepts of R-0 and S-0 variation have recently been reviewed and extended by Seneta (1976). He showed (page 94) that if f is R-0 varying, then (2) holds for any $k > h - 1$, where h is a constant depending on the values of a and m in (4).

Functions satisfying (1) are related to the Δ_λ-*functions* of Matuszewska (1962), who assumed that the inequality in (1) holds for all $\lambda \geqq 1$.

(iv) The right hand side of (4) implies (1), and if we assume it, we can rewrite Theorem 1 as:

THEOREM 2. *Assume* $f(x\lambda)/f(x) \leqq M < +\infty$ *for* $\lambda \in [1, a]$ *for some* $a > 1$ *and* $M \geqq 1$ *whenever* $x \geqq A$, *and suppose* $\int_0^\infty u^k f(u)du = +\infty$. *Then* $\liminf_{x\to+\infty}(x^{k+1}f(x)/\int_0^x u^k f(u)du) = 0$ *if and only if* $\liminf_{x\to+\infty} f(x\lambda)/f(x) \leqq \lambda^{-k-1}$, *for* $\lambda \geqq 1$.

PROOF OF THEOREM 2. If (4) holds, and $\lambda \in [a^{p-1}, a^p]$ for some $p > 1$, then as in Lemma A.1 of Seneta (1976 page 92), we can deduce that, for $x \geqq A$, $f(x\lambda)/f(x) \leqq M^p$. (Thus in fact f is a Δ_λ-function.) Furthermore, the conclusion of Lemma 3 now holds with $\lambda_0 = 1$. The conclusion of Theorem 2 is just the contrapositive of the conclusions of Lemmas 4 and 5, when λ_0 is put equal to 1.

To conclude our discussion we remark that one can always say something about the ratio $f(x\lambda)/f(x)$ in the following sense:

LEMMA 6. *Let f be positive, measurable and integrable on* $[a, \infty)$. *Then: if* $\int_a^\infty f(u)du = +\infty$, *we have* $\limsup_{x\to+\infty} f(x\lambda)/f(x) \geqq \lambda^{-1}$, *for* $\lambda \geqq 1$; *if* $\int_a^\infty f(u)du < +\infty$, *we have* $\liminf_{x\to+\infty} f(x\lambda)/f(x) \leqq \lambda^{-1}$, *for* $\lambda \geqq 1$.

PROOF. Suppose $\int_a^\infty f(u)du = +\infty$ and let $g(\lambda) = \limsup_{x\to+\infty} f(x\lambda)/f(x)$ for $\lambda \geqq 1$. Given $\lambda > 1$ and $\varepsilon > 0$ choose $x_0 = x_0(\varepsilon, \lambda) \geqq a$, so that $f(x\lambda) \leqq [g(\lambda) + \varepsilon]f(x)$ whenever $x \geqq x_0$. Then for $x \geqq x_0$,

$$\begin{aligned}
\int_a^x f(u)du &\leqq \int_a^{x\lambda} f(u)du = \lambda \int_{a/\lambda}^x f(u\lambda)du \\
&= \lambda \int_{a/\lambda}^{x_0} f(u\lambda)du + \lambda \int_{x_0}^x f(u\lambda)du \\
&\leqq o\left(\int_a^x f(u)du\right) + \lambda[g(\lambda) + \varepsilon] \int_{x_0}^x f(u)du \\
&\leqq o\left(\int_a^x f(u)du\right) + \lambda[g(\lambda) + \varepsilon] \int_a^x f(u)du,
\end{aligned}$$

so $g(\lambda) \geqq \lambda^{-1}$, as required. The other proof is similar. Using this lemma, we see that Theorem 2 holds even when $\int_0^\infty u^k f(u)du < +\infty$.

A result related to Lemma 6 is given by Seneta (1976) page 99: *if f is positive on* $[A, \infty)$, *bounded on finite intervals sufficiently far, and* $0 < \limsup_{x\to+\infty} f(x\lambda)/f(x) < +\infty$, *then* $\limsup_{x\to+\infty} f(x\lambda)/f(x) \geqq \lambda^p$ *for* $\lambda \geqq 1$ *for some* ρ.

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