THE THEORY OF SIMILARITY FOR LARGE-SCALE MOTIONS IN PLANETARY ATMOSPHERES

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Abstract. A general similarity theory for dynamics of planetary atmospheres, which gives results consistent with those of more complex theories, is briefly outlined.

1. One of the most important problems of contemporary meteorology is the theory of the general circulation of the atmosphere. This subject is currently attracting interest because of its relevance to problems of planetary physics in connection with the attainments of modern space technology. Attempts at calculation of characteristics of atmospheric circulation (as yet only for Mars) have already been undertaken by using an appropriate numerical model. However this method requires very laborious and extensive calculations, as well as very definite assumptions about a series of atmospheric characteristics, such as the absorptivity of the optically active constituents, knowledge of their relative concentration, etc. At present we do not possess such detailed knowledge. Another method is to obtain a few mean quantitative estimates on the basis of energetic and thermodynamic considerations and by the methods of the theory of similarity.

One such attempt was undertaken in Golitsyn (1968) where a formula for the mean rate of generation (and dissipation) of kinetic energy per unit mass by large scale motions was introduced:

$$\varepsilon = \left(\frac{k\delta T}{T_1}\right)\left(\frac{q_A}{4M}\right). \tag{1}$$

Here k is a numerical coefficient of the order of 0.1 (according to actual data) for earth's atmosphere, δT is a characteristic temperature difference on the surface or in the atmosphere of the planet, T_1 is the temperature of the most strongly heated regions, $q_A = q(1-A)$ where q is the solar constant for the planet and A is the planet's albedo, and $M = P_s/g$, the mass of a unit column of the atmosphere. For the characteristic scale L of synoptic processes Obukhov's scale $L_0 = c/l \approx c/\Omega$ (Obukhov, 1949) was adopted for rapidly rotating planets in Golitsyn (1968) (here c is the speed of sound, l is the Coriolis parameter), or, for a slowly rotating planet, the planetary radius. For the terrestrial planets, characteristic velocities $U \approx (\varepsilon L)^{1/3}$ and times $\tau \approx L^{2/3}\varepsilon^{-1/3}$ were estimated. For the calculation the value of 0.1 was adopted for k for all planets, and the values of δT and T_1 were taken from observations (Moroz, 1967).

However, in any closed theory of general circulation the quantities δT and T_1 should be determined, and not left to be assigned. We will now attempt to construct such a theory using the methods of similarity theory and dimensional analysis (Sedov, 1967).

2. Our discussion is based on the following physical principles: that the intensity of

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the circulation and its driving temperature difference are interconnected and self-consistent and that they are determined by the energy input to the atmosphere and by the mass of the atmosphere and its thermal properties.

The heat balance is established by long-wave radiation to space, since the atmosphere is considered to be gray – that is, the temperature of thermal radiation is determined by the energy input q_A. For the sake of simplicity we will not take into account the absorption of direct radiation by the atmosphere itself. This procedure is permissible for an atmosphere that is not very thick.*

We will first investigate the case of a non-rotating planet. The determining parameters in the equation describing the dynamic behavior of the atmosphere, averaged over height, are found to be the surface-averaged inflow of energy $q_A/4$ [gm sec^{-3}], the heat capacity per unit mass of the atmosphere c_p [cm^2 sec^{-2} deg^{-1}], the mass of a unit column of the atmosphere M [gm cm^{-2}], the radius of the planet r [cm] and the Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-5}$ gm sec^{-3} deg^{-4}. From these five parameters, a single nondimensional quantity can be formed:

$$\mathscr{M} = \sigma^{3/8} c_p^{-3/2} \left(\frac{q_A}{4}\right)^{5/8} r M^{-1}. \tag{2}$$

Under the adopted assumptions and constraints, the motions in the atmospheres of planets with equal values of the number $\mathscr{M}$ should be similar. The values of $\mathscr{M}$ for the terrestrial planets are 1.1×10^{-3} (earth), 3.4×10^{-2} (Mars, $p_s = 5$ mb), and 1.3×10^{-5} (Venus, $p_s = 90$ atm). These numbers are all seen to be small. Consequently, the dependence on $\mathscr{M}$ should not be significant. The smallness of $\mathscr{M}$ is equivalent to the largeness of M; thus in a first approximation the mass M can be neglected. This means that the atmospheres of the planets, with respect to some global circulation characteristics, are self-similar by mass.**

From the remaining four parameters we can form combinations with the dimensions of velocity, time, and energy. The first has the form:

$$w^{1/2} = c_p^{1/2} \sigma^{-1/8} (q_A/4)^{1/8} = C_e (\chi - 1)^{-1/2}. \tag{3}$$

* In general it is necessary to consider the nondimensional parameter l/H, where l is a characteristic mean free path of a photon, and H is the effective thickness of the atmosphere. This parameter can affect, for example, the quantity k in (1). However, since thermodynamics imposes some general restrictions that do not depend on the nature of the substance ($k < 1$!), we can hope that the estimates obtained will be fairly universal. These effects will be indirectly surveyed in Section 5 when the intensity of the circulation of Venus is considered.

** The author was asked many times why g, the gravity acceleration, does not enter the consideration. Drs. F. P. Bretherton and N. A. Phillips, to whom the author is very grateful, made the point especially clear stating that if $g = 0$ then no motion will arise through the inhomogeneous heating of an atmosphere which is the main cause of the circulation. The answer is this: if we add g then a new independent nondimensional quantity can be formed:

$$c_p \left(\frac{q_A}{4\sigma}\right)^{1/4} / gr \approx RT_e/\mu g r = H/r$$

which is the ratio of an atmosphere's scale height to the planetary radius. This ratio is usually very small unless g is large enough. This means the neglect of g is a new kind of self-similarity of circulations, i.e. independence of the exact value of g. Of course if $g \to 0$ the theory becomes invalid. The same kind of argument shows why an exact value, say, of the molecular viscosity is not essential, etc.

Recalling that for an ideal gas $c_p = \chi(\chi-1)^{-1}R/\mu$ where $\chi = c_p/c_v$, R is the universal gas constant, μ is the molecular weight, and that $(q_A/4\sigma)^{1/8} = T_e^{1/2}$, the effective radiational temperature of the atmosphere, we see that $w = C_e^2(\chi-1)^{-1}$ is the enthalpy and $C_e = (\chi-1)^{1/2}w^{1/2}$ is the velocity of sound (to within a multiplicative factor which takes into account the difference between T_e and the mean temperature of the atmosphere).

The quantity with the dimension of time

$$\tau_e = r/C_e = r(\chi - 1)^{1/2}c_p{}^{-1/2}(4\sigma/q_A)^{1/8} \tag{4}$$

is the relaxation time for atmospheric pressure or density disturbances on a global scale.

The quantity with dimensions of energy has the form

$$E = B\sigma^{1/8}c_p{}^{-1/2}(q_A/4)^{7/8}r^3, \tag{5}$$

where B is some numerical coefficient. This coefficient can change a little from planet to planet, even though there is still a series of factors, not considered here, that may have an effect on the intensity of atmospheric processes. Taking (3) and (4) into account, we write (5) as

$$E = [B(\chi - 1)^{1/2}/4\pi]q_A\pi r^2 \cdot r/C_e = [B(\chi - 1)^{1/2}/4\pi]Q_A\tau_e. \tag{6}$$

Thus the energy E is, to within a multiplicative constant, equal to the full power of the solar radiation Q_A that falls on the planet multipled by the relaxation time τ_e.

The total enthalpy of the atmosphere should depend in an explicit manner on M. It is natural to suppose that (5) determines the total kinetic energy of the atmosphere. The value of E/B, according to (5), is for earth 1.12×10^{27} erg and for Mars 0.97×10^{26} erg. The magnitude of the total kinetic energy of the terrestrial atmosphere varies from season to season and is equal to $6\text{–}9 \times 10^{27}$ erg (Borisenkov, 1963). According to calculations in Leovy and Mintz (1966), the total kinetic energy of Mars' atmosphere is $1.2\text{–}1.6 \times 10^{26}$ erg. Thus, to within a multiplicative constant of the order of unity, (5) does in fact determine the total kinetic energy. A series of consequences follows from this:

(1) the kinetic energy of a unit volume $\rho(U^2/2)$ does not depend on M or p_s;

(2) the mean velocity of atmospheric motions is

$$U = (E/2\pi r^2 M)^{1/2} = (B/2\pi)^{1/2}\sigma^{1/16}c_p{}^{-1/4}(q_A/4)^{7/16}(r/M)^{1/2}; \tag{7}$$

(3) the nondimensional parameter $\mathscr{M}$ is, to within multiplicative factor of the order of unity, equal to the square of the Mach number Ma:

$$\mathscr{M} = [2\pi(\chi - 1)/B]U^2/C_e^2 \approx \mathrm{Ma}^2; \tag{8}$$

(4) the time scale of atmospheric motions is of the order of

$$\tau_u \approx r/U \approx (2\pi/B)^{1/2}c_p{}^{1/4}\sigma^{-1/16}(q_A/4)^{-7/16}(rM)^{1/2}; \tag{9}$$

(5) the total rate of generation (dissipation) of kinetic energy in the entire atmosphere of a planet is on the order of

$$\epsilon \approx E/\tau_u \approx (B^3/2\pi)^{1/2}\sigma^{3/16}c_p{}^{-3/4}(q_A/4)^{21/16}r^{5/2}M^{-1/2}. \tag{10}$$

from which in the calculation for a unit mass we have

$$\varepsilon \approx \epsilon/4\pi r^2 M \approx \tfrac{1}{2}(B/2\pi)^{3/2}\sigma^{3/16}c_p^{-3/4}(q_A/4)^{21/16}r^{1/2}M^{-3/2}. \tag{11}$$

Comparing (11) and (1), we can write the efficiency of the atmosphere in the form

$$\eta = k\ \delta T/T_1 \approx \tfrac{1}{2}(B/2\pi)^{3/2}\sigma^{3/16}c_p^{-3/4}(q_A/4)^{5/16}(r/M)^{1/2} \approx \mathrm{Ma}. \tag{12}$$

Considering $T_1 \approx T_e$, we can evaluate the characteristic temperature difference

$$\delta T \approx \eta T_e/k \approx (2k)^{-1}(B/2\pi)^{3/2}\sigma^{-1/16}c_p^{-3/4}(q_A/4)^{9/16}(r/M)^{1/2}. \tag{13}$$

Sometimes we should introduce $\alpha = T_e/T_1$, and then in (13) it is necessary to substitute k for $k\alpha$. We note that Equation (1) does not appear as a result of the theory of similarity developed here, but follows from other considerations. It is also related to Equation (13).

3. We now include in the number of determining parameters the angular velocity of actual rotation of the planet, Ω (sec^{-1}). The quantity $\mathscr{M} \approx \mathrm{Ma}^2$, we let be small, as before. Then from the parameters σ, c_p, $q_A/4$, r, and Ω we can compose only one dimensionless combination

$$\lambda = (\chi - 1)^{-1/2}c_p^{-1/2}(4\sigma/q_A)^{1/8}\Omega r = \Omega(r/C_e) = \Omega\tau_e = r/L_0. \tag{14}$$

Formula (5) must now be multiplied by a function $f(\lambda)$, where $f(0) = 1$. From general considerations it follows only that $f(\lambda)$ must increase with an increase in λ. We cannot successfully obtain explicit formulas of the type (7) through (13), however the dependences $U \sim M^{-1/2}$, $T_u \sim M^{1/2}$, $\varepsilon \sim M^{-3/2}$, $\eta \sim M^{-1/2} \sim \delta T$ are preserved here, as is the independence of $\rho U^2/2$ from M. In the general case when $\mathscr{M} \gtrsim 1$, equation (5) must be multiplied by the function from the two dimensionless parameters $f(\mathscr{M}, \lambda)$. From the point of view of similarity, this case is the most complex and it is difficult to obtain here even qualitative relationships.

4. We will now demonstrate another derivation of the basic relationships of Section 2. We write the equal of the balance of heat for a spherical atmosphere, averaged over height in the most simplified form as

$$Mc_p u_i \frac{\partial T}{\partial x_i} = \sigma T_e^4, \tag{15}$$

i.e. the advection of heat by large scale motions is balanced by the atmosphere's cooling into space. To an order of magnitude the advection is $u_i\ \partial T/\partial x_i \approx U\delta T/(\pi r/2)$, where $U \approx (\varepsilon r)^{1/3}$. Then upon taking into account Equation (1), we obtain from (15)

$$\delta T \approx (\pi/2)^{1/2}k^{-1/4}\sigma^{-1/16}c_p^{-3/4}(q_A/4)^{9/16}(r/M)^{1/2}. \tag{16}$$

Putting δT in Equation (1), we determine ε, then U, and then E. Furthermore the structure of all the formulas obtained turns out to be same as that of the formulas in Section 2. Comparing them, we have

$$B \approx \pi^2 k^{1/2} \tag{17}$$

by which the theory of similarity and the concepts developed in Golitsyn (1968) are naturally married. From Equation (16), taking δT as fixed, we can obtain

$$k \approx \frac{\pi^2}{4}\,\delta T^{-4}\,\frac{(q_A/4)^{9/4}}{\sigma^{1/4} c_p^3}\left(\frac{r}{M}\right)^2. \tag{18}$$

Therefore the whole theory is determined with precision except for one empirical quantity, for which we can select B, k, or δT, or U, etc.

Since from thermodynamic considerations $k<1$ ($k=1$ only for an ideal heat engine in a Carnot cycle), then from (16) there follows the inequality

$$\delta T > \frac{\pi^{1/2}}{2}\,\frac{(q_A/4)^{9/16}}{\sigma^{1/16} c_p^{3/4}}\left(\frac{r}{M}\right)^{1/2}. \tag{19}$$

Similar inequalities can be obtained for U, ε (from above) and for τ_u (from below).

For temperature there is still one obvious inequality: $\delta T < T_1 \approx T_e$. This can give some information in the case of a planet with a very thin atmosphere (for example, the hypothetical atmosphere of Mercury), when the formal use of expression (16) can lead to its violation, that is to $\delta T > T_e$. In this case in Equation (13) instead of B it is necessary to put $Bf_1(\mathcal{M}) \approx \pi^2 k^{1/2} f_1(\mathcal{M})$ and then the inequality $\delta T < T_e$ gives the following estimate from above:

$$f_1(\mathcal{M}) < \mathcal{M}^{-1/3} \quad \text{for } \mathcal{M} \gtrsim 1. \tag{20}$$

5. We shall now see what it is possible to obtain using the dependences derived here for planets of the terrestrial type, if in the case of the earth and Mars we ignore the rotation of the planet. We take in all cases $k=0.1$, as it was evaluated in Golitsyn (1968) for the terrestrial atmosphere.

Results of calculations according to the formulas in Section 2 are given in Table I, together with the quantities U and δT observed for the earth, or computed in Leovy

TABLE I

Planet	ρ_s atm	U m/sec	U, obs. or comp.	δT K	δT, obs. or comp.	τ_u days	ε cm²/sec³
Earth	1	12	17	20	40	5	4
Mars	5×10^{-3}	50	40	100	110	1	~ 100
Venus	90	0.7	?	2	<15	100	3×10^{-4}

and Mintz (1966) for Mars. For these two planets the theoretical predictions are satisfactory. In the case of Venus, the most striking thing turns out to be the small magnitude of the calculated characteristic temperature difference. This agrees with the practical absence of a phase effect in the Venus radio emission, and also agrees with the results of radio interferometric measurements at 11 cm, about which Dr. David Morrison spoke at the present symposium. According to these measurements, there is no difference of temperature between the equator and poles on the surface of Venus within an error of less than 15 K.

Concerning the winds of Venus, we have only observations of the motions of ultraviolet clouds, the velocities of which reach 100 m/sec. This, obviously, is a mesopheric phenomenon, originating at a level where the pressure is of the order of 1 to 10 mb. Our value of the characteristic velocity relates to the entire thickness of the atmosphere. The motions of the deep atmosphere are so far unknown from experimental methods. However, the success of the theory in predicting the temperature difference gives faith in the correctness of the predicted order of magnitude of both the velocity and time scale τ_u.

If it turns out that all or most of the direct solar radiation is absorbed by the atmosphere of Venus, and does not reach the surface, then we are brought to the analogy with the oceans (Golitsyn, 1968; Goody and Robinson, 1966). We may think then, that this could lead to a reduction of k by roughly a factor of two or three orders of magnitude. In such a case the velocity is reduced by a factor of 3 to 5, and τ_u and δT are correspondingly increased.

A considerably more detailed discussion of the theory and its applications to the terrestrial planets as well as its possible extension to the case of giant planets can be found in Golitsyn (1970).

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