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GEOMETRIC PHASES IN THE QUANTISATION OF BOSONS AND FERMIONS

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Dedicated to Alan Carey, on the occasion of his 60th birthday

Abstract

After reviewing geometric quantisation of linear bosonic and fermionic systems, we study the holonomy of the projectively flat connection on the bundle of Hilbert spaces over the space of compatible complex structures and relate it to the Maslov index and its various generalisations. We also consider bosonic and fermionic harmonic oscillators parametrised by compatible complex structures and compare Berry's phase with the above holonomy.

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1. Introduction

In geometric quantisation, quantum Hilbert space is constructed from the classical phase space (a symplectic manifold) together with a choice of polarisation. An important question is whether the Hilbert spaces from different polarisations can be naturally identified. For linear bosonic systems, there is a projectively flat connection on the bundle of Hilbert spaces over the space of compatible linear complex structures (see [1]). This identifies vectors in various Hilbert spaces up to a phase.

In [8], parallel transport along geodesics in the space of polarisations was calculated and shown to agree with the Bogoliubov transformation (see [18, 19]) and other definitions of intertwining operators (see [11, 14]). The real Lagrangian subspaces are on the boundary (at infinity) of the space of complex structures. When the geodesic is extended to infinity, the parallel transport yields the Segal–Bargmann and Fourier transforms (see [8]).

For linear fermionic systems, prequantisation and quantisation were considered (see [9, 18]) and the space of compatible complex structures is a compact Hermitian symmetric space. The bundle of Hilbert spaces again admits a projectively flat connection whose curvature is proportional to the standard Kähler form on the base (see [23]). The parallel transport along the geodesics in the space of polarisations yields intertwining operators between various constructions of the spinor representation (see [23]).

In this paper, we study the holonomy of these projectively flat connections and explore its geometric significance. In the bosonic case, the holonomy along a geodesic triangle is related to the generalised Maslov index in [11]. When the vertices of the triangle approach three mutually transverse Lagrangian subspaces at infinity, the holonomy becomes the composition of three Fourier transformations, which is known to be related to the triple Maslov index of Kashiwara (see [10]). Thus, we get interesting formulae for the Maslov index and its generalisation in terms of integrations of curvature on a surface bounded by three geodesics. In the fermionic case, the holonomy along a geodesic triangle is related to the orthogonal counterpart of the Maslov index (see [12]) and we obtain similar results using the holonomy.

We also consider bosonic and fermionic harmonic oscillators whose Hamiltonians are parametrised by compatible complex structures. As the parameter changes adiabatically, the energy eigenstates acquire a geometric phase called Berry's phase (see [2]), which we study using its relation (see [20]) with the universal connection (see [13]). We find that Berry's phase on the vacuum vector is inverse to the holonomy of the projectively flat connection discussed above. However, the connection responsible for (nonabelian) Berry's phase when the energy eigenvalue is degenerate is not projectively flat.

The rest of the paper is organised as follows. In Section 2, we review the work on geometric quantisation of linear bosonic and fermionic systems. In Section 3, we study the holonomy of the projectively flat connection on the bundle of Hilbert spaces over the space of compatible complex structures and relate it to the triple Maslov index and its various generalisations. In Section 4, we consider bosonic and fermionic harmonic oscillators parametrised by compatible complex structures and compare Berry's phase with the holonomy in Section 3.

2. Quantisation of bosonic and fermionic systems

2.1. Quantisation of linear bosonic systems. Let (V, ω) be a symplectic vector space of dimension $2n$. The prequantum line bundle $\ell \to V$ is a complex line bundle with a connection whose curvature is $\omega/\sqrt{-1}$. The prequantum Hilbert space $\mathcal{H}_0$ is the space of L^2-integrable sections (with respect to the Liouville measure) of ℓ. It can be identified with $L^2(V, \mathbb{C})$ upon choosing a trivialisation of ℓ. A complex structure J on V is compatible with ω if $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot) > 0$. Each such J determines a complex polarisation: a complex Lagrangian subspace $V_J^{1,0}$ of $V^{\mathbb{C}}$. The corresponding quantum Hilbert space is

$$\mathcal{H}_J = \{\psi \in \mathcal{H}_0 \mid \nabla_x \psi = 0, \forall x \in V_J^{0,1}\}.$$

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The Heisenberg algebra (generated by V subject to the canonical commutation relation) acts on $\mathcal{H}_J$ by an irreducible representation.

The space $\mathcal{J}_\omega$ of compatible complex structures is a noncompact Hermitian symmetric space isomorphic to $\mathrm{Sp}(2n, \mathbb{R})/U(n)$. Fixing $J_0 \in \mathcal{J}_\omega$ and choosing a unitary basis of V, the space $\mathcal{J}_\omega$ can be identified with a bounded symmetric domain parametrised by $n \times n$ complex symmetric matrices Z such that $I - \bar{Z}Z > 0$. The subspace $V_J^{1,0}$ is the graph of Z under this basis. The natural Kähler form on $\mathcal{J}_\omega$ is

$$\sigma_\omega = -\frac{\sqrt{-1}}{4}\,\mathrm{tr}_{V_J^{1,0}}(dJ \wedge dJ) = \sqrt{-1}\,\mathrm{tr}((1 - Z\bar{Z})^{-1}\, dZ \wedge (I_n - \bar{Z}Z)^{-1}\, d\bar{Z}).$$

Since $\mathcal{H}_J$ is a subspace of $\mathcal{H}_0$ for each $J \in \mathcal{J}_\omega$, there is a bundle of Hilbert spaces $\mathcal{H} \to \mathcal{J}_\omega$ whose fibre over J is $\mathcal{H}_J$. The trivial connection on the product bundle $\mathcal{J}_\omega \times \mathcal{H}_0$ projects orthogonally to a natural, projectively flat connection on $\mathcal{H}$ whose curvature is (see [1])

$$\mathsf{F}^{\mathcal{H}} = (\sigma_\omega/2\sqrt{-1})\,\mathrm{id}_{\mathcal{H}} = -\tfrac{1}{8}\,\mathrm{tr}_{V_J^{1,0}}(dJ \wedge dJ)\mathrm{id}_{\mathcal{H}}, \tag{2.1}$$

where $\mathrm{id}_{\mathcal{H}}$ is the section of $\mathrm{End}(\mathcal{H})$ which is the identity operator on $\mathcal{H}_J$ at $J \in \mathcal{J}_\omega$. Parallel transport in the bundle $\mathcal{H}$ identifies, up to a phase, states on the quantum Hilbert spaces $\mathcal{H}_J$ constructed from various polarisations J.

Since $\mathcal{J}_\omega$ is contractible and nonpositively curved, there is a unique geodesic $\gamma_{J_1J_2}$ from J_1 to J_2 for any $J_1, J_2 \in \mathcal{J}_\omega$. The parallel transport $\mathcal{U}_{J_2J_1} : \mathcal{H}_{J_1} \to \mathcal{H}_{J_2}$ along $\gamma_{J_1J_2}$ was calculated in [8]. For example, the parallel transport of the coherent state

$$c^\alpha_{J_1}(x) = \exp[\sqrt{-1}\omega(\bar{\alpha}, x^{1,0}_{J_1}) - \tfrac{1}{4}\omega(x, J_1x)],$$

where $\alpha \in V^{1,0}_{J_1}$ is a parameter and $x = x^{1,0}_{J_1} + x^{0,1}_{J_1} \in V$ according to $V^{\mathbb{C}} = V^{1,0}_{J_1} \oplus V^{0,1}_{J_1}$ is (see [8, 23])

$$\begin{aligned}(\mathcal{U}^{\mathcal{H}}_{J_2J_1} c^\alpha_{J_1})(x) = {}& \left(\det \frac{J_1 + J_2}{2}\right)^{-1/4} e^{-\frac{1}{4}\omega(x, J_2x)} \\ & \cdot \exp\left[\frac{1}{2}\omega\left(x^{1,0}_{J_2} - \bar{\alpha}, \left(\frac{J_1 + J_2}{2}\right)^{-1}(x^{1,0}_{J_2} - \bar{\alpha})\right)\right].\end{aligned}$$

The operator $\mathcal{U}_{J_2J_1}$ is, up to a rescaling by a positive constant, the orthogonal projection from $\mathcal{H}_{J_1}$ to $\mathcal{H}_{J_2}$ in $\mathcal{H}_0$ (see [8]). Therefore, $\mathcal{U}_{J_2J_1}$ coincides with the Bogoliubov transformation defined in [18, 19]. It also agrees with the operator studied in [11, 14] that intertwines the two equivalent irreducible representations $\mathcal{H}_{J_1}$ and $\mathcal{H}_{J_2}$ of the Heisenberg algebra.

We now include metaplectic correction. Let $\mathcal{V} \to \mathcal{J}_\omega$ be a vector bundle whose fibre over $J \in \mathcal{J}_\omega$ is $V^{1,0}_J$. This is a subbundle of the product bundle $\mathcal{J}_\omega \times V^{\mathbb{C}}$ and the trivial connection on the latter projects to a connection on $\mathcal{V}$. Its curvature is

$$\begin{aligned}\mathsf{F}^{\mathcal{V}} &= -\frac{1}{4}(dJ \wedge dJ)|_{\mathcal{V}} \\ &= -\begin{pmatrix}1\\ Z\end{pmatrix}(1 - \bar{Z}Z)^{-1}\, d\bar{Z} \wedge (1 - Z\bar{Z})^{-1}\, dZ(1 - \bar{Z}Z)^{-1}(1, -\bar{Z}).\end{aligned} \tag{2.2}$$

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Consider the line bundle $\mathcal{K} = \det \mathcal{V}^*$. The fibre over $J \in \mathcal{J}_\omega$ is given by $\mathcal{K}_J = \bigwedge^n (V_J^{1,0})^*$. The induced connection on $\mathcal{K}$ is compatible with the Hermitian structure and its curvature is $\mathsf{F}^{\mathcal{K}} = \sqrt{-1}\,\sigma_\omega$. Since $\mathcal{J}_\omega$ is contractible, there is a unique line bundle $\sqrt{\mathcal{K}} \to \mathcal{J}_\omega$ such that $(\sqrt{\mathcal{K}})^{\otimes 2} = \mathcal{K}$. A half-form on $V_J^{1,0}$ is an element of $\sqrt{\mathcal{K}_J}$. The Hilbert space of half-form quantisation (with the polarisation J) is $\hat{\mathcal{H}}_J = \mathcal{H}_J \otimes \sqrt{\mathcal{K}_J}$. As J varies, the half-forms form a flat bundle $\hat{\mathcal{H}} = \mathcal{H} \otimes \sqrt{\mathcal{K}}$ over $\mathcal{J}_\omega$. That is, the curvature $\mathsf{F}^{\hat{\mathcal{H}}} = 0$ (see [8, 19]). Thus, all the fibres $\hat{\mathcal{H}}_J$ can be canonically identified. For any $J_1, J_2 \in \mathcal{J}_\omega$, there is a natural nondegenerate sesquilinear pairing between $\sqrt{\mathcal{K}_{J_1}}$ and $\sqrt{\mathcal{K}_{J_2}}$ and hence between $\hat{\mathcal{H}}_{J_1}$ and $\hat{\mathcal{H}}_{J_2}$. The parallel transport $\hat{\mathcal{U}}_{J_2 J_1} : \hat{\mathcal{H}}_{J_1} \to \hat{\mathcal{H}}_{J_2}$ is, in fact, the operator determined by the pairing between them (see [8]).

Next, we consider half-density quantisation. We associate to the vector space $V_J^{1,0}$ a real line $|\mathcal{K}_J|$ on which a linear transformation $A \in \mathrm{End}_{\mathbb{C}}(V_J^{1,0})$ acts as multiplication by $|\det A|^{-1}$. An element of $|\mathcal{K}_J|$ is called a density on $V_J^{1,0}$. A half-density on $V_J^{1,0}$ is an element of $\sqrt{|\mathcal{K}_J|}$ which is a line such that $(\sqrt{|\mathcal{K}_J|})^{\otimes 2} = |\mathcal{K}_J|$ and on which the linear transformation A acts by $|\det A|^{-1/2}$. The lines $|\mathcal{K}_J|$ (respectively $\sqrt{|\mathcal{K}_J|}$) form real line bundles $|\mathcal{K}|$ (respectively $\sqrt{|\mathcal{K}|}$) over $\mathcal{J}_\omega$, which are naturally flat. In a good open covering, the transition functions of $|\mathcal{K}|$ (respectively $\sqrt{|\mathcal{K}|}$) are the norms of those of $\mathcal{K}$ (respectively $\sqrt{\mathcal{K}}$).

For any $J \in \mathcal{J}_\omega$, the Hilbert space of half-density quantisation is $\tilde{\mathcal{H}}_J = \mathcal{H}_J \otimes \sqrt{|\mathcal{K}_J|}$. The half-densities form a bundle $\tilde{\mathcal{H}} = \mathcal{H} \otimes \sqrt{|\mathcal{K}|}$ of Hilbert spaces over $\mathcal{J}_\omega$. It has a natural projectively flat connection with curvature

$$\mathsf{F}^{\tilde{\mathcal{H}}} = \mathsf{F}^{\mathcal{H}} = \sigma_\omega / 2\sqrt{-1}.$$

For any $J_1, J_2 \in \mathcal{J}_\omega$, there is a natural nondegenerate pairing between $\sqrt{|\mathcal{K}_{J_1}|}$ and $\sqrt{|\mathcal{K}_{J_2}|}$ and hence between $\tilde{\mathcal{H}}_{J_1}$ and $\tilde{\mathcal{H}}_{J_2}$. The parallel transport $\tilde{\mathcal{U}}_{J_2 J_1} : \tilde{\mathcal{H}}_{J_1} \to \tilde{\mathcal{H}}_{J_2}$ is also the operator determined by the pairing between them.

A real Lagrangian subspace $L \subset V$ is a real polarisation selecting the sections of ℓ that are covariantly constant along L. Such a section can be identified with a complex-valued function on V/L. Let $\mathcal{K}_L = \bigwedge^n (V/L)^*$. We have, respectively, the spaces $\sqrt{\mathcal{K}_L}$ and $\sqrt{|\mathcal{K}_L|}$ of half-forms and half-densities on V/L, and the Hilbert spaces $\hat{\mathcal{H}}_L$ and $\tilde{\mathcal{H}}_L$ of half-form and half-density quantisation. The spaces $\hat{\mathcal{H}}_L$ and $\tilde{\mathcal{H}}_L$ have natural inner products and are irreducible representations of the Heisenberg algebra. If $J \in \mathcal{J}_\omega$, then there is a Segal–Bargmann transformation $\tilde{\mathcal{B}}_{JL} : \tilde{\mathcal{H}}_L \to \tilde{\mathcal{H}}_J$ (or $\hat{\mathcal{B}}_{JL} : \hat{\mathcal{H}}_L \to \hat{\mathcal{H}}_J$) that intertwines the two equivalent irreducible representations. Let $L_1, L_2 \subset V$ be two transverse Lagrangian subspaces. Then the intertwining operator $\tilde{\mathcal{F}}_{L_2 L_1} : \tilde{\mathcal{H}}_{L_1} \to \tilde{\mathcal{H}}_{L_2}$ (or $\hat{\mathcal{F}}_{L_2 L_1} : \hat{\mathcal{H}}_{L_1} \to \hat{\mathcal{H}}_{L_2}$) is a Fourier transformation (see [10]).

The real Lagrangian subspaces in V form the Shilov boundary $\mathcal{L}_\omega$ of $\mathcal{J}_\omega$ (as a bounded domain). The rest of the topological boundary consists of polarisations that are partly real and partly complex. For any $J_0 \in \mathcal{J}_\omega$ and $L \in \mathcal{L}_\omega$, there is a geodesic $\{J_t\}$ in $\mathcal{J}_\omega$ from J_0 such that $\lim_{t \to +\infty} J_t = L$. We have

$$\lim_{t \to +\infty} \tilde{\mathcal{U}}_{J_t J_0} = (\tilde{\mathcal{B}}_{J_0 L})^{-1}, \quad \lim_{t \to +\infty} \hat{\mathcal{U}}_{J_t J_0} = (\hat{\mathcal{B}}_{J_0 L})^{-1}$$

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(see [8] for half-form quantisation. The result for half-density quantisation is then straightforward). Two Lagrangian subspaces $L_+, L_- \in \mathcal{L}_\omega$ are transverse if and only if there is a geodesic $\{J_t\}$ in $\mathcal{J}_\omega$ such that $\lim_{t\to\pm\infty} J_t = L_\pm$ (see [8]). In this case,

$$\lim_{t\to+\infty} \tilde{\mathcal{U}}_{J_t J_{-t}} = \tilde{\mathcal{F}}_{L_+L_-}, \qquad \lim_{t\to+\infty} \hat{\mathcal{U}}_{J_t J_{-t}} = \hat{\mathcal{F}}_{L_+L_-}.$$

The above limits are taken in the sense of tempered distributions on V (see [8]).

2.2. Quantisation of linear fermionic systems. Let (V, g) be an oriented Euclidean vector space of dimension $2n$. Despite the absence of an honest prequantum line bundle, the prequantum Hilbert space can be taken to be $\mathcal{H}_0 = \bigwedge^n (V^{\mathbb{C}})^*$, on which covariant derivative operators act (see [9, 18, 23]). A compatible complex structure J on (V, g) is one that is compatible with the orientation of V and such that $g(J\cdot, J\cdot) = g(\cdot,\cdot)$. Each such J determines a polarisation: a maximally isotropic complex subspace $V_J^{1,0}$ of V. The corresponding quantum Hilbert space is

$$\mathcal{H}_J = \{\psi \in \mathcal{H}_0 \mid \nabla_x \psi = 0, \forall x \in V_J^{0,1}\}.$$

The Clifford algebra (generated by V subject to the canonical anticommutation relation) acts on $\mathcal{H}_J$ by an irreducible representation. In fact, up to a fermionic Gaussian factor, $\mathcal{H}_J$ agrees with the standard construction of the spinor representation (see [18, 23]).

The space $\mathcal{J}_g$ of compatible complex structures on (V, g) is a compact Hermitian symmetric space isomorphic to $\mathrm{SO}(2n)/U(n)$. Fixing $J_0 \in \mathcal{J}_g$ and choosing a unitary basis of V, the complement of the cut locus of J_0 in $\mathcal{J}_g$, which is an open dense subset, can be parametrised by $n \times n$ complex skew-symmetric matrices Z. Again, the subspace $V_J^{1,0}$ corresponds to the graph of Z. The natural Kähler form on $\mathcal{J}_g$ is

$$\sigma_g = \frac{\sqrt{-1}}{4} \operatorname{tr}_{V_J^{1,0}}(dJ \wedge dJ) = -\sqrt{-1}\, \operatorname{tr}((1 - Z\bar{Z})^{-1}\, dZ \wedge (I_n - \bar{Z}Z)^{-1}\, d\bar{Z}).$$

Since $\mathcal{H}_J$ is a subspace of $\mathcal{H}_0$ for each $J \in \mathcal{J}_g$, there is a bundle of Hilbert spaces $\mathcal{H} \to \mathcal{J}_g$ whose fibre over J is $\mathcal{H}_J$. The trivial connection on the product bundle $\mathcal{J}_g \times \mathcal{H}_0$ projects orthogonally to a natural connection on $\mathcal{H}$. Just like in the bosonic case, the connection is projectively flat and the curvature is (see [23])

$$\mathsf{F}^{\mathcal{H}} = (\sigma_g/2\sqrt{-1})\mathrm{id}_{\mathcal{H}} = \tfrac{1}{8} \operatorname{tr}_{V_J^{1,0}}(dJ \wedge dJ)\mathrm{id}_{\mathcal{H}}. \tag{2.3}$$

Parallel transport in the bundle $\mathcal{H}$ identifies, up to a phase, states in the quantum Hilbert spaces $\mathcal{H}_J$ from various polarisations J.

Unlike $\mathcal{J}_\omega$, the space $\mathcal{J}_g$ is compact and nonnegatively curved. Geodesics from J_1 to J_2, where $J_1, J_2 \in \mathcal{J}_g$, are not unique. However, if J_2 is not in the cut locus of J_1, then there is a unique length-minimising geodesic $\gamma_{J_2J_1}$ from J_1 to J_2. The parallel transport $\mathcal{U}_{J_2J_1} : \mathcal{H}_{J_1} \to \mathcal{H}_{J_2}$ along $\gamma_{J_1J_2}$ was calculated in [23]. For the coherent state

$$c^{\alpha}_{J_1}(\theta) = \exp\Big[-g(\theta^{1,0}_{J_1}, \bar{\alpha}) + \frac{\sqrt{-1}}{4} g(J_1\theta, \theta)\Big],$$

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where $\theta = \theta_{J_1}^{1,0} + \theta_{J_1}^{0,1}$ is a fermionic vector in V and $\alpha \in V_{J_1}^{1,0}$ is a fermionic parameter, the parallel transport from J_1 to J_2 is (see [23])

$$(\mathcal{U}_{J_1 J_0}^{\mathcal{H}} c_{J_0}^{\alpha})(\theta) = \left(\det \frac{J_1 + J_2}{2}\right)^{1/4} e^{(\sqrt{-1}/4) g(J_2\theta,\theta)} \cdot \exp\left[\frac{\sqrt{-1}}{2} g\left(\theta_{J_2}^{1,0} - \bar{\alpha}, \left(\frac{J_1 + J_2}{2}\right)^{-1} (\theta_{J_2}^{1,0} - \bar{\alpha})\right)\right].$$

The operator $\mathcal{U}_{J_2 J_1}$ is, up to a rescaling by a positive constant, the orthogonal projection from $\mathcal{H}_{J_1}$ to $\mathcal{H}_{J_2}$ in $\mathcal{H}_0$ (see [23]). Like the bosonic case, $\mathcal{U}_{J_2 J_1}$ coincides with the Bogoliubov transformation defined in [18]. It is the operator that intertwines the two equivalent irreducible representations $\mathcal{H}_{J_1}$ and $\mathcal{H}_{J_2}$ of the Clifford algebra.

Metaplectic correction of fermionic systems was introduced in [23]. Consider the line bundle $\mathcal{K}^{-1} = \det \mathcal{V}$ whose fibre over $J \in \mathcal{J}_g$ is $\mathcal{K}_J^{-1} = \bigwedge^n V_J^{1,0}$. The natural connection on $\mathcal{K}^{-1}$ is compatible with the Hermitian structure and its curvature is $\mathsf{F}^{\mathcal{K}^{-1}} = \sqrt{-1}\sigma_g$. Since $\mathcal{J}_g$ is simply connected and $c_1(\mathcal{K})$ is even, there is a unique line bundle $\sqrt{\mathcal{K}} \to \mathcal{J}_g$ such that $(\sqrt{\mathcal{K}})^{\otimes 2} = \mathcal{K}$ (see [23]). The Hilbert space of half-form quantisation (with the polarisation J) is $\hat{\mathcal{H}}_J = \mathcal{H}_J \otimes \sqrt{\mathcal{K}_J^{-1}}$ (see [23]). Notice the opposite power of $\mathcal{K}$ as in the bosonic case. When J varies, the half-form quantisations form a flat bundle $\hat{\mathcal{H}} = \mathcal{H} \otimes \sqrt{\mathcal{K}}$ over $\mathcal{J}_g$, that is, $\mathsf{F}^{\hat{\mathcal{H}}} = 0$ (see [23]). Thus, all the fibres $\hat{\mathcal{H}}_J$ (that is, the spinor representation spaces from various polarisations) can be canonically identified.

For any $J_1, J_2 \in \mathcal{J}_g$, the natural sesquilinear pairing between $\sqrt{\mathcal{K}_{J_1}}$ and $\sqrt{\mathcal{K}_{J_2}}$ is nondegenerate if and only if J_1 and J_2 are not in the cut locus of each other (see [23]). In this case, there is a sesquilinear pairing between $\hat{\mathcal{H}}_{J_1}$ and $\hat{\mathcal{H}}_{J_2}$. The parallel transport $\hat{\mathcal{U}}_{J_2 J_1} : \hat{\mathcal{H}}_{J_1} \to \hat{\mathcal{H}}_{J_2}$ along the length-minimising geodesic $\gamma_{J_2 J_1}$ from J_1 to J_2 is, in fact, the operator determined by the pairing between them (see [23]).

Half-density quantisation can also be established in the fermionic setting. Associated to the vector space $V_J^{1,0}$ is a real line $|\mathcal{K}_J^{-1}|$ on which a linear transformation $A \in \mathrm{End}_{\mathbb{C}}(V_J^{1,0})$ acts as multiplication by $|\det A|$. An element of $|\mathcal{K}_J^{-1}|$ is called a fermionic density of $V_J^{1,0}$. A fermionic half-density on $V_J^{1,0}$ is an element of $\sqrt{|\mathcal{K}_J^{-1}|}$. That is, a line such that $(\sqrt{|\mathcal{K}_J^{-1}|})^{\otimes 2} = |\mathcal{K}_J^{-1}|$ and on which the linear transformation A acts by $\sqrt{|\det A|}$. The lines $|\mathcal{K}_J^{-1}|$ (respectively $\sqrt{|\mathcal{K}_J^{-1}|}$) form real line bundles $|\mathcal{K}^{-1}|$ (respectively $\sqrt{|\mathcal{K}^{-1}|}$) over $\mathcal{J}_g$ which are naturally flat. For any $J \in \mathcal{J}_g$, the Hilbert space of half-density quantisations is

$$\tilde{\mathcal{H}}_J = \mathcal{H}_J \otimes \sqrt{|\mathcal{K}_J^{-1}|}.$$

The half-density quantisations form a bundle

$$\tilde{\mathcal{H}} = \mathcal{H} \otimes \sqrt{|\mathcal{K}^{-1}|}$$

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of Hilbert spaces over $\mathcal{J}_g$. It has a natural projectively flat connection with curvature

$$\mathsf{F}^{\tilde{\mathcal{H}}} = \mathsf{F}^{\mathcal{H}} = \sigma_g/2\sqrt{-1}.$$

When J_1 and J_2 are not in the cut locus of each other, there is a nondegenerate pairing between $\sqrt{|\mathcal{K}_{J_1}^{-1}|}$ and $\sqrt{|\mathcal{K}_{J_2}^{-1}|}$ and hence between $\tilde{\mathcal{H}}_{J_1}$ and $\tilde{\mathcal{H}}_{J_2}$. The parallel transport $\tilde{\mathcal{U}}_{J_2J_1} : \tilde{\mathcal{H}}_{J_1} \to \tilde{\mathcal{H}}_{J_2}$ along the length-minimising geodesic $\gamma_{J_2J_1}$ from J_1 to J_2 is also the operator determined by the pairing between them.

3. Holonomy of the bundle of Hilbert spaces

3.1. Bosonic systems: Maslov index and its generalisation. We recall that if L is a real Lagrangian subspace of V, then we have a Hilbert space $\tilde{\mathcal{H}}_L$ from half-density quantisation. For two transverse real Lagrangian subspaces $L_1, L_2 \in \mathcal{L}_\omega$, the Fourier transform operator $\tilde{\mathcal{F}}_{L_2L_1} : \tilde{\mathcal{H}}_{L_1} \to \tilde{\mathcal{H}}_{L_2}$ intertwines the two equivalent irreducible representations of the Heisenberg algebra. The operator $\tilde{\mathcal{F}}_{L_2L_1}$ is also the limit, in a certain sense, of the parallel transport in the bundle $\tilde{\mathcal{H}} \to \mathcal{J}_\omega$ along a geodesic in $\mathcal{J}_\omega$ extending to L_1 and L_2 (see [8]).

Suppose that there are three mutually transverse Lagrangian subspaces $L_1, L_2, L_3 \in \mathcal{L}_\omega$. Then we have (see [10])

$$\tilde{\mathcal{F}}_{L_1L_3} \circ \tilde{\mathcal{F}}_{L_3L_1} \circ \tilde{\mathcal{F}}_{L_2L_1} = \exp\Big[\frac{\sqrt{-1}\pi}{4}\alpha_\omega(L_1, L_2, L_3)\Big]\mathrm{id}_{\tilde{\mathcal{H}}_{L_1}}, \tag{3.1}$$

where $\alpha_\omega(L_1, L_2, L_3)$ is the triple Maslov index of Kashiwara (see [10]). It is defined to be the signature of the quadratic form

$$\omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$$

on $L_1 \oplus L_2 \oplus L_3$, where $x_i \in L_i$ $(i = 1, 2, 3)$. The triple Maslov index takes integer values in $[-n, n]$ and satisfies the properties (see [10]) that for any mutually transverse Lagrangian subspaces $L_1, L_2, L_3, L_4 \in \mathcal{L}_\omega$:

(a) $\alpha_\omega(gL_1, gL_2, gL_3) = \alpha_\omega(L_1, L_2, L_3)$ for all $g \in \mathrm{Sp}(V, \omega)$;
(b) $\alpha_\omega(L_1, L_2, L_3) = \alpha_\omega(L_2, L_3, L_1) = -\alpha_\omega(L_2, L_1, L_3)$;
(c) $\alpha_\omega(L_1, L_2, L_3) + \alpha_\omega(L_2, L_4, L_3) + \alpha_\omega(L_3, L_4, L_1) + \alpha_\omega(L_4, L_2, L_1) = 0$.

The last property is a cocycle condition on α_ω.

For each complex structure $J \in \mathcal{J}_\omega$, we have a quantum Hilbert space $\mathcal{H}_J$. Given any two complex structures $J_1, J_2 \in \mathcal{J}_\omega$, the parallel transport $\mathcal{U}_{J_2J_1} : \mathcal{H}_{J_1} \to \mathcal{H}_{J_2}$ in the bundle $\mathcal{H} \to \mathcal{J}_\omega$ along the geodesic $\gamma_{J_2J_1}$ from J_1 to J_2 is equal to the intertwining operator between the representations of the Heisenberg algebra on $\mathcal{H}_{J_1}$ and $\mathcal{H}_{J_2}$ (see [8]). We can use the same notation $\mathcal{U}_{J_2J_1}$ for the latter. For any three complex structures $J_1, J_2, J_3 \in \mathcal{J}_\omega$, we have (see [11])

$$\mathcal{U}_{J_1J_3} \circ \mathcal{U}_{J_3J_2} \circ \mathcal{U}_{J_2J_1} = \exp\Big[\frac{\sqrt{-1}\pi}{4}\alpha_\omega(J_1, J_2, J_3)\Big]\mathrm{id}_{\mathcal{H}_{J_1}}, \tag{3.2}$$

where $\alpha_\omega(J_1, J_2, J_3)$ is called the generalised Maslov index. The above formula also holds for half-density quantisation. Representing J_i by symmetric matrices Z_i (for $i = 1, 2, 3$), we have (see [11])

$$\begin{aligned}\alpha_\omega(J_1, J_2, J_3) \;=\; -\frac{2}{\pi}[&\arg\det(I_n - \bar{Z}_1 Z_2) \\ &+ \arg\det(I_n - \bar{Z}_2 Z_3) + \arg\det(I_n - \bar{Z}_3 Z_1)].\end{aligned} \tag{3.3}$$

Since $\mathcal{J}_\omega$ is contractible, the function 'arg' can be defined continuously so that $\arg\det(I_n - \bar{Z}Z') = 0$ when either Z or Z' is zero. We also note that the Bergman kernel function of the domain can be expressed explicitly in terms of $\det(I_n - \bar{Z}Z')$ (see [7]). The generalised Maslov index takes real values and satisfies the properties (see [11]) that for any $J_1, J_2, J_3, J_4 \in \mathcal{J}_\omega$:

(a) $\alpha_\omega(gJ_1g^{-1}, gJ_2g^{-1}, gJ_3g^{-1}) = \alpha_\omega(J_1, J_2, J_3)$ for all $g \in \mathrm{Sp}(V, \omega)$;
(b) $\alpha_\omega(J_1, J_2, J_3) = \alpha_\omega(J_2, J_3, J_1) = -\alpha_\omega(J_2, J_1, J_3)$;
(c) $\alpha_\omega(J_1, J_2, J_3) + \alpha_\omega(J_2, J_4, J_3) + \alpha_\omega(J_3, J_4, J_1) + \alpha_\omega(J_4, J_2, J_1) = 0$.

The last property means that α_ω is a 2-cocycle on the $\mathrm{Sp}(V, \omega)$-space $\mathcal{J}_\omega$ with values in $\mathbb{R}$ (see [11]).

Our observation is that (3.2) is the holonomy of the projectively flat bundle $\mathcal{H} \to \mathcal{J}_\omega$ along a loop that consists of three geodesics $\gamma_{J_2J_1}$, $\gamma_{J_3J_2}$ and $\gamma_{J_1J_3}$. Using the curvature (2.1), the holonomy is equal to $\exp[\sqrt{-1}/2 \int_\Delta \sigma_\omega]$, where Δ is an oriented surface bounded by the three geodesics. This implies that

$$\alpha_\omega(J_1, J_2, J_3) = \frac{2}{\pi}\int_\Delta \sigma_\omega. \tag{3.4}$$

It is clear that the equality holds modulo 8π. The additive constant is 0 by continuity when Δ shrinks to a point. A direct proof of this result is also possible. We write $\sigma_\omega = d\phi_\omega$, where

$$\phi_\omega = \sqrt{-1}(\bar{\partial} - \partial)\log\det(I_n - \bar{Z}Z).$$

It is easy to show that

$$\int_{\gamma_{J_2J_1}} \phi_\omega = -\arg\det(I_n - \bar{Z}_1 Z_2) \tag{3.5}$$

using, for example, the transformation of both sides under the symplectic group (see [6]). Formula (3.4) then follows from (3.3) and Stokes' theorem. Therefore, $|\alpha(J_1, J_2, J_3)| \le n$ by [6]. The properties of α_ω listed above also follow easily from (3.4).

We want to establish the analogue of (3.3) for the triple Maslov index $\alpha_\omega(L_1, L_2, L_3)$. A real Lagrangian subspace $L \in \mathcal{L}_\omega$, being on the Shilov boundary of $\mathcal{J}_\omega$, is represented by an $n \times n$ complex symmetric matrix that is unitary. Two Lagrangian subspaces L_1, L_2 are transverse if and only if their corresponding matrices Z_1, Z_2 satisfy $\det(I_n - \bar{Z}_1 Z_2) \ne 0$. The vanishing of this determinant implies the existence of a nonzero vector $v \in \mathbb{C}^n$ such that $Z_1 v = Z_2 v$, and hence the existence of a nonzero vector in $L_1 \cap L_2$. The converse is also true.

When the Lagrangian subspaces L_1, L_2 and L_3 (respectively parametrised by Z_1, Z_2 and Z_3) are mutually transverse, we claim that

$$\begin{aligned}\alpha_\omega(L_1, L_2, L_3) = -\frac{2}{\pi}[&\arg\det(I_n - \bar{Z}_1 Z_2) \\ &+ \arg\det(I_n - \bar{Z}_2 Z_3) + \arg\det(I_n - \bar{Z}_3 Z_1)].\end{aligned} \tag{3.6}$$

Consequently, the triple Maslov index can be expressed as the limit of the integration $\int_\Delta \sigma_\omega$ when the vertices of Δ approach the Shilov boundary from the interior $\mathcal{J}_\omega$. This result, together with its generalisation to Hermitian symmetric tube domains, appeared in [21, 22]. See also [4] for the case when $n = 1$ and [5] for the general case. However, the surface Δ itself (as well as its boundaries and vertices) moves in this limit procedure.

To show (3.6), we observe that both sides are invariant under the symplectic group $\mathrm{Sp}(V, \omega)$. Since $\mathrm{Sp}(V, \omega)$ acts transitively on transverse pairs of Lagrangian subspaces, we can assume, without loss of generality, that L_1 and L_2 are represented by $Z_1 = -I_n$ and $Z_2 = I_n$, respectively. To bring L_3 or Z_3 to a canonical form, we make a Cayley transform

$$Z \mapsto \Omega = \sqrt{-1}(I_n - Z)(I_n + Z)^{-1}.$$

The image of $\mathcal{J}_\omega$ is Siegel's upper-half space that consists of $n \times n$ complex symmetric matrices with a positive-definite imaginary part (see [15]). The Shilov boundary $\mathcal{L}_\omega$ is mapped to the set of real symmetric matrices plus a stratum of real codimension one at infinity. For example, the above L_1 and L_2 are represented by

$$\Omega_1 = \lim_{a \to +\infty} \sqrt{-1} a I_n$$

and $\Omega_2 = 0$, respectively. Note that Ω_3 is finite and invertible, since L_3 is transverse to both L_1 and L_2. A symplectic group element

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n, \mathbb{R})$$

acts on Ω by a fractional linear transformation

$$\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}.$$

The subgroup that preserves $Z_{1,2} = \mp I_n$ or Ω_1 and Ω_2 consists of elements such that $B = C = 0$ and $D = {}^T\!A^{-1}$. Using such an element, it is possible to bring Ω_3 to the form

$$\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}$$

or Z_3 to the form

$$\begin{pmatrix} \sqrt{-1} I_r & 0 \\ 0 & -\sqrt{-1} I_{n-r} \end{pmatrix},$$

where $0 \le r \le n$. By a simple calculation, both sides of (3.6) can be seen to be equal to $n - 2r$.

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It is possible to express the triple Maslov index $\alpha_\omega(L_1, L_2, L_3)$ as a generalised integral over a fixed surface. Let $\varDelta$ be a surface with boundary in $\mathcal{J}_\omega$. Suppose that $L_0 \in \mathcal{L}_\omega$ is in the boundary of $\varDelta$ (when $\mathcal{J}_\omega$ is regarded as a bounded domain). We say that $\varDelta$ is admissible at L_0 if:

(a) there are two geodesics γ_1 and γ_2 in $\mathcal{J}_\omega$ which are parametrised by t that is affine with respect to arc length on each geodesic. In addition, we must have

$$\lim_{t\to+\infty} \gamma_1(t) = \lim_{t\to+\infty} \gamma_2(t) = L_0;$$

and

(b) there exists $T > 0$ such that the boundary of $\varDelta$ in $\mathcal{J}_\omega$ contains $\gamma_1(t)$ and $\gamma_2(t)$ for all $t \geq T$ and such that $\varDelta$ contains $\varDelta_{L_0,T} = \bigcup_{t\geq T} \gamma_{L_0,t}$. Here $\gamma_{L_0,t}$ is the geodesic segment joining $\gamma_1(t)$ and $\gamma_2(t)$.

Let $L_0, \gamma_1(t)$ and $\gamma_2(t)$ be represented by symmetric matrices $Z_0, Z_1(t)$ and $Z_2(t)$, respectively. Then a straightforward calculation shows that

$$\begin{aligned}\lim_{t\to+\infty} [&\arg\det(I_n - \bar{Z}_0 Z_1(t)) + \arg\det(I_n - \overline{Z_1(t)} Z_2(t)) \\ &+ \arg\det(I_n - \overline{Z_2(t)} Z_0)] = 0.\end{aligned} \tag{3.7}$$

Roughly, this means that as $t \to +\infty$, the 'symplectic area' of $\varDelta_{L_0,t}$ (despite the lack of its definition as $\varDelta_{L_0,t}$ is unbounded) goes to 0. Now let $L_1, L_2, L_3 \in \mathcal{L}_\omega$ be three mutually transverse Lagrangian subspaces joined by geodesics $\gamma_{L_2L_1}, \gamma_{L_3L_2}$ and $\gamma_{L_1L_3}$ in $\mathcal{J}_\omega$. Suppose that $\varDelta$ is a surface in $\mathcal{J}_\omega$ bounded by these three geodesics, which make $\varDelta$ admissible at L_1, L_2 and L_3. For sufficiently large t, let

$$\varDelta(t) = \varDelta \setminus (\varDelta_{L_1,t} \cup \varDelta_{L_2,t} \cup \varDelta_{L_3,t}).$$

Then $\varDelta(t)$ is a bounded region in $\varDelta$ whose boundary is the union of six geodesic segments. Using (3.5), (3.6) and (3.7), we obtain

$$\alpha_\omega(L_1, L_2, L_3) = \frac{2}{\pi} \lim_{t\to+\infty} \int_{\varDelta(t)} \sigma_\omega.$$

In this way, the integration on the right-hand side is over a region $\varDelta(t)$ that expands (as $t \to +\infty$) in a fixed surface $\varDelta$. Alternatively, we can integrate σ_ω over the part of $\varDelta$ that is outside three horospheres that recede to L_1, L_2 and L_3, respectively, in the limit.

3.2. Fermionic systems: orthogonal analogue of Maslov index. For a fermionic system whose phase space is the Euclidean space (V, g) of dimension $2n$, the space $\mathcal{J}_g$ of polarisations is compact. The bundle of Hilbert spaces $\mathcal{H} \to \mathcal{J}_g$ admits a natural projectively flat connection.

Suppose that $J_1, J_2 \in \mathcal{J}_g$ and J_2 is not in the cut locus of J_1 (and *vice versa*). Let $\gamma_{J_2J_1}$ be the unique length-minimising geodesic from J_1 to J_2. Then the parallel transport $\mathcal{U}_{J_2J_1}$ along $\gamma_{J_2J_1}$ is the unitary intertwining operator between the equivalent representations of the Clifford algebra on $\mathcal{H}_{J_1}$ and $\mathcal{H}_{J_2}$ (see [23]). Suppose that $J_1, J_2, J_3 \in \mathcal{J}_g$ are such that J_i is not in the cut locus of J_j whenever $i \neq j$. Then, as in the bosonic case (3.2), we have the identity

$$\mathcal{U}_{J_1J_3} \circ \mathcal{U}_{J_3J_2} \circ \mathcal{U}_{J_2J_1} = \chi_g(J_1, J_2, J_3)\mathrm{id}_{\mathcal{H}_{J_1}}, \tag{3.8}$$

where $\chi_g(J_1, J_2, J_3) \in U(1)$. This follows from the irreducibility of $\mathcal{H}_{J_1}$ as a representation of the Clifford algebra. The phase χ_g itself was constructed in [12] as a 2-cocycle on the SO(V, g)-space $\mathcal{J}_g$ with values in $U(1)$. The phase satisfies the properties:

(a) $\chi_g(gJ_1g^{-1}, gJ_2g^{-1}, gJ_3g^{-1}) = \chi_g(J_1, J_2, J_3)$ for all $g \in \mathrm{SO}(V, g)$;
(b) $\chi_g(J_1, J_2, J_3) = \chi_g(J_2, J_3, J_1) = \chi_g(J_2, J_1, J_3)^{-1}$;
(c) $\chi_g(J_1, J_2, J_3)\chi_g(J_2, J_4, J_3)\chi_g(J_3, J_4, J_1)\chi_g(J_4, J_2, J_1) = 1$ for any $J_1, J_2, J_3, J_4 \in \mathcal{J}_g$ such that for $i \neq j$, J_i is not in the cut locus of J_j.

This is the orthogonal or spinorial counterpart of the generalised Maslov index in [11].

We observe that the phase $\chi_g(J_1, J_2, J_3)$ is the holonomy of the projectively flat connection on $\mathcal{H}$ along the three geodesics $\gamma_{J_2J_1}$, $\gamma_{J_3J_2}$ and $\gamma_{J_1J_3}$. Here and below we can replace $\mathcal{H}$ by the bundle $\tilde{\mathcal{H}} \to \mathcal{J}_g$ of half-density quantisation without changing the holonomy.

Since $\mathcal{J}_g$ is simply connected, there is an oriented surface Δ whose boundary consists of the three geodesics. Using the curvature (2.3), we get

$$\chi_g(J_1, J_2, J_3) = \exp\Big[\frac{\sqrt{-1}}{2}\int_\Delta \sigma_g\Big]. \tag{3.9}$$

We note that the exponent on the right-hand side is not well defined, as the integration over Δ changes when Δ is replaced by another surface with the same boundary but not homotopic to Δ. However, the exponential does not depend on the choice of Δ. This is consistent with the fact that σ_g is a closed 2-form whose periods are in $4\pi\mathbb{Z}$ (see [23]).

We can choose $J_0 \in \mathcal{J}_g$ such that J_1, J_2 and J_3 are in the complement of its cut locus. Then the phase $\chi_g(J_1, J_2, J_3)$ can be lifted to be real valued, that is,

$$\chi_g(J_1, J_2, J_3) = \exp\Big[\frac{\sqrt{-1}\pi}{4}\alpha_g(J_1, J_2, J_3)\Big],$$

where $\alpha_g(J_1, J_2, J_3) \in \mathbb{R}$ depends smoothly on J_1, J_2 and J_3. In addition, $\alpha_g(J_1, J_2, J_3) = 0$ whenever any two of J_1, J_2 and J_3 coincide. This is because the complement of a cut locus is contractible. We note, however, that α_g does depend on the choice of J_0. If we further choose Δ as a surface that lies entirely in the complement and whose boundary consists of the three geodesics joining J_1, J_2 and J_3, then formula (3.9) for $\chi_g(J_1, J_2, J_3)$ can be lifted to

$$\alpha_g(J_1, J_2, J_3) = \frac{2}{\pi}\int_\Delta \sigma_g, \tag{3.10}$$

an identity in $\mathbb{R}$ that is formally similar to (3.4).

When $J_1, J_2, J_3, J_4 \in \mathcal{J}_g$ are not in the cut locus of each other and we suppose that they are not in the cut locus of J_0 either, we have:

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(a) $\alpha_g(gJ_1g^{-1}, gJ_2g^{-1}, gJ_3g^{-1}) = \alpha_g(J_1, J_2, J_3)$ for all $g \in \mathrm{SO}(V, g)$;
(b) $\alpha_g(J_1, J_2, J_3) = \alpha_g(J_2, J_3, J_1) = -\alpha_g(J_2, J_1, J_3)$;
(c) $\alpha_g(J_1, J_2, J_3) + \alpha_g(J_2, J_4, J_3) + \alpha_g(J_3, J_4, J_1) + \alpha_g(J_4, J_2, J_1) = 0$.

In (a), $\alpha_g(gJ_1g^{-1}, gJ_2g^{-1}, gJ_3g^{-1})$ is defined because gJ_1g^{-1}, gJ_2g^{-1} and gJ_3g^{-1} are not in the cut locus of gJ_0g^{-1}.

Using the parametrisation of J_1, J_2 and J_3 by skew-symmetric matrices Z_1, Z_2 and Z_3, respectively, we can write the Kähler form on the complement of the cut locus as $\sigma_g = d\phi_g$, where

$$\phi_g = \sqrt{-1}(\partial - \bar{\partial}) \log \det(I_n - \bar{Z}Z).$$

So, $\log \det(I_n - \bar{Z}Z)$ is essentially the Kähler potential on the complement of a cut locus. As in the bosonic case, we get the formula (see also [12])

$$\alpha_g(L_1, L_2, L_3) = \frac{2}{\pi}[\arg \det(I_n - \bar{Z}_1Z_2) + \arg \det(I_n - \bar{Z}_2Z_3) + \arg \det(I_n - \bar{Z}_3Z_1)].$$

4. Comparison with Berry's phase

When the Hamiltonian of a quantum mechanical system undergoes an adiabatic change, an energy eigenstate of the initial Hamiltonian evolves to that of the new Hamiltonian, multiplied by a phase. If the change is in a cycle, the phase contains a dynamical part and Berry's geometric phase (see [2]). The latter was related to the holonomy of bundles over the space of parameters (see [16]). Berry's phase was generalised to the nonabelian setting when the energy levels are possibly degenerate (see [17]).

In [20], it was shown that nonabelian Berry's phase is related to universal connection (see [13]) in the following way. Let B be the space of parameters of the system. Assume that an energy eigenvalue varies smoothly over B and its degeneracy r does not change. This defines a map from B to the Grassmannian of r-planes in the quantum Hilbert space. Over the latter, there is a universal connection in the tautological vector bundle (see [13]) and Berry's phase is the holonomy of its pull-back along a cyclic path in B (see [20]).

The universal connection is defined by orthogonal projection of the trivial connection (see [13]). This is mathematically similar to the construction of the projectively flat connection in the bundle of Hilbert spaces over the space of polarisations (see Section 2). There is, however, an important conceptual difference. Although polarisations provide many ways to construct the quantum Hilbert space, they are not physical parameters of the system. On the contrary, our purpose was to show that physics is independent of the choice of polarisations by identifying the Hilbert spaces through parallel transport. However, Berry's phase is physical and can be measured experimentally. It occurs when the physical parameters of the system change, regardless of the polarisation chosen. It is the latter that concerns us now, even though the physical systems considered below are parametrised exactly by $\mathcal{J}_\omega$ and $\mathcal{J}_g$.

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We start with a linear bosonic system whose phase space is a symplectic vector space (V, ω) of dimension $2n$. We fix the quantum Hilbert space $\mathcal{H}_{\mathrm{b}}$. Consider the Hamiltonian of a harmonic oscillator

$$H_J = \tfrac{1}{2}\omega(\cdot, J\cdot) \in \mathrm{Sym}^2(V^*),$$

where $J \in \mathcal{J}_\omega$ is now a physical parameter. Its quantisation is a positive-definite selfadjoint operator $\hat{H}_J$ acting on $\mathcal{H}_{\mathrm{b}}$, which has a Fock space decomposition as follows. Recall that the creation and annihilation operators are the quantised linear functionals on $V_J^{1,0}$ and $V_J^{0,1}$, respectively.

The quantisation $\hat{f}$ of any polynomial $f \in \mathrm{Sym}(V_J^{1,0})^*$ acts on $\mathcal{H}_{\mathrm{b}}$ (operator ordering is not a problem here as the creation operators commute with each other). Let $|0\rangle_J$ be the vacuum state of $\hat{H}_J$ and let

$$\mathcal{H}_J^{(k)} = \mathrm{Sym}^k(V_J^{1,0})^*|0\rangle_J = \{\hat{f}|0\rangle_J \mid f \in \mathrm{Sym}^k(V_J^{1,0})^*\}$$

for any $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Each $\mathcal{H}_J^{(k)}$ is an eigenspace of $\hat{H}_J$ with energy $k + n/2$. We have $\dim_{\mathbb{C}} \mathcal{H}_J^{(k)} = \binom{n+k-1}{k}$ and, as a Hilbert space,

$$\mathcal{H}_{\mathrm{b}} = \bigoplus_{k\in\mathbb{N}} \mathcal{H}_J^{(k)}.$$

As the parameter $J \in \mathcal{J}_\omega$ varies, the eigenspaces $\mathcal{H}_J^{(k)}$ (for a fixed $k \in \mathbb{N}$) of $\hat{H}_J$ form a vector bundle $\mathcal{H}^{(k)}$ over $\mathcal{J}_\omega$. Each bundle $\mathcal{H}^{(k)}$ has a natural connection by orthogonal projection from the product bundle $\mathcal{J}_\omega \times \mathcal{H}_{\mathrm{b}}$. This is also the pull-back of the universal connection from the Grassmannian. Therefore, under an adiabatic cyclic evolution in $\mathcal{J}_\omega$, Berry's phase in the kth eigenspace is the holonomy of the bundle $\mathcal{H}^{(k)}$. Note that $\mathcal{H}^{(0)}$ is a line bundle whose fibre over $J \in \mathcal{J}_\omega$ is spanned by $|0\rangle_J \in \mathcal{H}_{\mathrm{b}}$. Its curvature is $\mathsf{F}^{(0)} = \sqrt{-1}/2\sigma_\omega$. Note the opposite sign from (2.1). Therefore, Berry's phase of the vacuum state is inverse to the holonomy considered in Section 3.1. The bundle $\mathcal{H}^{(0)}$ with its connection is isomorphic to $\sqrt{\mathcal{K}}$. For a general k, the bundle $\mathcal{H}^{(k)}$ with its connection is isomorphic to $\mathrm{Sym}^k(\mathcal{V}^*) \otimes \sqrt{\mathcal{K}}$. The latter has a connection induced from $\mathcal{V}$, whose curvature is given by (2.2). In particular, if $n = 1$, then $\mathcal{H}^{(k)} \cong \mathcal{K}^{\otimes(k+1/2)}$ and therefore its curvature is (see also [3])

$$\mathsf{F}^{(k)} = (k + \tfrac{1}{2})\sqrt{-1}\sigma_\omega.$$

However, unless $k = 0$ or $n = 1$ when $\mathcal{H}^{(k)}$ is a line bundle, the connection on $\mathcal{H}^{(k)}$ is not projectively flat.

The relation of Berry's phase of the vacuum and the holonomy of $\mathcal{H} \to \mathcal{J}_\omega$ can be explained as follows. When the parameter J of the Hamiltonian H_J varies in $\mathcal{J}_\omega$, if the polarisation is chosen as J itself, then the wave function (without metaplectic correction) of the vacuum is $c_J^0 = \exp(-H_J/2)$, which is real for each J. To be consistent, Berry's phase must cancel the nontrivial holonomy of the bundle $\mathcal{H}$. If metaplectic correction is included, then changing polarisation as above (or in any

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other way) has no effect on the holonomy. However, the wave function is now c_J^0 times an element of $\sqrt{\mathcal{K}_J}$. Therefore, with or without metaplectic correction, Berry's phase of the vacuum is inverse to the holonomy of the bundle $\mathcal{H}$.

Berry's phase also appears in fermionic systems. Suppose that the phase space is a Euclidean space (V, g) of dimension $2n$. Let $\mathcal{H}_{\mathrm{f}}$ be the quantum Hilbert space. We consider a fermionic harmonic oscillator whose Hamiltonian is

$$H_J = \tfrac{1}{2} g(J\cdot, \cdot) \in \bigwedge^2 (V^*),$$

where $J \in \mathcal{J}_g$ is again a physical parameter. Its quantisation $\hat{H}_J$ is a selfadjoint operator on $\mathcal{H}_{\mathrm{f}}$. As in the bosonic case, the creation and annihilation operators are the quantised linear functionals on $V_J^{1,0}$ and $V_J^{0,1}$, respectively. For any $f \in \wedge^k (V_J^{1,0})^*$, the quantum operator $\hat{f}$ acts on $\mathcal{H}_{\mathrm{f}}$. Let $|0\rangle_J$ be the vacuum state and, for any $k \in \mathbb{Z}$ with $0 \le k \le n$, let

$$\mathcal{H}_J^{(k)} = \bigwedge^k (V_J^{1,0})^* |0\rangle_J = \left\{ \hat{f}|0\rangle_J \,\middle|\, f \in \bigwedge^k (V_J^{1,0})^* \right\}.$$

Then each $\mathcal{H}_J^{(k)}$ is an eigenspace of $\hat{H}_J$ with energy $k - n/2$. We have $\dim_{\mathbb{C}} \mathcal{H}_J^{(k)} = \binom{n}{k}$ and

$$\mathcal{H}_{\mathrm{f}} = \bigoplus_{k=0}^{n} \mathcal{H}_J^{(k)}.$$

The eigenspaces $\mathcal{H}_J^{(k)}$ (for a fixed k) of $\hat{H}_J$ form a vector bundle $\mathcal{H}^{(k)}$ over $\mathcal{J}_g$. The natural connection on $\mathcal{H}^{(k)}$ by orthogonal projection from the product bundle $\mathcal{J}_g \times \mathcal{H}_{\mathrm{f}}$ coincides with the pull-back of the universal connection from the Grassmannian. Therefore, under an adiabatic cyclic evolution in $\mathcal{J}_g$, Berry's phase in the kth eigenspace is the holonomy of the bundle $\mathcal{H}^{(k)}$. In addition, $\mathcal{H}^{(0)}$ is a line bundle whose fibre over $J \in \mathcal{J}_g$ is spanned by $|0\rangle_J \in \mathcal{H}_{\mathrm{f}}$. Its curvature is $\mathsf{F}^{(0)} = \sqrt{-1}/2\sigma_g$. Note again that the sign is opposite to (2.3). So, Berry's phase of the vacuum state is inverse to the holonomy in Section 3.2. This can be explained in a similar way to the bosonic case. The line bundle $\mathcal{H}^{(0)}$ is isomorphic to $\sqrt{\mathcal{K}^{-1}}$. For $0 \le k \le n$, the bundle $\mathcal{H}^{(k)}$ with its connection is isomorphic to $\wedge^k(\mathcal{V}^*) \otimes \sqrt{\mathcal{K}^{-1}}$, which has a connection induced from $\mathcal{V}$. In particular, $\mathcal{H}^{(n)} \cong \sqrt{\mathcal{K}}$, which is a line bundle that is dual to $\mathcal{H}^{(0)}$ and its curvature is $\mathsf{F}^{(n)} = -\sqrt{-1}/2\sigma_g$. In general cases (for $k \neq 0, n$), the connection on $\mathcal{H}^{(k)}$ is not projectively flat.

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