ON THE SOLUTION OF SOME AXISYMMETRIC BOUNDARY VALUE PROBLEMS BY MEANS OF INTEGRAL EQUATIONS. IV. THE ELECTROSTATIC POTENTIAL DUE TO A SPHERICAL CAP BETWEEN TWO INFINITE CONDUCTING PLANES

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1. Introduction

This paper is a sequel to previous papers (**1**, **2**, **3**) on the solution of axisymmetric potential problems for circular disks and spherical caps by means of integral equations and applies the methods developed in these papers to the electrostatic potential problem for a perfectly conducting thin spherical cap or circular disk between two infinite earthed conducting planes.

Representations of the potentials due to distributions of charge over a spherical cap and a circular disk as contour integrals have been given by the author (**1**) and Green and Zerna (**4**). By means of these results an expression for the potential due to a cap or disk between two earthed planes as a sum of contour integrals is found by the method of images, the condition that the potential be a given function on the cap or disk leading to a Fredholm integral equation of the second kind for an unknown function occurring in this potential. This equation can be solved by iteration when the planes are a large distance apart and the capacity of the cap or disk found. The solution for a cap is given in sections 2 to 5 and the solution for a disk stated in section 6.

In a previous paper (**3**) the problem of a disk between two planes is considered in the general case when the potential is given on the disk and the planes. The method of this paper, whilst not so general as that of (**3**), gives a simpler derivation of the governing Fredholm equation of the problem and is applicable to most cases of interest.

2. The Potential due to a Spherical Cap between Two Earthed Planes

We consider a thin spherical cap Σ of radius a and semi-angle α, placed between two infinite earthed parallel conducting planes at a distance $2f$ apart, the axis of symmetry of the cap being normal to the planes. For simplicity the centre 0 of Σ is taken to be equidistant from the planes, though the method of this paper can be applied whatever the position of 0. We suppose the cylindrical and spherical polar coordinates of a point referred to 0 as origin and the axis of symmetry $0z$ of Σ as polar axis to be (z, ϖ, ϕ) and (r, θ, ϕ), the cap being given by $r = a$ $(0 \leqq \theta \leqq \alpha)$ and the two planes by $z = \pm f$ $(f > a)$.

If $V_0(z, \varpi)$ is the potential due to an axisymmetric distribution of electric charge in infinite unbounded space, the singularities of the distribution lying in the region $-f < z < f$ though not on the surface $r = a$ $(0 \leqq \theta \leqq \alpha)$, we require the potential $V(z, \varpi)$, defined in the region $-f \leqq z \leqq f$, which has the same singularities as $V_0(z, \varpi)$ in this region and which is zero on the planes $z = \pm f$. On the cap we suppose it given as a known even function $f_0(\theta)$ symmetric about the axis $0z$.

We write $V(z, \varpi)$ as

$$V(z, \varpi) = V_0(z, \varpi) + V_1(z, \varpi) + U(r, \theta), \quad \text{.................}(2.1)$$

where $V_1(z, \varpi) + U(r, \theta)$ is the perturbation potential which must be added to $V_0(z, \varpi)$ in order that the conditions on $V(z, \varpi)$ be satisfied. This perturbation potential is written as the sum of two potentials, the first of which, $V_1(z, \varpi)$, is chosen to have no singularities in the region between the planes and to be such that on the planes $z = \pm f$

$$V_0(z, \varpi) + V_1(z, \varpi) = 0.$$

Such a function is readily constructed by the method of images as

$$V_1(z, \varpi) = \sum_{n=0}^{\infty} [-V_0(-z+(4n+2)f, \varpi) - V_0(-z-(4n+2)f, \varpi) \\ + V_0(z+(4n+4)f, \varpi) + V_0(z-(4n+4)f, \varpi)] \text{............}(2.2)$$

The remaining part of the perturbation potential, $U(r, \theta)$, is then zero on the planes $z = \pm f$ whilst on the cap $r = a$ $(0 \leqq \theta \leqq \alpha)$ it satisfies the condition

$$U(a, \theta) = f(\theta) \quad (0 \leqq \theta \leqq \alpha), \quad \text{..........................}(2.3)$$

where

$$f(\theta) = f_0(\theta) - V_0(a \cos \theta, a \sin \theta) - V_1(a \cos \theta, a \sin \theta)$$

and is an even function. In addition $U(r, \theta)$ is continuously differentiable everywhere in the region $-f \leqq z \leqq f$ except possibly on the edge of Σ and is $0(r^{-1})$ at a large distance r from Σ.

The potential of a distribution of charge on Σ can be represented as a contour integral in the form (1)

$$\frac{a}{2} \int_{-\alpha}^{\alpha} \frac{g(\eta) d\eta}{(r^2 e^{i\eta} + a^2 e^{-i\eta} - 2ar \cos \theta)^{\frac{1}{2}}}. \quad \text{.....................}(2.4)$$

If for this charge distribution we construct its image system in the planes $z = \pm f$ under the condition that these planes be at zero potential, the potential due to the distribution on Σ together with the image distribution is the potential $U(r, \theta)$ we require and can be expressed as

$$U(r, \theta) = \frac{a}{2} \sum_{n=-\infty}^{\infty} \left[\int_{-\alpha}^{\alpha} \frac{g(\eta) d\eta}{(r_{2n}^2 e^{i\eta} + a^2 e^{-i\eta} - 2ar_{2n} \cos \theta_{2n})^{\frac{1}{2}}} \right. \\ \left. - \int_{-\alpha}^{\alpha} \frac{g(\eta) d\eta}{(r_{2n-1}^2 e^{i\eta} + a^2 e^{-i\eta} + 2ar_{2n-1} \cos \theta_{2n-1})^{\frac{1}{2}}} \right], \text{...........}(2.5)$$

where (r_n, θ_n, ϕ), $(n = 0, \pm 1, \pm 2, \ldots)$, are the spherical polar coordinates of the point (r, θ, ϕ) referred to $0z$ as polar axis and the point 0_n, whose cylindrical polar coordinates referred to 0 are $(2nf, 0, 0)$, as origin. We thus have that

$$r_n^2 = r^2+4n^2f^2-4nfr\cos\theta, \quad r_n\cos\theta_n = r\cos\theta-2nf,$$
$$r_n\sin\theta_n = r\sin\theta, \quad (n=0, \pm 1, \pm 2, \ldots), \qquad (2.6)$$

where we understand (r_0, θ_0, ϕ) as (r, θ, ϕ). The first integral in (2.5) represents a charge distribution on the surface $r_{2n} = a$ $(0 \leqq \theta_{2n} \leqq \alpha)$ and the second a distribution on the surface $r_{2n-1} = a$ $(\pi-\alpha \leqq \theta_{2n-1} \leqq \pi)$. The function $g(\eta)$ is taken to be real, continuous and even. Further we have that

$$(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}} = \rho e^{\frac{1}{2}i\tau} \text{ for } r \geqq a,$$
$$= \rho e^{-\frac{1}{2}i\tau} \text{ for } r<a, \qquad (2.7)$$

with
$$\rho \geqq 0, \quad 0 \leqq \tau \leqq \pi \quad \text{for} \quad 0 \leqq \eta \leqq \alpha,$$
$$-\pi<\tau \leqq 0 \quad \text{for} \quad -\alpha \leqq \eta < 0.$$

When $r = a$, we have that

$$(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}} = a(2\cos\eta-2\cos\theta)^{\frac{1}{2}} \quad (\theta \geqq |\eta|),$$

while

$$(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}} = \pm ia(2\cos\theta-2\cos\eta)^{\frac{1}{2}} \quad (\theta<|\eta|),$$

if $r \to a$ through values greater than a,

$$= \mp ia(2\cos\theta-2\cos\eta)^{\frac{1}{2}} \quad (\theta<|\eta|),$$

if $r \to a$ through values less than a, $\qquad (2.8)$

the upper sign holding for $0 \leqq \eta \leqq \alpha$ and the lower for $-\alpha \leqq \eta < 0$. The integral in (2.5) involving this square root is to be interpreted as a Cauchy integral at the point $r = a$ $(\theta = 0)$. Similarly we have that

$$(r_{2n}^2e^{i\eta}+a^2e^{-i\eta}-2ar_{2n}\cos\theta_{2n})^{\frac{1}{2}} = \rho_{2n}e^{\frac{1}{2}i\tau_{2n}} \text{ for } r_{2n}>a,$$
$$(n = \pm 1, \pm 2, \ldots), \qquad (2.9)$$

with
$$\rho_{2n} \geqq 0, \quad 0 \leqq \tau_{2n} \leqq \pi \quad \text{for} \quad 0 \leqq \eta \leqq \alpha,$$
$$-\pi<\tau_{2n} \leqq 0 \quad \text{for} \quad -\alpha \leqq \eta < 0,$$

similar expressions holding for the remaining square roots in (2.5).

The potential $U(r, \theta)$ defined by (2.5) is $0(r^{-1})$ for large r and is real since $g(\eta)$ is an even function. Further it follows from the proofs given for the electrostatic potential problem for a single cap (1) that $U(r, \theta)$ is continuously differentiable everywhere except on the edge $r = a$ $(\theta = \alpha)$ of Σ. We next show that the condition (2.3) satisfied by $U(r, \theta)$ on Σ leads to a Fredholm integral equation determining $g(\eta)$.

3. The Integral Equation for $g(\eta)$

In order to obtain the equation determining $g(\eta)$ we find the value of $U(r, \theta)$ as (r, θ, ϕ) tends to a point (a, θ, ϕ) on the cap Σ. We first consider the contribution to $U(a, \theta)$ of the term in (2.5) representing a charge distribution on Σ, this term being given by (2.4). It follows from a proof given in the potential problem for a single cap (1) that the integral (2.4) is continuous for normal approach to Σ and hence from (2.7) and (2.8) its value on Σ is found as

$$\int_0^\theta \frac{g(\eta)d\eta}{(2\cos\eta-2\cos\theta)^{\frac{1}{2}}}\ (0\leqq\theta\leqq\alpha).$$

Making the substitutions $t = \tan\frac{1}{2}\theta$, $x = \tan\frac{1}{2}\eta$, and writing

$$g(\eta)\cos\tfrac{1}{2}\eta\equiv G(\tan\tfrac{1}{2}\eta), \qquad \text{.......(3.1)}$$

we obtain (2.4) as

$$(1+t^2)^{\frac{1}{2}}\int_0^t \frac{G(x)dx}{(t^2-x^2)^{\frac{1}{2}}}. \qquad \text{.......(3.2)}$$

The limits of the remaining integrals in (2.5) as the point (r, θ, ϕ) tends to the point (a, θ, ϕ) of Σ are simply their values at (a, θ, ϕ). If we consider the integral

$$I_{2n}=\frac{a}{2}\int_{-\alpha}^{\alpha}\frac{g(\eta)d\eta}{(r_{2n}^2e^{i\eta}+a^2e^{-i\eta}-2ar_{2n}\cos\theta_{2n})^{\frac{1}{2}}},$$

for $n = 1, 2, \ldots$, and note from (2.6) that on $r = a$ we have

$$r_{2n}^2=a^2+16n^2f^2-8naf\cos\theta,\quad r_{2n}\cos\theta_{2n}=a\cos\theta-4nf,$$

the value of I_{2n} on the cap is

$$I_{2n}=\frac{a}{2}\int_{-\alpha}^{\alpha}\frac{g(\eta)d\eta}{[(a^2+16n^2f^2-8naf\cos\theta)e^{i\eta}+a^2e^{-i\eta}-2a(a\cos\theta-4nf)]^{\frac{1}{2}}}.$$

Making the substitutions $t = \tan\frac{1}{2}\theta$, $y = \tan\frac{1}{2}\eta$, $t_\alpha = \tan\frac{1}{2}\alpha$, and using (3.1), we obtain after some manipulation of the denominator of the integrand

$$I_{2n}=\frac{a}{2}(1+t^2)^{\frac{1}{2}}\int_{-t_\alpha}^{t_\alpha}\frac{G(y)dy}{[(2nf+iy(2nf-a))^2-t^2(2nfy-i(a+2nf))^2]^{\frac{1}{2}}}.$$

We now want to express I_{2n} as an integral of the form (3.2). This we do by representing its integrand other than $G(y)$ as a Jacobi integral (5), the square root being interpreted in accordance with (2.9), so that

$$\begin{aligned}I_{2n}&=\frac{a(1+t^2)^{\frac{1}{2}}}{2\pi}\int_{-t_\alpha}^{t_\alpha}G(y)\int_0^\pi\frac{d\psi dy}{2nf+iy(2nf-a)+t\cos\psi(2nfy-i(a+2nf)}\\&=\frac{a(1+t^2)^{\frac{1}{2}}}{2\pi}\int_{-t_\alpha}^{t_\alpha}G(y)\int_0^\pi\frac{2nf(1+yt\cos\psi)d\psi dy}{4n^2f^2(1+yt\cos\psi)^2+(y(2nf-a)-t\cos\psi(2nf+a))^2}\\&=\frac{a(1+t^2)^{\frac{1}{2}}}{\pi}\int_0^t\frac{dx}{(t^2-x^2)^{\frac{1}{2}}}\int_{-t_\alpha}^{t_\alpha}\frac{G(y)2nf(1+xy)dy}{4n^2f^2(1+xy)^2+(x(2nf+a)-y(2nf-a))^2},\end{aligned}$$

.........(3.3)

since $G(y)$ is an even function of y, $g(\eta)$ being an even function of η.

In the same way we can show that, when $n = -1, -2, \ldots,$

$$I_{2n} = -\frac{a(1+t^2)^{\frac{1}{2}}}{\pi}\int_0^t \frac{dx}{(t^2-x^2)^{\frac{1}{2}}}\int_{-t_\alpha}^{t_\alpha} \frac{G(y)2nf(1+xy)dy}{4n^2f^2(1+xy)^2+(x(2nf+a)-y(2nf-a))^2}, \quad \ldots\ldots\ldots(3.4)$$

and also obtain similar expressions for the remaining integrals in (2.5).

The boundary condition (2.3) on Σ can be written as

$$U = F(t)(1+t^2)^{\frac{1}{2}}, \quad r = a(0 \leqq t \leqq t_\alpha),$$

where

$$F(\tan \tfrac{1}{2}\theta) = f(\theta)\cos \tfrac{1}{2}\theta, \quad \ldots\ldots\ldots(3.5)$$

so, on combining the limits of the integrals in (2.5) given by (3.2), (3.3) and (3.4), we obtain

$$\int_0^t \frac{H(x)dx}{(t^2-x^2)^{\frac{1}{2}}} = F(t) \quad (0 \leqq t \leqq t_\alpha), \quad \ldots\ldots\ldots(3.6)$$

where

$$G(x) + \frac{1}{\pi}\int_{-t_\alpha}^{t_\alpha} K(x, y)G(y)dy = H(x), \quad (-t_\alpha \leqq x \leqq t_\alpha). \quad \ldots\ldots\ldots(3.7)$$

In this last equation the kernel $K(x, y)$ is given by

$$\begin{aligned} K(x, y) = a\sum_{n=1}^{\infty}\Bigg[& \frac{2nf(1+xy)}{4n^2f^2(1+xy)^2+(x(2nf+a)-y(2nf-a))^2} \\ & + \frac{2nf(1+xy)}{4n^2f^2(1+xy)^2+(x(2nf-a)-y(2nf+a))^2} \\ & - \frac{(2n-1)f+a+((2n-1)f-a)xy}{((2n-1)f+a+((2n-1)f-a)xy)^2+(2n-1)^2f^2(x-y)^2} \\ & - \frac{(2n-1)f-a+((2n-1)f+a)xy}{((2n-1)f-a+((2n-1)f+a)xy)^2+(2n-1)^2f^2(x-y)^2}\Bigg]. \end{aligned}$$

Equation (3.6) is a Volterra integral equation of the first kind determining $H(x)$. It is in fact a form of Abel's equation, its solution being given by Green and Zerna (**4**) as

$$H(x) = \frac{2}{\pi}\frac{d}{dx}\int_0^x \frac{tF(t)dt}{(x^2-t^2)^{\frac{1}{2}}} \quad (0 \leqq x \leqq t_\alpha). \quad \ldots\ldots\ldots(3.8)$$

It now follows that, since $f(\theta)$ and hence $F(t)$ are even functions, $H(x)$ is an even function, so that in (3.7) $H(x) = H(-x)$ for $-t_\alpha \leqq x < 0$. Equation (3.7) is then a Fredholm integral equation of the second kind determining $G(x)$ when $H(x)$ is known. If we substitute

$$x = \tan \tfrac{1}{2}\xi, \quad y = \tan \tfrac{1}{2}\eta,$$

write

$$H(\tan \tfrac{1}{2}\xi) = h(\xi)\cos \tfrac{1}{2}\xi, \quad \ldots\ldots\ldots(3.9)$$

and use (3.1), the Fredholm equation determining $g(\xi)$ is obtained as

$$g(\xi)+\frac{1}{\pi}\int_{-\alpha}^{\alpha} k(\xi,\eta)g(\eta)d\eta = h(\xi) \quad (-\alpha\leqq\xi\leqq\alpha), \qquad \text{.........(3.10)}$$

where

$$\begin{aligned} k(\xi,\eta) = \frac{a}{2}\sum_{n=1}^{\infty}\Bigg[&\frac{2nf\cos\frac{1}{2}(\xi-\eta)}{4n^2f^2+4naf\sin\frac{1}{2}(\xi-\eta)\sin\frac{1}{2}(\xi+\eta)+a^2\sin^2\frac{1}{2}(\xi+\eta)} \\ &+\frac{2nf\cos\frac{1}{2}(\xi-\eta)}{4n^2f^2-4naf\sin\frac{1}{2}(\xi-\eta)\sin\frac{1}{2}(\xi+\eta)+a^2\sin^2\frac{1}{2}(\xi+\eta)} \\ &-\frac{(2n-1)f\cos\frac{1}{2}(\xi-\eta)+a\cos\frac{1}{2}(\xi+\eta)}{(2n-1)^2f^2+2(2n-1)af\cos\frac{1}{2}(\xi-\eta)\cos\frac{1}{2}(\xi+\eta)+a^2\cos^2\frac{1}{2}(\xi+\eta)} \\ &-\frac{(2n-1)f\cos\frac{1}{2}(\xi-\eta)-a\cos\frac{1}{2}(\xi+\eta)}{(2n-1)^2f^2-2(2n-1)af\cos\frac{1}{2}(\xi-\eta)\cos\frac{1}{2}(\xi+\eta)+a^2\cos^2\frac{1}{2}(\xi+\eta)}\Bigg]. \end{aligned}$$

.........(3.11)

From equation (3.10) it follows that $g(\xi)$ is even, real and continuous if $h(\xi)$ is continuous. This last condition is satisfied provided $f(\theta)$ is continuously differentiable for $0\leqq\theta\leqq\alpha$. Further, by a proof similar to that given in (3) for the potential problem for two spherical caps (3.10) can be shown to determine a unique function $g(\xi)$.

4. Alternative Expressions for $k(\xi, \eta)$

We first give an integral representation for the kernel $k(\xi, \eta)$ of equation (3.10) and then express $k(\xi, \eta)$ as an infinite series in powers of a/f, this latter expansion being needed in the iterative solution of (3.10) for the case when the distance between the planes is large compared with the radius of the cap.

To obtain an integral representation of $k(\xi, \eta)$ we use results given in (6) to express each term of the series in (3.11) as an infinite integral and find that

$$\begin{aligned} k(\xi,\eta) = a\sum_{n=1}^{\infty}\Bigg[&\int_0^{\infty}\exp[-2nfu\cos\tfrac{1}{2}(\xi-\eta)]\cos[2nfu\sin\tfrac{1}{2}(\xi-\eta)] \\ &\qquad\times\cos[au\sin\tfrac{1}{2}(\xi+\eta)]du \\ &-\int_0^{\infty}\exp[-(2n-1)fu\sin\tfrac{1}{2}|\xi-\eta|]\sin[(2n-1)fu\cos\tfrac{1}{2}(\xi-\eta)] \\ &\qquad\times\cos[au\cos\tfrac{1}{2}(\xi+\eta)]du \\ = a\int_0^{\infty}\Bigg[&\frac{\cos(au\sin\frac{1}{2}(\xi+\eta))[\cos(2fu\sin\frac{1}{2}(\xi-\eta))-\exp(-2fu\cos\frac{1}{2}(\xi-\eta))]}{2(\cosh(2fu\cos\frac{1}{2}(\xi-\eta))-\cos(2fu\sin\frac{1}{2}(\xi-\eta)))} \\ &-\frac{\cos(au\cos\frac{1}{2}(\xi+\eta))\sin(fu\cos\frac{1}{2}(\xi-\eta))\cosh(fu\sin\frac{1}{2}(\xi-\eta))}{(\cosh(2fu\sin\frac{1}{2}(\xi-\eta))-\cos(2fu\cos\frac{1}{2}(\xi-\eta)))}\Bigg]du. \end{aligned}$$

...(4.1)

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We next express $k(\xi, \eta)$ as an infinite series in powers of a/f by means of results given by Whittaker and Watson (7). If we write

$$S_1 = \frac{a}{2}\sum_{n=1}^{\infty}\left[\frac{2nf\cos\frac{1}{2}(\xi-\eta)}{4n^2f^2+4naf\sin\frac{1}{2}(\xi-\eta)\sin\frac{1}{2}(\xi+\eta)+a^2\sin^2\frac{1}{2}(\xi+\eta)} - \frac{\cos\frac{1}{2}(\xi-\eta)}{2nf}\right]$$

and express each term of this series as a power series in $a/2nf$, we find that

$$S_1 = \tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^r\left(\frac{a}{2nf}\right)^{r+1}\sin^r\tfrac{1}{2}(\xi+\eta)\sin\left[\tfrac{1}{2}(r+1)(\pi-\xi+\eta)\right]$$

$$= \tfrac{1}{2}\sum_{r=1}^{\infty}(-1)^r\left(\frac{a}{2f}\right)^{r+1}\sin^r\tfrac{1}{2}(\xi+\eta)\sin\left[\tfrac{1}{2}(r+1)(\pi-\xi+\eta)\right]\zeta(r+1),$$

where $\zeta(r+1)$ is Riemann's zeta-function (**8***a*), defined by

$$\zeta(r+1) = \sum_{n=1}^{\infty}\frac{1}{n^{r+1}}, \quad r>0.$$

Similarly we find that

$$S_2 = \frac{a}{2}\sum_{n=1}^{\infty}\left[\frac{2nf\cos\frac{1}{2}(\xi-\eta)}{4n^2f^2-4naf\sin\frac{1}{2}(\xi-\eta)\sin\frac{1}{2}(\xi+\eta)+a^2\sin^2\frac{1}{2}(\xi+\eta)} - \frac{\cos\frac{1}{2}(\xi-\eta)}{2nf}\right]$$

$$= \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{a}{2f}\right)^{r+1}\sin^r\tfrac{1}{2}(\xi+\eta)\sin\left[\tfrac{1}{2}(r+1)(\pi-\xi+\eta)\right]\zeta(r+1),$$

so that

$$S_1+S_2 = \sum_{r=1}^{\infty}(-1)^r\left(\frac{a}{2f}\right)^{2r+1}\sin^{2r}\tfrac{1}{2}(\xi+\eta)\cos\left[(r+\tfrac{1}{2})(\xi-\eta)\right]\zeta(2r+1). \quad \ldots(4.2)$$

Further, if we write

$$S_3 = \frac{a}{2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)f\cos\frac{1}{2}(\xi-\eta)+a\cos\frac{1}{2}(\xi+\eta)}{(2n-1)^2f^2+2(2n-1)af\cos\frac{1}{2}(\xi-\eta)\cos\frac{1}{2}(\xi+\eta)+a^2\cos^2\frac{1}{2}(\xi+\eta)} - \frac{\cos\frac{1}{2}(\xi-\eta)}{(2n-1)f}\right],$$

we have that

$$S_3 = \tfrac{1}{2}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(-1)^r\left(\frac{a}{(2n-1)f}\right)^{r+1}\cos^r\tfrac{1}{2}(\xi+\eta)\cos\left[\tfrac{1}{2}(r+1)(\xi-\eta)\right]$$

$$= \tfrac{1}{2}\sum_{r=1}^{\infty}(-1)^r\left(\frac{a}{f}\right)^{r+1}\cos^r\tfrac{1}{2}(\xi+\eta)\cos\left[\tfrac{1}{2}(r+1)(\xi-\eta)\right](1-2^{-r-1})\zeta(r+1),$$

since

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{r+1}} = (1-2^{-r-1})\zeta(r+1), \quad r>0.$$

Finally we find that

$$S_4 = \frac{a}{2}\sum_{n=1}^{\infty}\left[\frac{(2n-1)f\cos\frac{1}{2}(\xi-\eta)-a\cos\frac{1}{2}(\xi+\eta)}{(2n-1)^2f^2-2(2n-1)af\cos\frac{1}{2}(\xi-\eta)\cos\frac{1}{2}(\xi+\eta)+a^2\cos^2\frac{1}{2}(\xi+\eta)} - \frac{\cos\frac{1}{2}(\xi-\eta)}{(2n-1)f}\right]$$

$$= \frac{1}{2}\sum_{r=1}^{\infty}\left(\frac{a}{f}\right)^{r+1}\cos^r\tfrac{1}{2}(\xi+\eta)\cos\left[\tfrac{1}{2}(r+1)(\xi-\eta)\right](1-2^{-r-1})\zeta(r+1),$$

so that

$$S_3+S_4 = \sum_{r=1}^{\infty}\left(\frac{a}{2f}\right)^{2r+1}\cos^{2r}\tfrac{1}{2}(\xi+\eta)\cos\left[(r+\tfrac{1}{2})(\xi-\eta)\right](2^{2r+1}-1)\zeta(2r+1). \qquad \text{.........(4.3)}$$

From (3.11) we have that

$$k(\xi,\eta) = S_1+S_2-S_3-S_4-\frac{a}{f}\cos\tfrac{1}{2}(\xi-\eta)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n},$$

and, since (8*b*)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \log 2,$$

we find on using (4.2) and (4.3) that

$$\begin{aligned} k(\xi,\eta) = &-\frac{a}{f}(\log 2)\cos\tfrac{1}{2}(\xi-\eta) \\ &+\sum_{r=1}^{\infty}\left(\frac{a}{2f}\right)^{2r+1}\cos\left[(r+\tfrac{1}{2})(\xi-\eta)\right]\zeta(2r+1) \\ &\times\left[(-1)^r\sin^{2r}\tfrac{1}{2}(\xi+\eta)-(2^{2r+1}-1)\cos^{2r}\tfrac{1}{2}(\xi+\eta)\right] \qquad \text{.................. (4.4)} \\ = &-\frac{a}{f}(\log 2)\cos\left[\tfrac{1}{2}(\xi-\eta)\right] \\ &-\frac{1}{8}\left(\frac{a}{f}\right)^3\zeta(3)\cos\left[\tfrac{3}{2}(\xi-\eta)\right](4+3\cos(\xi+\eta))+0\left(\frac{a^5}{f^5}\right). \qquad \text{...............(4.5)} \end{aligned}$$

5. The Cap Maintained at a Constant Potential

When the cap is maintained at a constant potential U_0 and

$$V_0(z,\varpi) = V_1(z,\varpi) = 0,$$

equation (2.3) gives

$$U(a,\theta) = f_0(\theta) = U_0 \quad (0 \leqq \theta \leqq \alpha).$$

From equations (3.5), (3.8) and (3.9) we then find that

$$h(\xi) = \frac{2U_0}{\pi}\cos\tfrac{1}{2}\xi,$$

so that in this case the integral equation to be solved for $g(\xi)$ is

$$g(\xi)+\frac{1}{\pi}\int_{-\alpha}^{\alpha}k(\xi,\eta)g(\eta)d\eta = \frac{2U_0}{\pi}\cos\tfrac{1}{2}\xi \quad (-\alpha \leqq \xi \leqq \alpha). \qquad \text{...........(5.1)}$$

The capacity C of the cap is found as

$$U_0 C = a \int_0^\alpha g(\eta) \cos \tfrac{1}{2}\eta \, d\eta. \qquad \text{......(5.2)}$$

In general it is probably necessary to solve equation (5.1) numerically using the expression (4.1) for $k(\xi, \eta)$. However, when the distance between the planes is large compared with the radius of the cap, we can solve (5.1) by iteration to obtain an approximate solution for $g(\xi)$. The appropriate expression to use for $k(\xi, \eta)$ is then (4.4), which, if we neglect terms in $(a/f)^5$ and higher powers, reduces to (4.5). Writing $\sigma = a/f$, we thus find that

$$\begin{aligned}
g(\xi) = {} & \frac{2U_0}{\pi} \cos \tfrac{1}{2}\xi \Bigg[1 + \frac{\sigma}{\pi}(\alpha + \sin\alpha)\log 2 + \frac{\sigma^2}{\pi^2}(\alpha+\sin\alpha)^2(\log 2)^2 \\
& + \frac{\sigma^3}{\pi^3}(\alpha+\sin\alpha)^3(\log 2)^3 + \frac{\sigma^3\zeta(3)}{32\pi}(3\sin 2\alpha + 2\sin 3\alpha) + \frac{\sigma^4}{\pi^4}(\alpha+\sin\alpha)^4(\log 2)^4 \\
& + \frac{\sigma^4\zeta(3)\log 2}{32\pi^2}\bigg(9\alpha\sin 2\alpha + 6\alpha\sin 3\alpha + 10 + \frac{25}{2}\cos\alpha \\
& \qquad - 5\cos 2\alpha - \frac{25}{2}\cos 3\alpha - 5\cos 4\alpha\bigg)\Bigg] \\
& + \frac{U_0\zeta(3)}{2\pi^2}\cos\tfrac{3}{2}\xi \Bigg[\sigma^3(2\sin\alpha + \sin 2\alpha) \\
& + \frac{\sigma^4\log 2}{2\pi}(4\alpha\sin\alpha + 2\alpha\sin 2\alpha + 2 + \cos\alpha - 2\cos 2\alpha - \cos 3\alpha)\Bigg] \\
& + \frac{3U_0\zeta(3)}{8\pi^2}\cos\tfrac{5}{2}\xi\left[\sigma^3(\alpha+\sin\alpha) + \frac{\sigma^4\log 2}{\pi}(\alpha+\sin\alpha)^2\right] + 0(\sigma^5). \qquad \text{......(5.3)}
\end{aligned}$$

The capacity of the cap is then found from (5.2) as

$$\begin{aligned}
\frac{C}{a} = {} & \frac{1}{\pi}(\alpha+\sin\alpha) + \frac{\sigma}{\pi^2}(\alpha+\sin\alpha)^2\log 2 + \frac{\sigma^2}{\pi^3}(\alpha+\sin\alpha)^3(\log 2)^2 \\
& + \frac{\sigma^3}{\pi^2}\Bigg[\frac{\zeta(3)}{16}\bigg(3\alpha\sin 2\alpha + 2\alpha\sin 3\alpha + 5 + \frac{11}{2}\cos\alpha - 3\cos 2\alpha \\
& \qquad - \frac{11}{2}\cos 3\alpha - 2\cos 4\alpha\bigg) + \frac{1}{\pi^2}(\alpha+\sin\alpha)^4(\log 2)^3\Bigg] \\
& + \frac{\sigma^4}{\pi^3}\Bigg[\frac{\zeta(3)\log 2}{16}\bigg(2\alpha^2(3\sin 2\alpha + 2\sin 3\alpha) + 2\alpha(5 + 7\cos\alpha - 2\cos 2\alpha - 7\cos 3\alpha \\
& \qquad - 3\cos 4\alpha) + 13\sin\alpha + 11\sin 2\alpha - \sin 3\alpha - \frac{11}{2}\sin 4\alpha - 2\sin 5\alpha\bigg) \\
& \qquad + \frac{1}{\pi^2}(\alpha+\sin\alpha)^5(\log 2)^4\Bigg] + 0(\sigma^5). \qquad \text{......(5.4)}
\end{aligned}$$

When $\alpha = \pi$, the cap becomes a sphere of radius a and the expression (2.5) then holds for the region between the planes for which $r \geqq a$. The expression (5.3) for $g(\xi)$ now becomes

$$g(\xi) = \frac{2U_0}{\pi} \cos \tfrac{1}{2}\xi[1+\sigma \log 2+(\sigma \log 2)^2+(\sigma \log 2)^3+(\sigma \log 2)^4]$$
$$+ \frac{3U_0\zeta(3)}{8\pi} \cos \tfrac{3}{2}\xi[\sigma^3+\sigma^4 \log 2]+0(\sigma^5),$$

whilst the capacity C of the sphere is obtained from (5.4) as

$$\frac{C}{a} = 1+\sigma \log 2+(\sigma \log 2)^2+(\sigma \log 2)^3+(\sigma \log 2)^4+0(\sigma^5),$$

in agreement with the results of Rigby (**9**) and Hurst (**10**).

In conclusion it may be noted that the various terms in the expression (2.5) for the potential $U(r, \theta)$ can be expanded as series of spherical harmonics and an alternative form for $U(r, \theta)$ obtained.

6. The Potential due to a Circular Disk between Two Earthed Planes

The method of the previous sections can also be applied to the problem of determining the potential due to a circular disk parallel to and lying between two earthed planes. We consider only the case when the disk is midway between the planes and state corresponding results to those obtained for a spherical cap.

If (z, ϖ, ϕ) are the cylindrical polar coordinates of a point referred to the centre of the disk, radius c, as origin and the axis of the disk as z-axis, the disk is given by $z = 0\,(0 \leqq \varpi \leqq c)$ and the planes by $z = \pm f$. If $V_0(z, \varpi)$ is the potential due to an axisymmetric distribution of charge in infinite unbounded space, the singularities of the distribution lying in the region $-f<z<f$ but not on the surface $z = 0\,(0 \leqq \varpi \leqq c)$, the potential $V(z, \varpi)$, defined in the region $-f<z<f$, which has the same singularities as $V_0(z, \varpi)$ in this region, is zero on the planes $z = \pm f$ and is a known even function $f_0(\varpi)$ on the disk, is

$$V(z, \varpi) = V_0(z, \varpi)+V_1(z, \varpi)+U(z, \varpi),$$

where $V_1(z, \varpi)$ is given by (2.2) and

$$U(z, \varpi) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-c}^{c} \frac{g(t)dt}{(\varpi^2+(z-2fn+it)^2)^{\frac{1}{2}}}. \qquad \text{(6.1)}$$

In this expression for $U(z, \varpi)$ $g(t)$ is real, continuous and even and the square roots are interpreted similarly to (2.7), (2.8) and (2.9). The function $U(z, \varpi)$ can be shown to be real, harmonic and continuously differentiable at all points except those on the rim $z = 0\ (\varpi = c)$ of the disk. Further it is $0(r^{-1})$ at a large distance r from the disk.

On the disk

$$V(0, \varpi) = f_0(\varpi) \quad (0 \leqq \varpi \leqq c),$$

so that $U(z, \varpi)$ satisfies the condition

$$U(0, \varpi) = f(\varpi) = f_0(\varpi)-V_0(0, \varpi)-V_1(0, \varpi) \quad (0 \leqq \varpi \leqq c),$$

$f_0(\varpi)$ and $f(\varpi)$ being known continuously differentiable even functions. This condition gives as the integral equation determining $g(t)$

$$g(t)-\frac{1}{\pi}\int_{-c}^{c} k(t-s)g(s)ds = h(t) \quad (-c \leqq t \leqq c), \quad \text{..............(6.2)}$$

where the kernel of the equation is given by

$$k(t-s) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}2nf}{4n^2f^2+(t-s)^2}$$
$$= \int_0^\infty (1-\tanh fu)\cos(t-s)u\,du, \quad \text{..................(6.3)}$$

and $h(t)$ by

$$h(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\varpi f(\varpi)d\varpi}{(t^2-\varpi^2)^{\frac{1}{2}}} \quad (0 \leqq t \leqq c)$$

with $h(t) = h(-t)(-c \leqq t < 0)$, $h(t)$ being an even function. Equation (6.2) is the equation obtained in Cooke's solution of this problem by the method of dual integral equations (**11**).

When the disk is maintained at a constant potential U_0, we have that $f(\varpi) = U_0$, so that

$$h(t) = \frac{2U_0}{\pi},$$

and the capacity C of the disk is given by

$$U_0 C = \int_0^c g(t)dt.$$

To obtain an approximate solution of (6.2) by iteration when the distance between the planes is large compared with the radius of the disk we expand the kernel $k(t-s)$ as a power series in $(t-s)/f$ obtaining

$$k(t-s) = \frac{1}{f}\sum_{n=0}^{\infty}(-1)^n \frac{a_{2n}}{2n!}\left(\frac{t-s}{f}\right)^{2n},$$

where

$$a_{2n} = \int_0^\infty (1-\tanh v)v^{2n}dv$$
$$= \int_0^\infty \frac{2v^{2n}}{e^{2v}+1}\,dv$$
$$= \log 2, \quad n = 0,$$
$$= 2^{-4n}(2^{2n}-1)(2n!)\zeta(2n+1), \quad n>0,$$

use being made of an integral expression for the zeta-function (**8***a*, equation (5)). If we neglect terms in $(t-s)^4/f^5$ and higher powers, we have that

$$k(t-s) = \frac{\log 2}{f} - \frac{3\zeta(3)(t-s)^2}{16f^3} + 0\left(\frac{c^4}{f^5}\right)$$

and from (6.2) find as an approximate solution for $g(t)$

$$g(t) = \frac{2U_0}{\pi} + \frac{4U_0 c}{\pi^2 f}\log 2 + \frac{8U_0 c^2}{\pi^3 f^2}(\log 2)^2 + \frac{U_0 c}{f^3}\left[\frac{16c^2}{\pi^4}(\log 2)^3 - \frac{\zeta(3)}{4\pi^2}(3t^2+c^2)\right] + \frac{U_0 c^2}{f^4}\left[\frac{32c^2}{\pi^5}(\log 2)^4 - \frac{3\zeta(3)}{2\pi^3}\log 2(c^2+t^2)\right] + 0\left(\frac{c^5}{f^5}\right).$$

The capacity C of the disk is then found as

$$C = \frac{2c}{\pi}\left[1 + \frac{2}{\pi}\left(\frac{c}{f}\right)\log 2 + \frac{4}{\pi^2}\left(\frac{c}{f}\right)^2(\log 2)^2 + \left(\frac{c}{f}\right)^3\left(\frac{8}{\pi^3}(\log 2)^3 - \frac{\zeta(3)}{4\pi}\right) + \left(\frac{c}{f}\right)^4\left(\frac{16}{\pi^4}(\log 2)^4 - \frac{\zeta(3)}{\pi^2}\log 2\right) + 0\left(\frac{c^5}{f^5}\right)\right].$$

It may be noted that, if the integrands of the various integrals in (6.1) are expressed as integrals involving Bessel functions, $U(z, \varpi)$ can be expressed as a Hankel integral.

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