J. Aust. Math. Soc. **104** (2018), 380–402
doi:10.1017/S1446788717000143

A NOTE ON MARCINKIEWICZ INTEGRALS ASSOCIATED TO SURFACES OF REVOLUTION

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(Received 18 November 2016; accepted 25 April 2017; first published online 14 August 2017)

Communicated by C. Meaney

Abstract

We establish the bounds of Marcinkiewicz integrals associated to surfaces of revolution generated by two polynomial mappings on Triebel–Lizorkin spaces and Besov spaces when their integral kernels are given by functions $\Omega \in H^1(S^{n-1}) \cup L(\log^+ L)^{1/2}(S^{n-1})$. Our main results represent improvements as well as natural extensions of many previously known results.

2010 *Mathematics subject classification*: primary 42B20; secondary 42B25, 47G10.

Keywords and phrases: Marcinkiewicz integral, rough kernel, surfaces of revolution, Triebel–Lizorkin space, Besov space.

1. Introduction

It is well known that the Marcinkiewicz integral operator is the most typical representative of the Littlewood–Paley g-function and L^p bounds, for these operators play a key role in the study of smoothness properties of functions and behavior of integral transformations, such as Poisson integrals, singular integrals and, more generally, singular Radon transforms. In recent years, the investigation on the bounds for Marcinkiewicz integral operators on Triebel–Lizorkin spaces and Besov spaces has attracted the attention of many authors (see [19, 29–31] for example). In this paper we focus on this topic. More precisely, we will establish the bounds for Marcinkiewicz integral operators associated to surfaces of revolution generated by two polynomial mappings on the above function spaces under the rather weak size conditions on the integral kernels.

This work was partially supported by the NSF of China (number 11526122), by the Scientific Research Foundation of Shandong University of Science and Technology for Recruited Talents (number 2015RCJJ053), by the Research Award Fund for Outstanding Young Scientists of Shandong Province (number BS2015SF012) and by the Support Program for Outstanding Young Scientific and Technological Top-notch Talents of the College of Mathematics and Systems Science (number Sxy2016K01).

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space. For $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$ $(p \neq \infty)$, we define the homogeneous Triebel–Lizorkin spaces $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and homogeneous Besov spaces $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$ by

$$\dot{F}_\alpha^{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} = \Big\| \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} |\Psi_i * f|^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\} \tag{1.1}$$

and

$$\dot{B}_\alpha^{p,q}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} = \Big(\sum_{i\in\mathbb{Z}} 2^{-i\alpha q} \|\Psi_i * f\|_{L^p(\mathbb{R}^n)}^q \Big)^{1/q} < \infty \Big\}, \tag{1.2}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, $\widehat{\Psi_i}(\xi) = \phi(2^i\xi)$ for $i \in \mathbb{Z}$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ satisfies these conditions: $0 \leq \phi(x) \leq 1$; $\operatorname{supp}(\phi) \subset \{x : 1/2 \leq |x| \leq 2\}$; $\phi(x) > c > 0$ if $3/5 \leq |x| \leq 5/3$. It is well known that $\dot{F}_0^{p,2}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and

$$\dot{F}_\alpha^{p,p}(\mathbb{R}^n) = \dot{B}_\alpha^{p,p}(\mathbb{R}^n) \quad \forall \alpha \in \mathbb{R} \text{ and } 1 < p < \infty. \tag{1.3}$$

See [14, 15, 27] for more properties of $\dot{F}_\alpha^{p,q}(\mathbb{R}^n)$ and $\dot{B}_\alpha^{p,q}(\mathbb{R}^n)$. The inhomogeneous versions of Triebel–Lizorkin spaces and Besov spaces, which are denoted by $F_\alpha^{p,q}(\mathbb{R}^n)$ and $B_\alpha^{p,q}(\mathbb{R}^n)$, respectively, are obtained by adding the term $\|\Theta * f\|_{L^p(\mathbb{R}^n)}$ to the right-hand side of (1.1) or (1.2) with $\sum_{i\in\mathbb{Z}}$ replaced by $\sum_{i\geq 1}$, where $\Theta \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}(\hat{\Theta}) \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$, $\hat{\Theta}(x) > c > 0$ if $|x| \leq 5/3$. The following properties are well known (see [14] or [15], for example): for any $1 < p, q < \infty$ and $\alpha > 0$,

$$F_\alpha^{p,q}(\mathbb{R}^n) \sim \dot{F}_\alpha^{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{F_\alpha^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}; \tag{1.4}$$

$$B_\alpha^{p,q}(\mathbb{R}^n) \sim \dot{B}_\alpha^{p,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \quad \text{and} \quad \|f\|_{B_\alpha^{p,q}(\mathbb{R}^n)} \sim \|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^n)} + \|f\|_{L^p(\mathbb{R}^n)}. \tag{1.5}$$

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $d, m \geq 1$ and $\Gamma_{\Phi,\Psi} = \{(\Phi(y), \Psi(|y|)) : y \in \mathbb{R}^n\}$ be surfaces generated by two suitable mappings $\Phi : \mathbb{R}^n \to \mathbb{R}^d$ and $\Psi : [0, \infty) \to \mathbb{R}^m$. Suppose $\Omega \in L^1(\mathrm{S}^{n-1})$ satisfies the cancellation condition

$$\int_{\mathrm{S}^{n-1}} \Omega(u)\, d\sigma(u) = 0. \tag{1.6}$$

For a complex number $\rho = \tau + i\vartheta$ $(\tau, \vartheta \in \mathbb{R}$ with $\tau > 0)$, we define the parametric Marcinkiewicz integral operator $\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}$ along the surface $\Gamma_{\Phi,\Psi}$ by

$$\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f(u, v) = \Big(\int_0^\infty \Big| \frac{1}{t^\rho} \int_{|y|\leq t} \frac{h(|y|)\Omega(y')}{|y|^{n-\rho}} f(u - \Phi(y), v - \Psi(|y|))\, dy \Big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{1.7}$$

where $y' = y/|y|$ for any nonzero vector $y \in \mathbb{R}^n$, $(u, v) \in \mathbb{R}^d \times \mathbb{R}^m = \mathbb{R}^{d+m}$, $f \in \mathcal{S}(\mathbb{R}^{d+m})$ (the Schwartz class) and $h \in \Delta_1(\mathbb{R}_+)$. Here $\Delta_\gamma(\mathbb{R}_+)$ for $\gamma > 0$ denotes the set of all measurable functions h defined on $\mathbb{R}_+ := (0, \infty)$ satisfying

$$\|h\|_{\Delta_\gamma(\mathbb{R}_+)} := \sup_{R>0} \Big(R^{-1} \int_0^R |h(t)|^\gamma\, dt \Big)^{1/\gamma} < \infty.$$

Note that $L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}^+) \subsetneq \Delta_{\gamma_1}(\mathbb{R}_+) \subsetneq \Delta_{\gamma_2}(\mathbb{R}_+)$ for any $0 < \gamma_2 < \gamma_1 < \infty$.

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When $\Psi(t) \equiv (0, \ldots, 0) \in \mathbb{R}^m$, the operator $\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}$ essentially reduces to the lower-dimensional Marcinkiewicz integral operator $\mathscr{M}_{h,\Omega,\Phi,\rho}$, which is given by

$$\mathscr{M}_{h,\Omega,\Phi,\rho}f(x) = \bigg(\int_0^\infty \bigg|\frac{1}{t^\rho}\int_{|y|\le t}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}f(x-\Phi(y))\,dy\bigg|^2\frac{dt}{t}\bigg)^{1/2}.$$

For the sake of simplicity, we denote $\mathscr{M}_{h,\Omega,\Phi,\rho} = \mathscr{M}_{h,\Omega}$ if $n = d$, $\rho = 1$ and $\Phi(y) = y$. When $h(t) \equiv 1$, the operator $\mathscr{M}_{h,\Omega}$ reduces to the classical Marcinkiewicz integral operator $\mathscr{M}_\Omega$. Over the last several years the L^p mapping properties of $\mathscr{M}_\Omega$ have been studied by many authors. For example, see [8] for the case $\Omega \in H^1(\mathrm{S}^{n-1})$ (the Hardy space on the unit sphere; see [6, 26]), [1] for the case $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$, [3] for the case $\Omega \in B_q^{(0,-1/2)}(\mathrm{S}^{n-1})$ (the Block space generated by q-block), [5] for the case $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ (the Grafakos–Stefanov function class; see [16]). For relevant results on parametric Marcinkiewicz integral operator $\mathscr{M}_{h,\Omega,\rho}$ and other integral operators with rough kernels, we refer the readers to [10, 17, 18, 21, 24], among others. Recently, the investigation of the boundedness of the Marcinkiewicz integral operator on the Triebel–Lizorkin spaces has also attracted the attention of many authors. In 2009, Zhang and Chen [30] proved that $\mathscr{M}_\Omega$ is bounded on the $F_\alpha^{p,q}(\mathbb{R}^n)$ for $0 < \alpha < 1$ and $1 < p, q < \infty$ if $\Omega \in H^1(\mathrm{S}^{n-1})$. Later on, the above authors [31] showed that $\mathscr{M}_{h,\Omega}$ is bounded on $F_\alpha^{p,q}(\mathbb{R}^n)$ for $0 < \alpha < 1$ and $1 + (n+1)/(n+2-1/r) < p, q < 2 + (1-1/r)/(n+1)$ if $\Omega \in L^r(\mathrm{S}^{n-1})$ for some $r > 1$ and $h \in L^\infty(\mathbb{R}_+)$. Very recently, Yabuta [29] investigated the Triebel–Lizorkin space boundedness of Marcinkiewicz integrals associated to certain surfaces under the integral kernels given by $\Omega \in H^1(\mathrm{S}^{n-1}) \cup L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$.

We notice that the following inclusion relations are valid:

$$\begin{gathered}
L(\log^+ L)^{\beta_1}(\mathrm{S}^{n-1}) \subsetneq L(\log^+ L)^{\beta_2}(\mathrm{S}^{n-1}) \quad \forall \beta_1 > \beta_2 > 0;\\
\bigcup_{q>1} L^q(\mathrm{S}^{n-1}) \subsetneq L(\log^+ L)^{\beta}(\mathrm{S}^{n-1}) \subsetneq H^1(\mathrm{S}^{n-1}) \quad \forall \beta \ge 1;\\
L(\log^+ L)^{\beta}(\mathrm{S}^{n-1}) \not\subseteq H^1(\mathrm{S}^{n-1}) \not\subseteq L(\log^+ L)^{\beta}(\mathrm{S}^{n-1}) \quad \forall 0 < \beta < 1;\\
\bigcup_{q>1} L^q(\mathrm{S}^{n-1}) \subsetneq \bigcap_{\beta>1} \mathscr{F}_\beta(\mathrm{S}^{n-1}) \not\subseteq L\log^+ L(\mathrm{S}^{n-1});\\
\bigcap_{\beta>1} \mathscr{F}_\beta(\mathrm{S}^{n-1}) \not\subseteq H^1(\mathrm{S}^{n-1}) \not\subseteq \bigcup_{\beta>1} \mathscr{F}_\beta(\mathrm{S}^{n-1});\\
B_q^{(0,v)}(\mathrm{S}^{n-1}) \subset H^1(\mathrm{S}^{n-1}) + L(\log^+ L)^{1+v}(\mathrm{S}^{n-1}) \quad \forall q > 1, v > -1.
\end{gathered} \tag{1.8}$$

When $\rho = m = 1$, $h(t) \equiv 1$, $d = n$ and $\Phi(y) = y$, the operator $\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}$ becomes the classical Marcinkiewicz integral operator associated to surfaces of revolution $\Gamma_\Psi = \{(y, \Psi(|y|)) : y \in \mathbb{R}^n\}$, denoted by $\mathscr{M}_{\Omega,\Psi}$. In 2002, Ding *et al.* [9] proved that $\mathscr{M}_{\Omega,\Psi}$ is bounded on $L^p(\mathbb{R}^{n+1})$ provided that $\Omega \in H^1(\mathrm{S}^{n-1})$ and the following maximal operator

$$\mathcal{M}_\Psi g(u, v) = \sup_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |g(u-s, v-\Psi(s))|\,ds$$

is bounded on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$. Subsequently, Fan and Sato [13] gave an improvement of the above result. For relevant results on the operator $\mathscr{M}_{\Omega,\Psi}$ we refer the readers to [11, 28].

In this paper we aim to establish some new results concerning the Triebel–Lizorkin space boundedness for parametric Marcinkiewicz integral operators associated to certain surfaces of revolution. Before establishing our main results, let us introduce some notation. We denote by $\mathscr{A}_n$ the class of polynomials of n variables with real coefficients. For $N \geq 1$, let $\mathscr{A}_{n,N}$ be the collection of polynomials in $\mathscr{A}_n$ which have degrees not exceeding N, and let $V_{n,N}$ be the collection of polynomials in $\mathscr{A}_{n,N}$ which are homogeneous of degree N.

Our main results can be formulated as follows.

Theorem 1.1. *Let $\Psi(y) = \mathcal{Q}(\varphi(|y|))$ with $\mathcal{Q} = (Q_1, \ldots, Q_m) \in (\mathscr{A}_1)^m$ and $\varphi \in \mathfrak{F}$, where $\mathfrak{F}$ is the set of all positive increasing $C^1(\mathbb{R}_+)$ functions ϕ such that $t\phi'(t) \geq C_\phi \phi(t)$ and $\phi(2t) \leq c_\phi \phi(t)$ for all $t > 0$ and some $C_\phi, c_\phi > 0$. Let $\Omega \in L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ satisfy (1.6) and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$. Suppose that one of the following conditions holds:*

(i) *$n = d$, $\Phi(y) = P(\varphi(|y|)) \bigotimes y$ with $P = (P_1, \ldots, P_n) \in (\mathscr{A}_1)^n$;*
(ii) *$\Phi(y) = \mathscr{P}(\varphi(|y|)y')$ with $\mathscr{P} = (P_1, \ldots, P_d) \in (\mathscr{A}_n)^d$.*

Then for $\alpha \in (0, 1)$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$, there exists a constant $C > 0$ such that

$$\|\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})} \leq C\|\Omega\|_{L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})},$$

where $\mathscr{R}_\gamma$ is the set of all interiors of the convex hull of three squares $(\frac{1}{2}, \frac{1}{2} + 1/\max\{2, \gamma'\})^2$, $(\frac{1}{2} - 1/\max\{2, \gamma'\}, \frac{1}{2})^2$ and $(1/2\gamma, 1 - 1/2\gamma)^2$. The constant $C = C_{\rho,n,d,p,q,\varphi,\alpha,\max_{1\leq l\leq d}\deg(P_l),\deg(\mathcal{Q})}$ is independent of the coefficients of P_i and Q_j for $1 \leq i \leq d$ and $1 \leq j \leq m$.

Theorem 1.2. *Let $\varphi \in \mathfrak{F}$, $\Phi(y) = \mathscr{P}(\varphi(|y|)y')$ with $\mathscr{P} = (P_1, \ldots, P_d) \in (\mathscr{A}_n)^d$ and $\Psi(y) = \mathcal{Q}(\varphi(|y|))$ with $\mathcal{Q} = (Q_1, \ldots, Q_m) \in (\mathscr{A}_1)^m$. Suppose that $\Omega \in H^1(\mathrm{S}^{n-1})$ satisfies (1.6) and $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma > 1$. Then for $\alpha \in (0, 1)$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$, there exists a constant $C > 0$ such that*

$$\|\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})} \leq C\|\Omega\|_{H^1(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})},$$

where $\mathscr{R}_\gamma$ is given as in Theorem 1.1 and $C = C_{\rho,n,d,p,q,\varphi,\alpha,\deg(\mathscr{P}),\deg(\mathcal{Q})}$ is independent of the coefficients of P_i and Q_j for $1 \leq i \leq d$ and $1 \leq j \leq m$.

Remark 1.3. Some remarks follow.

(i) Note that $\mathscr{R}_{\gamma_2} \subsetneq \mathscr{R}_{\gamma_1}$ for $\gamma_1 > \gamma_2 > 1$ and $\mathscr{R}_\infty = (0, 1)^2$. It follows that the operator $\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}$ is bounded on $\dot{F}^{p,q}_\alpha(\mathbb{R}^{d+m})$ for $\alpha \in (0, 1)$ and $1 < p, q < \infty$ if Ω, Φ, Ψ are given as in Theorems 1.1 or 1.2 and $h \in L^\infty(\mathbb{R}_+)$.
(ii) There are some model examples for the class $\mathfrak{F}$, such as t^α $(\alpha > 0)$, $t^\beta \ln(1 + t)$ $(\beta \geq 1)$, $t \ln\ln(e + t)$, real-valued polynomials P on $\mathbb{R}$ with positive coefficients and $P(0) = 0$ and so on. It was shown in [22] that for any $\varphi \in \mathfrak{F}$ there exists $B_\varphi > 1$ such that $\varphi(2t) \geq B_\varphi \varphi(t)$.

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(iii) By employing methods as in the proof of [4, Theorem 2.3] and using some estimates about Fourier transform of measures appeared in the proofs of Theorems 1.1–1.2, one can obtain that $\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}$ is bounded on $L^p(\mathbb{R}^{d+m})$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ if h, Ω, Φ, Ψ are given as in Theorems 1.1 or 1.2.

(iv) We remark that the corresponding results on singular integrals along surfaces of revolution have been established by us in [20]. Moreover, the questions concerning the $\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})$ and $\dot{B}_\alpha^{p,q}(\mathbb{R}^{d+m})$ bounds for $\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}$ with Φ, Ψ being as in Theorem 1.1 and $\Omega \in \mathscr{F}_\beta(\mathrm{S}^{n-1})$ have been answered by us in [19].

(v) It should be also pointed out that Theorem 1.2 also holds for $\Omega \in \bigcup_{1<r<\infty} B_r^{(0,-1/2)}(\mathrm{S}^{n-1})$ by (1.8) and Theorem 1.1.

Observe that

$$|\Delta_\zeta(\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f)(x)| \le |\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}\Delta_\zeta(f)(x)| \quad \forall x, \zeta \in \mathbb{R}^{d+m}. \tag{1.9}$$

Combining (1.9) with (iii) of Remark 1.3 and [19, Theorem 4.1] yields the following theorem.

Theorem 1.4. *Under the same conditions as in Theorems 1.1 and 1.2, these operators are bounded on $\dot{B}_\alpha^{p,q}(\mathbb{R}^{d+m})$ for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.*

By the properties (1.4) and (1.5), (iii) of Remark 1.3 and Theorems 1.1 to 1.4, we get the following results immediately.

Theorem 1.5. *Under the same conditions as in Theorems 1.1 to 1.4, these operators are bounded on $F_\alpha^{p,q}(\mathbb{R}^{d+m})$ and $B_\alpha^{p,q}(\mathbb{R}^{d+m})$.*

Remark 1.6. It should be pointed out that our main results are new, even in the special case $\rho = 1$, $h(t) \equiv 1$, $n = d$, $m = 1$ and $\Phi(y) = y$ or $\Psi(|y|) = |y|$.

The paper is organized as follows. Section 2 is devoted to presenting some auxiliary lemmas. In Section 3 we shall prove Theorem 1.1. The proof of Theorem 1.2 will be given in Section 4. It should be pointed out that the main method employed in this paper is a combination of ideas and arguments from [2, 12, 19, 29]. Particularly, in proving Theorem 1.1, the key decomposition of $L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})$ following from [2] will be needed. On the other hand, Theorem 1.2 is proved by applying some ideas and techniques following from [12]. Throughout the paper, we denote by p' the conjugate index of p, which satisfies $1/p + 1/p' = 1$. The letter C or c, sometimes with certain parameters, will stand for positive constants: not necessarily the same one at each occurrence, but independent of the essential variables. In what follows, we set $\mathfrak{R}_d = \{\xi \in \mathbb{R}^d : 1/2 < |\xi| \le 1\}$. Let $\triangle_\zeta(f)$ be the difference of f for an arbitrary function f defined on $\mathbb{R}^d$ and $\zeta \in \mathbb{R}^d$, that is, $\triangle_\zeta(f)(x) = f(x+\zeta) - f(x)$. We also use the conventions $\sum_{j\in\emptyset} a_j = 0$ and $\prod_{j\in\emptyset} a_j = 1$.

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2. Preliminary lemmas

Let us begin with the following lemma of van der Corput type.

LEMMA 2.1 [25]. *Let $l \in \mathbb{N}\backslash\{0\}$, $\mu_1, \dots, \mu_l \in \mathbb{R}$, and $d_1, \dots, d_l$ be distinct positive real numbers. Let $\psi \in C^1([0,1])$. Then there exists a constant $C > 0$ independent of $\mu_1, \dots, \mu_l$ such that*

$$\bigg|\int_\delta^\tau e^{i(\mu_1 t^{d_1}+\cdots+\mu_l t^{d_l})}\psi(t)\,dt\bigg| \le C|\mu_1|^{-\epsilon}\Big(|\psi(\tau)| + \int_\delta^\tau |\psi'(t)|\,dt\Big)$$

holds for $0 \le \delta < \tau \le 1$ and $\epsilon = \min\{1/d_1, 1/l\}$.

The following results are two vector-valued norm inequalities of maximal operators.

LEMMA 2.2 [19]. *Let $\mathrm{M}_{(d)}$ be the Hardy–Littlewood maximal operator on $\mathbb{R}^d$ and $\mathrm{M}_{\mathscr{P}}$ denote the Hardy–Littlewood maximal operator supported by polynomial mappings $\mathscr{P}$ defined by $\mathrm{M}_{\mathscr{P}} f(x) = \sup_{r>0} (1/r^n) \int_{|y|\le r} |f(x - \mathscr{P}(y))|\,dy$, where $\mathscr{P} = (P_1, P_2, \dots, P_d) \in (\mathscr{A}_n)^d$.*

(i) *For $1 < p, q, r < \infty$, it holds that*

$$\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|\mathrm{M}_{(d)} g_{j,\zeta,k}|^2\bigg)^{1/2}\bigg\|_{L^r(\Re_d)}^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^d)} \le C\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\bigg)^{1/2}\bigg\|_{L^r(\Re_d)}^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^d)}.$$

(ii) *For $1 < p, q, r < \infty$, it holds that*

$$\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\|\mathrm{M}_{\mathscr{P}} f_{j,\zeta}\|_{L^r(\Re_d)}^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^d)} \le C\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\Re_d)}^q\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^d)},$$

where $C > 0$ is independent of the coefficients of P_j for $1 \le j \le d$.

Let h, Ω, ρ be given as in (1.7). For a suitable mapping $\Gamma : \mathbb{R}^n \to \mathbb{R}^d$, we define the measures $\{\sigma_{h,\Omega,\Gamma,t}\}_{t\in\mathbb{R}_+}$ and $\{|\sigma_{h,\Omega,\Gamma,t}|\}_{t\in\mathbb{R}_+}$ respectively by

$$\widehat{\sigma_{h,\Omega,\Gamma,t}}(x) = \frac{1}{t^\rho}\int_{t/2<|y|\le t} e^{-2\pi i x\cdot\Gamma(y)}\frac{h(|y|)\Omega(y')}{|y|^{n-\rho}}\,dy \tag{2.1}$$

and

$$|\widehat{\sigma_{h,\Omega,\Gamma,t}}|(x) = \frac{1}{t^\rho}\int_{t/2<|y|\le t} e^{-2\pi i x\cdot\Gamma(y)}\frac{|h(|y|)\Omega(y')|}{|y|^{n-\rho}}\,dy. \tag{2.2}$$

The following lemma is a refined estimate of a vector-valued inequality, which plays a key role in the proofs of our main results.

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Lemma 2.3. *Let* $\Gamma(y) = (P_1(\varphi(|y|))a_1(y'), \ldots, P_d(\varphi(|y|))a_d(y'))$ *and* $v \geq 1$, *where* $(P_1, \ldots, P_d) \in (\mathscr{A}_n)^d$ *and* $\varphi \in \mathfrak{F}$. *Suppose that* $\Omega \in L^1(\mathrm{S}^{n-1})$ *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$. *Then for* $(1/p, 1/q, 1/r)$ *belonging to the interior of the convex hull of three cubes* $(\frac{1}{2}, 1/2 + 1/\max\{2, \gamma'\})^3$, $(\frac{1}{2} - 1/\max\{2, \gamma'\}, \frac{1}{2})^3$ *and* $(1/2\gamma, 1 - 1/2\gamma)^3$, *there exists a constant* $C > 0$ *such that*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}|^2\,\frac{dt}{t}\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq Cv^{1/2}\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}, \tag{2.3}$$

where $C > 0$ *is independent of* v, Ω *and the coefficients of* P_j *for* $1 \leq j \leq d$.

Proof. Define the maximal operator $\sigma^*_{h,\Omega,\Gamma}(f)(x) = \sup_{t>0}\||\sigma_{h,\Omega,\Gamma,t}| * f(x)|$. We first show that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\|\sigma^*_{h,\Omega,\Gamma}(f_{j,\zeta})\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{2.4}$$

for any $\gamma' < p, q, r < \infty$. By a change of variable and Hölder's inequality,

$$\begin{aligned}\||\sigma_{h,\Omega,\Gamma,t}| * f(x)| &\leq \int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}|f(x-\Gamma(r\theta))||\Omega(\theta)|\,d\sigma(\theta)|h(r)|\,\frac{dr}{r}\\ &\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/\gamma}\Big(\int_{\mathrm{S}^{n-1}}\int_{t/2}^{t}|f(x-\Gamma(r\theta))|^{\gamma'}\,\frac{dr}{r}|\Omega(\theta)|\,d\sigma(\theta)\Big)^{1/\gamma'}.\end{aligned} \tag{2.5}$$

By a change of variable again and the properties of φ,

$$\begin{aligned}\int_{t/2}^{t}|f(x-\Gamma(r\theta))|^{\gamma'}\,\frac{dr}{r} &= \int_{\varphi(t/2)}^{\varphi(t)}|f(x-\Gamma(\varphi^{-1}(s)\theta))|^{\gamma'}\,\frac{ds}{\varphi^{-1}(s)\varphi'(\varphi^{-1}(s))}\\ &\leq \frac{1}{C_\varphi}\int_{\varphi(t/2)}^{\varphi(t)}|f(x-\Gamma(\varphi^{-1}(s)\theta))|^{\gamma'}\,\frac{ds}{s}\\ &\leq C(\varphi)\frac{1}{\varphi(t)}\int_{|s|\leq\varphi(t)}|f(x-\Gamma(\varphi^{-1}(s)\theta))|^{\gamma'}\,ds,\end{aligned}$$

which together with (2.5) implies that

$$\sigma^*_{h,\Omega,\Gamma}(f)(x) \leq C(\varphi)\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/\gamma}\Big(\int_{\mathrm{S}^{n-1}}\sup_{t>0}\frac{1}{t}\int_{|s|\leq t}|f(x-\Gamma(\varphi^{-1}(s)\theta))|^{\gamma'}\,ds|\Omega(\theta)|\,d\sigma(\theta)\Big)^{1/\gamma'}. \tag{2.6}$$

Note that $\Gamma(\varphi^{-1}(s)\theta) = (P_1(s)a_1(\theta), \ldots, P_d(s)a_d(\theta))$. Using Minkowski's inequality along with (2.6) and invoking (ii) of Lemma 2.2, we get (2.4).

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We now prove (2.3) by considering the following three cases.

Case 1 ($1 < \gamma \le \infty$). We get from (2.4) that

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}\sup_{t\in[2^{kv},2^{(k+1)v}]}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}\big|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ \le \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sigma^*_{h,\Omega,\Gamma}\Big(\sup_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big)\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ \le C\|\Omega\|_{L^1(\mathbb{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sup_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\end{aligned} \tag{2.7}$$

for any $\gamma' < p, q, r < \infty$. On the other hand, by duality, Hölder's inequality, Fubini's theorem and (2.4), we have that for $1 < p, q, r < \gamma$, there exists a sequence of positive functions $\{f_{j,\zeta}\}_{j,\zeta}$ with $\|\{f_{j,\zeta}\}\|_{L^{p'}(\mathbb{R}^d,\ell^{q'}(L^{r'}(\mathfrak{R}_d)))} = 1$ such that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}\big|\frac{dt}{t}\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\\ &= \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}(x)\big|\frac{dt}{t}f_{j,\zeta}(x)\,d\zeta\,dx\\ &\le \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^d}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}(x)|\int_{2^{kv}}^{2^{(k+1)v}}\big||\sigma_{h,\Omega,\Gamma,t}|*\widetilde{f_{j,\zeta}}(-x)\big|\frac{dt}{t}\,d\zeta\,dx\\ &\le v\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sigma^*_{h,\Omega,\Gamma}(\widetilde{f_{j,\zeta}})\Big\|^{q'}_{L^{r'}(\mathfrak{R}_d)}\Big)^{1/q'}\Big\|_{L^{p'}(\mathbb{R}^d)}\\ &\le Cv\|\Omega\|_{L^1(\mathbb{S}^{n-1})}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|\Big\|^q_{L^r(\mathfrak{R}_d)}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)},\end{aligned} \tag{2.8}$$

where $\widetilde{f_{j,\zeta}}(x) = f_{j,\zeta}(-x)$. Thus, the interpolation between (2.7) and (2.8) yields that (2.3) holds for $(1/p, 1/q, 1/r)$ belonging to the interior of the cube $(1/2\gamma, 1-1/2\gamma)^3$.

Case 2 ($1 < \gamma \le 2$). By Hölder's inequality,

$$\begin{aligned}&\big||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}(x)\big|\\ &\le \int_{t/2<|y|\le t}|g_{j,\zeta,k}(x-\Gamma(y))|\frac{|h(y)\Omega(y')|}{|y|^n}\,dy\\ &\le \Big(\int_{t/2<|y|\le t}|g_{j,\zeta,k}(x-\Gamma(y))|^2\frac{|h(y)|^{2-\gamma}|\Omega(y')|}{|y|^n}\,dy\Big)^{1/2}\Big(\int_{t/2<|y|\le t}\frac{|h(y)|^{\gamma}|\Omega(y')|}{|y|^n}\,dy\Big)^{1/2}\\ &\le C\|\Omega\|^{1/2}_{L^1(\mathbb{S}^{n-1})}(|\sigma_{|h|^{2-\gamma},\Omega,\Gamma,t}|*|g_{j,\zeta,k}|^2(x))^{1/2}.\end{aligned}$$

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It follows that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\||\sigma_{h,\Omega,\Gamma,t}|*g_{j,\zeta,k}|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$

$$\leq C\|\Omega\|_{L^1(\mathrm{S}^{n-1})}^{1/2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\sigma_{|h|^{2-\gamma},\Omega,\Gamma,t}|*|g_{j,\zeta,k}|^2\frac{dt}{t}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}. \quad (2.9)$$

Note that $|h|^{2-\gamma}\in\Delta_{\gamma/(2-\gamma)}(\mathbb{R}_+)$. Using (2.9) and (2.8) with γ,p,q,r replaced by $\gamma/(2-\gamma),p/2,q/2,r/2$, respectively, we have (2.3) for $(1/p,1/q,1/r)$ belonging to the interior of the cube $(\frac{1}{2}-1/\gamma',\frac{1}{2})^3$. By duality we see that (2.3) holds for $(1/p,1/q,1/r)$ belonging to the interior of the cube $(\frac{1}{2},\frac{1}{2}+1/\gamma')^3$. Interpolating these two cases, we know that (2.3) holds for $(1/p,1/q,1/r)$ belonging to the interior of the convex hull of two cubes $(\frac{1}{2}-1/\gamma',\frac{1}{2})^3$ and $(\frac{1}{2},\frac{1}{2}+1/\gamma')^3$. Note that in this case the interior of the cubes $(1/2\gamma,1-1/2\gamma)^3$ is contained in the interior of the convex hull of two cubes $(\frac{1}{2}-1/\gamma',\frac{1}{2})^3$ and $(\frac{1}{2},\frac{1}{2}+1/\gamma')^3$.

Case 3 ($\gamma\geq 2$). Note that $\Delta_\gamma(\mathbb{R}_+)\subset\Delta_2(\mathbb{R}_+)$ for $\gamma\geq 2$. Interpolation between cases 1 and 2 give us that (2.3) holds for $(1/p,1/q,1/r)$ belonging to the interior of the convex hull of three cubes $(1/2\gamma,1-1/2\gamma)^3$, $(0,\frac{1}{2})^3$ and $(\frac{1}{2},1)^3$. This completes the proof of Lemma 2.3. □

The following lemma gives some useful characterizations of Triebel–Lizorkin spaces and Besov spaces, which come from [29].

LEMMA 2.4 [29].

(i) *Let* $0<\alpha<1$, $1<p<\infty$, $1<q\leq\infty$ *and* $1\leq r<\min\{p,q\}$. *Then*

$$\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^d)}\approx\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_d}|\Delta_{2^{-l}\zeta}(f)|^r\,d\zeta\Big)^{q/r}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.$$

(ii) *Let* $0<\alpha<1$, $1\leq p<\infty$, $1\leq q\leq\infty$ *and* $1\leq r\leq p$. *Then*

$$\|f\|_{\dot{B}_\alpha^{p,q}(\mathbb{R}^d)}\approx\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big\|\Big(\int_{\mathfrak{R}_d}|\Delta_{2^{-l}\zeta}(f)|^r\,d\zeta\Big)^{1/r}\Big\|_{L^p(\mathbb{R}^d)}^q\Big)^{1/q}.$$

We end this section by presenting the following lemma, which is the heart of our proofs.

LEMMA 2.5. *Let* $v\geq 1$, $\Lambda\in\mathbb{N}\backslash\{0\}$ *and* $\{\sigma_{s,t}:t\in\mathbb{R}_+,1\leq s\leq\Lambda\}$ *be a family of Borel measures on* $\mathbb{R}^d$. *We also denote by* $|\sigma_{s,t}|$ *the total variation of* $\sigma_{s,t}$. *For* $1\leq s\leq\Lambda$, *let* $\delta_s,\beta_s>0$, $M_s\in\mathbb{N}\backslash\{0\}$ *and* $L_s:\mathbb{R}^d\to\mathbb{R}^{M_s}$ *be linear transformations. Suppose that* $\varphi\in\mathfrak{F}$ *and there exist* $p_0,q_0>1$, $1<r_0<\min\{p_0,q_0\}$ *and* $C,A>0$ *independent of* v *such that the following conditions are satisfied for* $1\leq s\leq\Lambda$, $t\in\mathbb{R}_+$, $\xi\in\mathbb{R}^d$ *and* $\{g_{l,\zeta,k}\}_{l,\zeta,k}\in L^{p_0}(\mathbb{R}^d,\ell^{q_0}(L^{r_0}(\ell^2)))$:

(i) $\sigma_{0,t}=0$;

(ii) $|\widehat{\sigma_{s,t}}(\xi)-\widehat{\sigma_{s-1,t}}(\xi)|\leq CA\varphi(t)^{\delta_s}|L_s(\xi)|$;

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(iii) $|\widehat{\sigma_{s,t}}(\xi)| \le CA\min\{1, \varphi(t)^{\delta_s}|L_s(\xi)|\}^{-\beta_s/\nu}$;

(iv)

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\Re_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k\nu}}^{2^{(k+1)\nu}}\||\sigma_{s,t}|*g_{l,\zeta,k}|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)} \le CA\nu^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^{r_0}(\Re_d)}^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)}.$$

Then for $\alpha \in (0,1)$ and $(1/p, 1/q) \in L_1L_2\backslash\{(1/p_0, 1/q_0)\}$, there exists a constant $C > 0$ independent of A and ν such that

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\int_0^\infty|\sigma_{\Lambda,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \le CA\nu^{1/2}\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)}, \tag{2.10}$$

where L_1L_2 is the line segment from L_1 to L_2 with $L_1 = (\frac{1}{2}, \frac{1}{2})$ and $L_2 = (1/p_0, 1/q_0)$.

Proof. For any $1 \le s \le \Lambda$, let $l_s = \mathrm{rank}(L_s) \le \min\{d, M_s\}$. By [12, Lemma 6.1], there are two nonsingular linear transformations $H_s : \mathbb{R}^{l_s} \to \mathbb{R}^{l_s}$ and $G_s : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$|H_s\pi_{l_s}^d G_s\xi| \le |L_s(\xi)| \le M_s|H_s\pi_{l_s}^d G_s\xi|, \tag{2.11}$$

where $\xi \in \mathbb{R}^d$ and $\pi_{l_s}^d$ is a projection operator from $\mathbb{R}^d$ to $\mathbb{R}^{l_s}$. We can choose a function $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi(t) \equiv 1$ for $|t| \le 1/2$ and $\psi(t) \equiv 0$ for $|t| > 1$. For $1 \le s \le \Lambda$, we define the family of measures $\{\tau_{s,t}\}_{t\in\mathbb{R}_+}$ by

$$\widehat{\tau_{s,t}}(\xi) = \widehat{\sigma_{s,t}}(\xi)\prod_{j=s+1}^{\Lambda}\psi(|\varphi(t)^{\delta_j}H_j\pi_{l_j}^d G_j\xi|) - \widehat{\sigma_{s-1,t}}(\xi)\prod_{j=s}^{\Lambda}\psi(|\varphi(t)^{\delta_j}H_j\pi_{l_j}^d G_j\xi|). \tag{2.12}$$

Equation (2.12) together with our assumption $\sigma_{0,t} = 0$ implies that

$$\sigma_{\Lambda,t} = \sum_{s=1}^{\Lambda}\tau_{s,t}.$$

It follows that

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\int_0^\infty|\sigma_{\Lambda,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \le \sum_{s=1}^{\Lambda}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\int_0^\infty|\tau_{s,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.$$

Therefore, to prove (2.10), it suffices to show that

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\int_0^\infty|\tau_{s,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \le CA\nu^{1/2}\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)} \tag{2.13}$$

for any $1 \le s \le \Lambda$, $\alpha \in (0,1)$ and $(1/p, 1/q) \in L_1L_2\backslash\{(1/p_0, 1/q_0)\}$, where $C > 0$ is independent of A, ν.

Next we prove (2.13). Fix $0 < \alpha < 1$. By straightforward calculations, and our assumptions (i), (ii), (2.11) and (2.12), we obtain that for any $1 \le s \le \Lambda$,

$$|\widehat{\tau_{s,t}}(\xi)| \le CA\min\{1, \varphi(t)^{\delta_s}|L_s(\xi)|, (\varphi(t)^{\delta_s}|L_s(\xi)|)^{-\beta_s}\}^{1/\nu}. \tag{2.14}$$

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Since $\varphi \in \mathfrak{F}$, by (ii) of Remark 1.3 we obtain that there exists $B_\varphi > 1$ such that $\varphi(2t) \geq B_\varphi \varphi(t)$ for all $t > 0$. Fix $1 \leq s \leq \Lambda$. Let $\eta_0 \in C^\infty(\mathbb{R})$ be an even function satisfying $0 \leq \eta_0(t) \leq 1$, $\eta_0(0) = 1$ and $\eta_0(t) = 0$ for $|t| \geq 1$. Set $\eta(\xi) = 1$ for $|\xi| \leq 1$, $\eta(\xi) = \eta_0((|\xi| - 1)/(a - 1))$, where $a = B_\varphi^{\nu\delta_s} > 1$. Then, η satisfies $\chi_{|\xi|\leq 1}(\xi) \leq \eta(\xi) \leq \chi_{|\xi|\leq a}(\xi)$ and $|\partial^\alpha \eta(\xi)| \leq c_\alpha (a - 1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$, where c_α is independent of a. Let $a_k = \varphi(2^{-k\nu})^{-\delta_s}$. Define the sequence of functions $\{\psi_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^d$ by

$$\psi_k(\xi) = \eta(a_{k+1}^{-1}\xi) - \eta(a_k^{-1}\xi), \quad \xi \in \mathbb{R}^d.$$

Observe that:

(a) $\operatorname{supp}(\psi_k) \subset \{a_k \leq |\xi| \leq a a_{k+1}\}$;
(b) $\operatorname{supp}(\psi_k) \cap \operatorname{supp}(\psi_j) = \emptyset$ for $|j - k| \geq 2$;
(c) $\sum_{k\in\mathbb{Z}} \psi_k(\xi) = 1$ for every $\xi \in \mathbb{R}^d \backslash \{0\}$.

Define the function Ψ_k by $\widehat{\Psi_k}(\xi) = \psi_k(\xi)$. It was shown in [29] that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_k * f_{j,\zeta}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq C\Big(\frac{B_\varphi^{\nu\delta_s}}{B_\varphi^{\nu\delta_s} - 1}\Big)^{d+2}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}. \tag{2.15}$$

Since ψ_k is radial, we shall use the convention $\psi_k(|\xi|) = \psi_k(\xi)$ for $\zeta \in \mathbb{R}^d$. Define the multiplier operator $S_{k,s}$ on $\mathbb{R}^d$ by

$$\widehat{S_{k,s}f}(\xi) = \psi_k(|H_s \pi_{l_s}^d G_s \xi|)\hat{f}(\xi).$$

We shall prove

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{k,s}f_{j,\zeta}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \leq C\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{2.16}$$

for $1 < p, q, r < \infty$, where $C > 0$ depends only on φ and d. Let G_s^{-1} and H_s^{-1} denote the inverse transforms of linear transformations G_s and H_s, respectively. Define U_s by $U_s = G_s^{-1}(H_s^{-1} \otimes \delta_{\mathbb{R}^{d-l_s}})$, where $\delta_{\mathbb{R}^{d-l_s}}$ denotes the Dirac delta function on $\mathbb{R}^{d-l_s}$. Obviously, U_s is a nonsingular linear transformation on $\mathbb{R}^d$. Let $y = (y^1, y^2)$ with $y^1 = (y_1, y_2, \ldots, y_{l_s})$ and $y^2 = (y_{l_s+1}, y_{l_s+2}, \ldots, y_d)$. One can easily check that

$$S_{k,s}f(x) = |U_s|\Psi_k \otimes \delta_{\mathbb{R}^{d-l_s}} * f^{U_s}(U_s^t x), \tag{2.17}$$

where $f^{U_s}(\xi) = |U_s|^{-1} f((U_s^t)^{-1}\xi)$ and U_s^t denotes the transpose of U_s. It follows from (2.15) and (2.17) that

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{k,s}f_{j,\zeta}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^p \\
&\quad \leq \int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}\||U_s|\Psi_k \otimes \delta_{\mathbb{R}^{d-l_s}} * f_{j,\zeta}^{U_s}(U_s^t x)|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{p/q}\,dx \\
&\quad = |U_s|^{p-1}\int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|\Psi_k \otimes \delta_{\mathbb{R}^{d-l_s}} * f_{j,\zeta}^{U_s}(y)|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{p/q}\,dy
\end{aligned}$$

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$$= |U_s|^{p-1}\int_{\mathbb{R}^{d-l_s}}\int_{\mathbb{R}^{l_s}}\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|[\Psi_k * f_{j,\zeta}^{U_s}(\cdot,y^2)](y^1)|^2\Big)^{1/2}\Big\|_{L^r(\Re_d)}^q\Big)^{p/q}dy^1\,dy^2$$
$$\le C|U_s|^{p-1}\Big(\frac{B_\varphi^{\nu\delta_s}}{B_\varphi^{\nu\delta_s}-1}\Big)^{p(d+2)}\int_{\mathbb{R}^d}\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}^{U_s}(y)\|_{L^r(\Re_d)}^q\Big)^{p/q}dy$$
$$\le C\Big(\frac{B_\varphi^{\delta_s}}{B_\varphi^{\delta_s}-1}\Big)^{p(d+2)}\Big\|\Big(\sum_{j\in\mathbb{Z}}\|f_{j,\zeta}\|_{L^r(\Re_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^p$$

for all $1 < p, q, r < \infty$, which yields (2.16).

We get by Minkowski's inequality that

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\int_0^\infty|\tau_{s,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$
$$=\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k\nu}}^{2^{(k+1)\nu}}\Big|\tau_{s,t}*\sum_{j\in\mathbb{Z}}S_{j-k,s}\Delta_{2^{-l}\zeta}(f)\Big|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}$$
$$\le\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k\nu}}^{2^{(k+1)\nu}}|\tau_{s,t}*S_{j-k,s}\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.$$

Define the mixed norm $\|\cdot\|_{E_\alpha^{p,q}}$ for measurable functions on $\mathbb{R}^d\times\Re_d\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{R}_+$ by

$$\|g\|_{E_\alpha^{p,q}} := \Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\sum_{k\in\mathbb{Z}}\int_0^\infty|g(x,\zeta,l,k,t)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}.$$

For any $j\in\mathbb{Z}$, let

$$V_{j,s}(f)(x,\zeta,l,k,t) := \tau_{s,t}*S_{j-k,s}\Delta_{2^{-l}\zeta}(f)(x)\chi_{[2^{k\nu},2^{(k+1)\nu})}(t).$$

Thus we have

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\Re_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k\nu}}^{2^{(k+1)\nu}}|\tau_{s,t}*\Delta_{2^{-l}\zeta}(f)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}\le\sum_{j\in\mathbb{Z}}\|V_{j,s}(f)\|_{E_\alpha^{p,q}}. \tag{2.18}$$

We want to show that there exists a constant $C > 0$ independent of ν such that

$$\|V_{j,s}(f)\|_{E_\alpha^{p_0,q_0}}\le CA\nu^{1/2}\|f\|_{\dot F_\alpha^{p_0,q_0}(\mathbb{R}^d)} \tag{2.19}$$

and

$$\|V_{j,s}(f)\|_{E_\alpha^{2,2}}\le CA\nu^{1/2}B_\varphi^{-c|j|}\|f\|_{\dot F_\alpha^{2,2}(\mathbb{R}^d)}, \tag{2.20}$$

where $c > 0$ is independent of ν. In fact, by interpolating between (2.19) and (2.20) we have that for $(1/p, 1/q)\in L_1L_2\backslash\{(1/p_0, 1/q_0)\}$, there exists $\theta\in(0,1]$ such that

$$\|V_{j,s}(f)\|_{E_\alpha^{p,q}}\le CA\nu^{1/2}B_\varphi^{-c\theta|j|}\|f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^d)}, \tag{2.21}$$

where $C > 0$ is independent of ν. Inequality (2.21) together with (2.18) yields (2.13).

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Below we shall prove (2.19) and (2.20). For $1 \le s \le \Lambda$, let Φ^s be a radial function in $\mathcal{S}(\mathbb{R}^{l_s})$ defined by $\widehat{\Phi^s}(x) = \psi(|x|)$, where $x \in \mathbb{R}^{l_s}$ and ψ is as in (2.12). Define J_s and X_s by

$$J_s f(x) := f(G_s^t(H_s^t \otimes \mathrm{id}_{\mathbb{R}^{d-l_s}})x) \quad \text{and} \quad X_s f(x) = \sup_{k\in\mathbb{Z}} \sup_{t\in[2^{kv},2^{(k+1)v}]} |X_{k,t;s} f(x)|,$$

where

$$X_{k,t;s} f(x) = J_s^{-1}((\Phi_{k,t;s} \otimes \delta_{\mathbb{R}^{d-l_s}}) * J_s f)(x) \quad \text{and} \quad \Phi_{k,t;s}(x^0) = (\varphi(t)^{\gamma_s})^{-l_s} \Phi^s(\varphi(t)^{-\gamma_s} x^0),$$

where $x^0 \in \mathbb{R}^{l_s}$. One can easily check that

$$|X_s f(x)| \le C_s[J_s^{-1} \circ (\mathrm{M}_{(l_s)} \otimes \mathrm{id}_{\mathbb{R}^{d-l_s}}) \circ J_s](f)(x), \tag{2.22}$$

where $x = (x^0, x^1) \in \mathbb{R}^{l_s} \times \mathbb{R}^{d-l_s}$. Combining (2.22) with (i) of Lemma 2.2 yields that

$$\begin{aligned}
&\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}} |X_s g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^p \\
&\quad\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}} |[J_s^{-1} \circ (\mathrm{M}_{(l_s)} \otimes \mathrm{id}_{\mathbb{R}^{d-l_s}}) \circ J_s](g_{l,\zeta,k})|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^p \\
&\quad\le C|J_s| \int_{\mathbb{R}^{d-l_s}} \int_{\mathbb{R}^{l_s}} \Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}} |(\mathrm{M}_{(l_s)}[(J_s g_{l,\zeta,k}(\cdot, x^1))](x^0)^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{p/q} dx^0\, dx^1 \\
&\quad\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)}^p
\end{aligned} \tag{2.23}$$

for any $1 \le s \le \Lambda$ and $1 < p, q, r < \infty$. Define $X^s f = X_s \circ X_{s+1} \circ \cdots \circ X_\Lambda f$ for $1 \le s \le \Lambda$. We get from (2.23) that

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}} |X^s g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \le C\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}} |g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\mathfrak{R}_d)}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^d)} \tag{2.24}$$

for any $1 \le s \le \Lambda$ and $1 < p, q, r < \infty$. By the definition of $X_{k,t;s}$ and (2.12),

$$\tau_{s,t} * f = \sigma_{s,t} * (X_{k,t;s+1} \circ X_{k,t;s+2} \circ \cdots \circ X_{k,t;\Lambda} f) - \sigma_{s-1,t} * (X_{k,t;s} \circ X_{k,t;s+1} \circ \cdots \circ X_{k,t;\Lambda} f).$$

It follows that

$$\int_{2^{kv}}^{2^{(k+1)v}} |\tau_{1,t} * f|^2 \frac{dt}{t} \le \int_{2^{kv}}^{2^{(k+1)v}} \||\sigma_{1,t}| * X^2 f|^2 \frac{dt}{t} \tag{2.25}$$

and

$$\int_{2^{kv}}^{2^{(k+1)v}} |\tau_{s,t} * f|^2 \frac{dt}{t} \le 2\Big(\int_{2^{kv}}^{2^{(k+1)v}} \||\sigma_{s,t}| * X^{s+1} f|^2 \frac{dt}{t} + \int_{2^{kv}}^{2^{(k+1)v}} \||\sigma_{s-1,t}| * X^s f|^2 \frac{dt}{t}\Big) \tag{2.26}$$

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for any $2 \le s \le \Lambda$. From (2.24) to (2.26) and assumption (iv), one can get

$$\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*g_{l,\zeta,k}|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^{q_0}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)}$$
$$\le CA v^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{l,\zeta,k}|^2\Big)^{1/2}\Big\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)} \tag{2.27}$$

for $1 \le s \le \Lambda$. Inequality (2.27) together with (2.16) and (i) of Lemma 2.4 implies that

$$\|V_{j,s}(f)\|_{E^{p_0,q_0}_\alpha} \le CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq_0\alpha}\Big\|\Big(\sum_{k\in\mathbb{Z}}|S_{j-k,s}\triangle_{2^{-l}\zeta}(f)|^2\Big)^{1/2}\Big\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)}$$
$$\le CAv^{1/2}\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq_0\alpha}\|\triangle_{2^{-l}\zeta}(f)\|^{q_0}_{L^{r_0}(\mathfrak{R}_d)}\Big)^{1/q_0}\Big\|_{L^{p_0}(\mathbb{R}^d)}$$
$$\le CAv^{1/2}\|f\|_{\dot F^{p_0,q_0}_\alpha(\mathbb{R}^d)},$$

which gives (2.19).

On the other hand, by (2.14), Hölder's inequality, Minkowski's inequality, Fubini's theorem, Plancherel's theorem and (ii) of Lemma 2.4,

$$\|V_{j,s}(f)\|^2_{E^{2,2}_\alpha} = \int_{\mathbb{R}^d}\sum_{l\in\mathbb{Z}}2^{2l\alpha}\Big(\int_{\mathfrak{R}_d}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}|\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)(x)|^2\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^2dx$$
$$\le C\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}\int_{2^{kv}}^{2^{(k+1)v}}\int_{\mathbb{R}^d}|\tau_{s,t}*S_{j-k,s}\triangle_{2^{-l}\zeta}(f)(x)|^2\,dx\,\frac{dt}{t}\,d\zeta$$
$$\le C\sum_{l\in\mathbb{Z}}2^{2l\alpha}\int_{\mathfrak{R}_d}\sum_{k\in\mathbb{Z}}\int_{E_{j-k,s}}\int_{2^{kv}}^{2^{(k+1)v}}|\widehat{\tau_{s,t}}(x)|^2\frac{dt}{t}|\widehat{\triangle_{2^{-l}\zeta}(f)}(x)|^2\,dx\,d\zeta$$
$$\le CA^2vB_\varphi^{-2c|j|}\|f\|^2_{\dot B^{2,2}_\alpha(\mathbb{R}^d)}, \tag{2.28}$$

where $E_{j-k,s} = \{x\in\mathbb{R}^d : \varphi(2^{(k-j)v})^{-\delta_s} \le |H_s\pi^d_{l_s}G_s\xi| \le B_\varphi^{v\delta_s}\varphi(2^{(k-j-1)v})^{-\delta_s}\}$ and $C, c > 0$ are independent of v. Combining (2.28) with (1.3) yields (2.20) and finishes the proof of Lemma 2.5. □

3. Proof of Theorem 1.1

In this section we shall prove Theorem 1.1. Let $\mathscr{R}_\gamma$ be given as in Theorem 1.1 and $\sigma_{h,\Omega,\Gamma,t}$ and $|\sigma_{h,\Omega,\Gamma,t}|$ defined as in (2.1) and (2.2), respectively. Employing the notation in [2], let $E_0 = \{y'\in S^{n-1} : |\Omega(y')| < 2\}$ and $E_v = \{y'\in S^{n-1} : 2^v < |\Omega(y')| \le 2^{v+1}\}$ for $v\in\mathbb{N}\backslash\{0\}$. Let $N(\Omega) = \{v\in\mathbb{N}\backslash\{0\} : \sigma(E_v) > 2^{-4v}\}$ and $\Omega_0 = \Omega - \sum_{v\in N(\Omega)}\Omega_v$, where $\Omega_v = \Omega\chi_{E_v} - \sigma(S^{n-1})^{-1}\int_{E_v}\Omega(y')\,d\sigma(y')$. One can easily check that

$$\int_{S^{n-1}}\Omega_v(y')\,d\sigma(y') = 0;$$

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$$\|\Omega_\nu\|_{L^1(\mathrm{S}^{n-1})} \le C\|\Omega\|_{L^1(E_\nu)} \quad \forall \nu \in N(\Omega) \cup \{0\}; \tag{3.1}$$

$$\|\Omega_\nu\|_{L^2(\mathrm{S}^{n-1})} \le C2^{2\nu}\|\Omega\|_{L^1(E_\nu)} \quad \forall \nu \in N(\Omega) \cup \{0\}; \tag{3.2}$$

$$\sum_{\nu \in N(\Omega)\cup\{0\}} (\nu+1)^{1/2}\|\Omega\|_{L^1(E_\nu)} \le C\|\Omega\|_{L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})}; \tag{3.3}$$

$$\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f(x,y) \le \sum_{\nu \in N(\Omega)\cup\{0\}} \mathscr{M}_{h,\Omega_\nu,\Phi,\Psi,\rho} f(x,y) \quad \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^m. \tag{3.4}$$

Inequality (3.4) together with (1.9), Minkowski's inequality and (i) of Lemma 2.4 yields that

$$\begin{aligned}
&\|\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f\|_{\dot F_\alpha^{p,q}(\mathbb{R}^{d+m})} \\
&\quad \le C\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\Re_{d+m}} \triangle_{2^{-l}\zeta}(\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f)\, d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})} \\
&\quad \le C\Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\Re_{d+m}} \mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}(\triangle_{2^{-l}\zeta}(f))\, d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})} \\
&\quad \le C \sum_{\nu \in N(\Omega)\cup\{0\}} \Big\|\Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\Big(\int_{\Re_{d+m}} \mathscr{M}_{h,\Omega_\nu,\Phi,\Psi,\rho}(\triangle_{2^{-l}\zeta}(f))\, d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})}
\end{aligned} \tag{3.5}$$

for $\alpha \in (0,1)$ and $(p,q) \in (1,\infty)^2$.

Proof of Theorem 1.1. We shall prove Theorem 1.1 by considering the following two parts.

Part 1. Let $n = d$ and $\Phi(y) = P(\varphi(|y|)) \bigotimes y$ with $P = (P_1,\dots,P_n) \in (\mathscr{A}_1)^n$. There exist $\mathcal{N} \in \mathbb{N}$, some integers $0 = d_0 < d_1 < d_2 < \cdots < d_{N_1} = \max_{1\le i\le n} \deg(P_i)$ and $\{a_{i,j} : 1 \le i \le n, 0 \le j \le N_1\}$ such that $(a_{1,j},\dots,a_{n,j}) \ne (0,\dots,0)$ for all $1 \le j \le N_1$ and

$$(P_1(t),\dots,P_n(t)) = \Big(\sum_{j=0}^{N_1} a_{1,j}t^{d_j},\dots,\sum_{j=0}^{N_1} a_{n,j}t^{d_j}\Big).$$

For $0 \le s \le N_1$ and $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, define the linear transformation $\mathcal{L}_s : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ by $\mathcal{L}_s(x,y) = (a_{1,s}x_1,\dots,a_{n,s}x_n)$, where $x = (x_1,\dots,x_n)$. For any $0 \le s \le N_1$, let

$$\mathcal{P}_s(t,x) = \Big(\sum_{j=0}^{s} a_{1,j}t^{d_j}x_1,\dots,\sum_{j=0}^{s} a_{n,j}t^{d_j}x_n\Big).$$

For $t \in \mathbb{R}_+$, $\nu \in N(\Omega) \cup \{0\}$ and $0 \le s \le N_1$, we denote $\sigma_{s,t}^\nu$ by $\sigma_{h,\Omega_\nu,\Gamma_s,t}$ and $|\sigma_{s,t}^\nu|$ by $|\sigma_{h,\Omega_\nu,\Gamma_s,t}|$ with $\Gamma_s(y) = (\mathcal{P}_s(\varphi(|y|),y'),\Psi(\varphi(|y|)))$. By (1.6), one can easily check that

$$\sigma_{0,t}^\nu = 0. \tag{3.6}$$

We also verify easily that

$$|\widehat{\sigma_{s,t}^\nu}(\xi,\eta)| \le C\|\Omega_\nu\|_{L^1(\mathrm{S}^{n-1})} \le C\|\Omega\|_{L^1(E_\nu)}. \tag{3.7}$$

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By a change of variable, (3.1) and Hölder's inequality,

$$\begin{aligned}
&|\widehat{\sigma^{\nu}_{s,t}}(\xi,\eta) - \widehat{\sigma^{\nu}_{s-1,t}}(\xi,\eta)| \\
&\quad\le \int_{t/2<|y|\le t} |e^{-2\pi i\xi\cdot\mathcal{P}_s(\varphi(|y|),y')} - e^{-2\pi i\xi\cdot\mathcal{P}_{s-1}(\varphi(|y|),y')}| \frac{|h(|y|)\Omega_\nu(y')|}{|y|^n}\,dy \\
&\quad\le C\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}} |\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)\cdot y'||\Omega_\nu(y')|\,d\sigma(y')|h(t)|\,\frac{dt}{t} \\
&\quad\le C\|\Omega\|_{L^1(E_\nu)}|\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)|;
\end{aligned}\tag{3.8}$$

$$\begin{aligned}
|\widehat{\sigma^{\nu}_{s,t}}(\xi,\eta)| &= \Big|\frac{1}{t^{\rho}}\int_{t/2}^{t}\int_{\mathrm{S}^{n-1}}\Omega_\nu(y')e^{-2\pi i(\xi\cdot\mathcal{P}_s(\varphi(r),y')+\eta\cdot\Psi(\varphi(r)))}\,d\sigma(y')h(r)\,\frac{dr}{r^{1-\rho}}\Big| \\
&\le C\|h\|_{\Delta_\gamma(\mathbb{R}_+)}\Big(\int_{t/2}^{t}\Big|\int_{\mathrm{S}^{n-1}}\Omega_\nu(y')e^{-2\pi i(\xi\cdot\mathcal{P}_s(\varphi(r),y')+\eta\cdot\Psi(\varphi(r)))}\,d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'} \\
&\le C\Big(\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{\mathrm{S}^{n-1}}\Omega_\nu(y')e^{-2\pi i\xi\cdot\mathcal{P}_s(r,y')}\,d\sigma(y')\Big|^{\gamma'}\frac{dr}{\varphi'(\varphi^{-1}(r))\varphi^{-1}(r)}\Big)^{1/\gamma'} \\
&\le C\Big(\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{\mathrm{S}^{n-1}}\Omega_\nu(y')e^{-2\pi i\xi\cdot\mathcal{P}_s(r,y')}\,d\sigma(y')\Big|^{\gamma'}\frac{dr}{r}\Big)^{1/\gamma'} \\
&\le C\|\Omega_\nu\|_{L^1(\mathrm{S}^{n-1})}^{\max\{0,1-2/\gamma'\}}\Big(\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{\mathrm{S}^{n-1}}e^{-2\pi i\mathcal{P}_s(r,y')\cdot\xi}\Omega_\nu(y')\,d\sigma(y')\Big|^{2}\frac{dr}{r}\Big)^{\min\{2,\gamma'\}/2\gamma'}.
\end{aligned}\tag{3.9}$$

A change of variable together with (3.2), Lemma 2.1 and Hölder's inequality implies that

$$\begin{aligned}
&\int_{\varphi(t/2)}^{\varphi(t)}\Big|\int_{\mathrm{S}^{n-1}}e^{-2\pi i\mathcal{P}_s(r,y')\cdot\xi}\Omega_\nu(y')\,d\sigma(y')\Big|^2\frac{dr}{r} \\
&\quad\le \int_{c_\varphi^{-1}}^{1}\Big|\int_{\mathrm{S}^{n-1}}e^{-2\pi i\mathcal{P}_s(\varphi(t)r,y')\cdot\xi}\Omega_\nu(y')\,d\sigma(y')\Big|^2\frac{dr}{r} \\
&\quad= \int_{c_\varphi^{-1}}^{1}\iint_{(\mathrm{S}^{n-1})^2}e^{-2\pi i(\mathcal{P}_s(\varphi(t)r,y')-\mathcal{P}_s(\varphi(t)r,\theta))\cdot\xi}\Omega_\nu(y')\overline{\Omega_\nu(\theta)}\,d\sigma(y')\,d\sigma(\theta)\,\frac{dr}{r} \\
&\quad\le \iint_{(\mathrm{S}^{n-1})^2}\Big|\int_{c_\varphi^{-1}}^{1}e^{-2\pi i(\mathcal{P}_s(\varphi(t)r,y')-\mathcal{P}_s(\varphi(t)r,\theta))\cdot\xi}\,\frac{dr}{r}\Big||\Omega_\nu(y')\overline{\Omega_\nu(\theta)}|\,d\sigma(y')\,d\sigma(\theta) \\
&\quad\le \iint_{(\mathrm{S}^{n-1})^2}\min\{1,|\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)\cdot(y'-\theta)|^{-1/d_s}\}|\Omega_\nu(y')\overline{\Omega_\nu(\theta)}|\,d\sigma(y')\,d\sigma(\theta) \\
&\quad\le \|\Omega_\nu\|_{L^2(\mathrm{S}^{n-1})}^2|\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)|^{-1/4d_s} \\
&\qquad\times\Big(\iint_{(\mathrm{S}^{n-1})^2}|\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)\cdot(y'-\theta)|^{-1/2d_s}\}\,d\sigma(y')\,d\sigma(\theta)\Big)^{1/2} \\
&\quad\le C2^{4\nu}\|\Omega\|_{L^1(E_\nu)}^2|\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)|^{-1/4d_s},
\end{aligned}$$

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which together with (3.9) and (3.1) yields that

$$|\widehat{\sigma_{s,t}^{\nu}}(\xi,\eta)| \leq C\|\Omega\|_{L^1(E_\nu)} 2^{2\min\{2,\gamma'\}\nu/\gamma'} |\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)|^{-\min\{2,\gamma'\}/8d_s\gamma'}.$$

Combining this inequality with (3.7) yields that

$$|\widehat{\sigma_{s,t}^{\nu}}(\xi,\eta)| \leq C\|\Omega\|_{L^1(E_\nu)} \min\{1, |\varphi(t)^{d_s}\mathcal{L}_s(\xi,\eta)|\}^{-\min\{2,\gamma'\}/8d_s\gamma'(\nu+1)}. \tag{3.10}$$

On the other hand, by Lemma 2.3, (3.1) and Hölder's inequality,

$$\begin{aligned}
&\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\bigg(\int_{\mathfrak{R}_{d+m}}\bigg(\sum_{k\in\mathbb{Z}}\int_{2^{k(\nu+1)}}^{2^{(k+1)(\nu+1)}} \||\sigma_{s,t}^{\nu}| * g_{j,\zeta,k}|^2 \frac{dt}{t}\bigg)^{1/2} d\zeta\bigg)^{q}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\leq \bigg\|\bigg(\sum_{j\in\mathbb{Z}}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}\int_{2^{k(\nu+1)}}^{2^{(k+1)(\nu+1)}} \||\sigma_{s,t}^{\nu}| * g_{j,\zeta,k}|^2 \frac{dt}{t}\bigg)^{1/2}\bigg\|_{L^r(\mathfrak{R}_{d+m})}^{q}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\leq C(\nu+1)^{1/2}\|\Omega\|_{L^1(E_\nu)}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\bigg\|\bigg(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\bigg)^{1/2}\bigg\|_{L^r(\mathfrak{R}_{d+m})}^{q}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^{d+m})}
\end{aligned} \tag{3.11}$$

for $1 \leq s \leq N_1$ and $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of three cubes $(\frac{1}{2}, \frac{1}{2} + 1/\max\{2,\gamma'\})^3$, $(1/2 - 1/\max\{2,\gamma'\}, \frac{1}{2})^3$ and $(1/2\gamma, 1-1/2\gamma)^3$. Then by (3.6), (3.8), (3.10), (3.11) and Lemma 2.5 we have

$$\begin{aligned}
&\bigg\|\bigg(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\bigg(\int_{\mathfrak{R}_{n+m}}\bigg(\int_0^\infty |\sigma_{N_1,t}^{\nu} * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t}\bigg)^{1/2} d\zeta\bigg)^{q}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\leq C(\nu+1)^{1/2}\|\Omega\|_{L^1(E_\nu)}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})}
\end{aligned} \tag{3.12}$$

for $\alpha \in (0,1)$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$. We get by Minkowski's inequality that

$$\begin{aligned}
\mathscr{M}_{h,\Omega_\nu,\Phi,\Psi,\rho} f(x,y) &= \bigg(\int_0^\infty \bigg|\sum_{k=-\infty}^{0} 2^{k\rho}\sigma_{N_1,2^k t}^{\nu} * f(x,y)\bigg|^2 \frac{dt}{t}\bigg)^{1/2}\\
&\leq \sum_{k=-\infty}^{0} 2^{k\tau}\bigg(\int_0^\infty |\sigma_{N_1,2^k t}^{\nu} * f(x,y)|^2 \frac{dt}{t}\bigg)^{1/2}\\
&\leq \frac{1}{1-2^{-\tau}}\bigg(\int_0^\infty |\sigma_{N_1,t}^{\nu} * f(x,y)|^2 \frac{dt}{t}\bigg)^{1/2}.
\end{aligned} \tag{3.13}$$

Inequality (3.13) together with (3.3), (3.5) and (3.12) yields that

$$\begin{aligned}
&\|\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})}\\
&\quad\leq C\sum_{\nu\in N(\Omega)\cup\{0\}}\bigg\|\bigg(\sum_{l\in\mathbb{Z}} 2^{lq\alpha}\bigg(\int_{\mathfrak{R}_{d+m}}\bigg(\int_0^\infty |\sigma_{N_1,t}^{\nu} * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t}\bigg)^{1/2} d\zeta\bigg)^{q}\bigg)^{1/q}\bigg\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\leq C\sum_{\nu\in N(\Omega)\cup\{0\}}(\nu+1)^{1/2}\|\Omega\|_{L^1(E_\nu)}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})}\\
&\quad\leq C\|\Omega\|_{L(\log^+ L)^{1/2}(\mathrm{S}^{n-1})}\|f\|_{\dot{F}_\alpha^{p,q}(\mathbb{R}^{d+m})}
\end{aligned} \tag{3.14}$$

for $\alpha \in (0,1)$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$. This proves part (i) of Theorem 1.1.

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Part 2. Let $\Phi(y) = \mathcal{P}(\varphi(|y|)y')$ with $\mathcal{P} = (P_1, \ldots, P_d) \in (\mathscr{A}_n)^d$. Following from [12], there are $N_2 \in \mathbb{N}$, some integers $0 < l_1 < l_2 < \cdots < l_{N_2} \leq \max_{1\leq j\leq d} \deg(P_j)$ and polynomials $P_j^s \in V_{n,l_s}, R_j \in \mathscr{A}_1$ with $\deg(R_j) \leq \max_{1\leq j\leq d} \deg(P_j)$ for $1 \leq s \leq N_2, 1 \leq j \leq d$, such that

$$\Phi(x) = \sum_{s=1}^{\Lambda} \mathcal{P}^s(x) + \mathcal{R}(|x|),$$

where $\mathcal{P}^s = (P_1^s, \ldots, P_d^s)$ and $\mathcal{R} = (R_1, \ldots, R_d)$. For each $s \in \{1, \ldots, N_2\}$, there is at least one $j \in \{1, \ldots, d\}$ such that $P_j^s \neq 0$. For $j = 1, \ldots, d$ and $1 \leq s \leq \Lambda$, write

$$P_j^s(x) = \sum_{|\beta|=l_s} b_{sj\beta} x^\beta = \sum_{i=1}^{d(s)} b'_{sji} x^{\beta(s,i)},$$

where $d(s) = \dim(V_{n,l_s})$. For $1 \leq s \leq N_2$, define the linear transformation $\mathcal{I}_s : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^{d(s)}$ by $\mathcal{I}_s(\xi, \eta) = (\sum_{j=1}^d b'_{sj1}\xi_j, \ldots, \sum_{j=1}^d b'_{sjd(s)}\xi_j)$, where $\xi = (\xi_1, \ldots, \xi_d)$. For $0 \leq s \leq N_2$, we define $\mathcal{P}_s$ by

$$\mathcal{P}_s(x) = \mathcal{R}(|x|) + \sum_{u=1}^{s} \mathcal{P}^u(x).$$

For $t \in \mathbb{R}_+$, $\nu \in N(\Omega) \cup \{0\}$ and $0 \leq s \leq N_2$, we denote $\sigma_{s,t}^\nu$ by $\sigma_{h,\Omega_\nu,\Gamma_s,t}$ and $|\sigma_{s,t}^\nu|$ by $|\sigma_{h,\Omega_\nu,\Gamma_s,t}|$ with $\Gamma_s(y) = (\mathcal{P}_s(\varphi(|y|)y'), \Psi(\varphi(|y|)))$. Obviously,

$$|\widehat{\tau_{s,t}^\nu}(\xi, \eta)| \leq C\|\Omega_\nu\|_{L^1(\mathrm{S}^{n-1})} \leq C\|\Omega\|_{L^1(E_\nu)}. \tag{3.15}$$

We easily get by (1.6) that

$$\tau_{s,t}^\nu = 0. \tag{3.16}$$

By arguments similar to those for deriving (3.8) and (3.9),

$$|\widehat{\tau_{s,t}^\nu}(\xi, \eta) - \widehat{\tau_{s-1,t}^\nu}(\xi, \eta)| \leq C\|\Omega\|_{L^1(E_\nu)}|\varphi(t)^{l_s}\mathcal{I}_s(\xi, \eta)|, \tag{3.17}$$

$$|\widehat{\tau_{s,t}^\nu}(\xi, \eta)| \leq C\|\Omega_\nu\|_{L^1(\mathrm{S}^{n-1})}^{\max\{0,1-2/\gamma'\}}\bigg(\int_{\varphi(t/2)}^{\varphi(t)}\bigg|\int_{\mathrm{S}^{n-1}} e^{-2\pi i\mathcal{P}_s(ry')\cdot\xi}\Omega_\mu(y')\,d\sigma(y')\bigg|^2\frac{dr}{r}\bigg)^{\min\{2,\gamma'\}/2\gamma'}. \tag{3.18}$$

By an argument similar to that for [12, Corollary 4.3] with $\epsilon = (8l_\eta)^{-1}$ and $p = 2$, there exists $C > 0$ such that

$$\bigg(\int_1^{c_\varphi}\bigg|\int_{\mathrm{S}^{n-1}} \Omega_\nu(y')e^{-2\pi i\mathcal{P}_s(\varphi(t/2)ry')\cdot\xi}\,d\sigma(y')\bigg|^2\frac{dr}{r}\bigg)^{1/2} \leq C\|\Omega_\nu\|_{L^2(\mathrm{S}^{n-1})}|\varphi(t/2)^{l_s}L_s(\xi)|^{-1/8l_s}. \tag{3.19}$$

By a change of variable and the properties of φ, we get from (3.2) and (3.19) that

$$\begin{aligned}\int_{\varphi(t/2)}^{\varphi(t)}&\bigg|\int_{\mathrm{S}^{n-1}} e^{-2\pi i\mathcal{P}_s(ry')\cdot\xi}\Omega_\nu(y')\,d\sigma(y')\bigg|^2\frac{dr}{r}\\ &\leq \int_1^{c_\varphi}\bigg|\int_{\mathrm{S}^{n-1}} e^{-2\pi i\mathcal{P}_s(\varphi(t/2)ry')\cdot\xi}\Omega_\nu(y')\,d\sigma(y')\bigg|^2\frac{dr}{r}\\ &\leq C2^{2\nu}\|\Omega\|_{L^1(E_\nu)}|\varphi(t)^{l_s}\mathcal{I}_s(\xi, \eta)|^{-1/8l_s}.\end{aligned}$$

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This together with (3.1), (3.15) and (3.18) yields that

$$|\widehat{\tau_{s,t}^{\nu}}(\xi,\eta)| \leq C\|\Omega\|_{L^1(E_\nu)} \min\{1, |\varphi(t)^{l_s} \mathcal{I}_s(\xi,\eta)|\}^{-\min\{2,\gamma'\}/16 l_s \gamma'(\nu+1)}. \tag{3.20}$$

Using Lemma 2.3 and Hölder's inequality we obtain

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big(\int_{\Re_{d+m}}\Big(\sum_{k\in\mathbb{Z}}\int_{2^{k(\nu+1)}}^{2^{(k+1)(\nu+1)}} \||\tau_{s,t}^{\nu}| * g_{j,\zeta,k}|^2 \frac{dt}{t}\Big)^{1/2} d\zeta\Big)^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})} \\ \leq C(\nu+1)^{1/2}\|\Omega\|_{L^1(E_\mu)}\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_{j,\zeta,k}|^2\Big)^{1/2}\Big\|_{L^r(\Re_{d+m})}^q\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})} \end{aligned} \tag{3.21}$$

for $1 \leq s \leq N_2$ and $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of three cubes $(\frac{1}{2}, \frac{1}{2} + 1/\max\{2,\gamma'\})^3$, $(\frac{1}{2} - 1/\max\{2,\gamma'\}, \frac{1}{2})^3$ and $(1/2\gamma, 1 - 1/2\gamma)^3$. The rest of the proof follows from (3.16), (3.17), (3.20) and (3.21), and arguments similar to those used in deriving (3.14). This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\Omega \in H^1(\mathrm{S}^{n-1})$ satisfy (1.6). By the well-known atomic decomposition of Hardy space (see [6, 7]), there exist $\{c_\kappa\} \subset \mathbb{C}$ and H^1 regular atoms $\{\Omega_\kappa\}$ such that $\Omega = \sum_\kappa c_\kappa \Omega_\kappa$ and $\sum_\kappa |c_\kappa| \approx \|\Omega\|_{H^1(\mathrm{S}^{n-1})}$. Here each Ω_κ satisfies the following conditions: for some $\varepsilon \in \mathrm{S}^{n-1}$ and $\varsigma \in (0,2]$,

$$\begin{gathered}\mathrm{supp}(\Omega_\kappa) \subset \mathrm{S}^{n-1} \cap B(\varepsilon,\varsigma) \quad \text{where } B(\varepsilon,\varsigma) = \{y \in \mathbb{R}^n : |y - \varepsilon| < \varsigma\}; \\ \|\Omega_\kappa\|_{L^\infty(\mathrm{S}^{n-1})} \leq \varsigma^{-n+1}; \\ \int_{\mathrm{S}^{n-1}} \Omega_\kappa(y)\, d\sigma(y) = 0.\end{gathered}$$

Then by Minkowski's inequality we have

$$\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho} f(x,y) \leq \sum_\kappa \mathscr{M}_{h,\Omega_\kappa,\Phi,\Psi,\rho} f(x,y) \quad \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^m. \tag{4.1}$$

Without loss of generality we may assume that $\mathrm{supp}(\Omega_\kappa) \subset \mathrm{S}^{n-1} \cap B(\varepsilon,\varsigma)$ with $0 < \varsigma < 1/4$ and $\varepsilon = \mathbf{e} = (0,\dots,0,1)$. Below we give some notation, which is the same as in [12]. In what follows, we use $x = (\tilde{x}, x_n)$ with $\tilde{x} = (x_1,\dots,x_{n-1})$. Then there are $N_3 \in \mathbb{N}$, some integers $0 < l_1 < l_2 < \cdots < l_{N_3} \leq \max_{1\leq j\leq d} \deg(P_j)$ and polynomials $P_j^u \in V_{n,l_u}$, $R_j \in \mathscr{A}_1$ with $\deg(R_j) \leq \max_{1\leq j\leq d} \deg(P_j)$ for $1 \leq u \leq N_3$, $1 \leq j \leq d$, such that

$$\mathcal{P}(x) = \sum_{u=1}^{N_3} \mathcal{P}^u(x) + \mathcal{R}(|x|),$$

where $\mathcal{P}^u = (P_1^u, P_2^u, \dots, P_d^u)$ and $\mathcal{R} = (R_1, R_2, \dots, R_d)$. For $j = 1,\dots,d$, denote $P_j^u(x) = \sum_{|\beta|=l_u} b_{uj\beta} x^\beta$. For $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| = l$, we choose $\eta_{l,\alpha}(\cdot) \in \mathscr{A}_{n-1}$ such that

$$|x^\alpha - \eta_{l,\alpha}(\tilde{x})| \leq C\varsigma^{4(n-1)} \quad \text{for } x \in \mathrm{S}^{n-1} \cap B(\mathbf{e},\varsigma).$$

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For each $u \in \{1, \dots, N_3\}$, $j \in \{1, \dots, d\}$, we define $q_j^u \in \mathscr{A}_{n-1}$ by

$$q_j^u(\tilde{x}) = \sum_{|\beta|=l_u} b_{uj\beta}\eta_{l_u,\beta}(\tilde{x}),$$

and set $q^u(\tilde{x}) = (q_1^u(\tilde{x}), q_2^u(\tilde{x}), \dots, q_d^u(\tilde{x}))$. Fix each $u \in \{1, \dots, N_3\}$; there are positive integers $v(u), 0 < h_{u,1} < \cdots < h_{u,v(u)}$, and polynomials $\{W_{j\eta}^u : j = 1, \dots, d; \eta = 1, \dots, v(u)\} \subset \mathscr{A}_{n-1}$ such that:

(i) for $j \in \{1, \dots, d\}, \eta \in \{1, \dots, v(u)\}$, $W_{j\eta}^u(\cdot)$ is homogeneous of degree $h_{u,\eta}$;
(ii) for each $\eta \in \{1, \dots, v(u)\}$, there exists at least one $j \in \{1, \dots, d\}$ such that $W_{j\eta}^u \neq 0$;
(iii) for each $j \in \{1, \dots, d\}$, there is a $v_j^u \in \mathbb{R}$ such that $q_j^u(\tilde{x}) = \sum_{\eta=1}^{v(u)} W_{j\eta}^u(\tilde{x}) + v_j^u$.

For $u \in \{1, \dots, N_3\}$ and $\eta \in \{1, \dots, v(u)\}$, we define $\mathcal{R}^u(x)$ and $\mathcal{W}^{u,\eta}(\tilde{x})$ by

$$\mathcal{R}^u(x) = \mathcal{R}(|x|) + \sum_{u \le k \le \mathcal{N}} |x|^{l_k}(v_1^k, \dots, v_d^k) + \sum_{1 \le k \le u-1} Q^k(x),$$

and

$$\mathcal{W}^{u,\eta}(\tilde{x}) = (W_{1\eta}^u(\tilde{x}), \dots, W_{d\eta}^u(\tilde{x})).$$

Let $M(0) = 0$, $M(u) = \sum_{k=1}^u [v(k) + 1]$ for $1 \le u \le N_3$, and define $\Gamma_0, \Gamma_1, \dots, \Gamma_{M(N_3)}$ by

$$\Gamma_{M(u-1)+\theta}(x) = \mathcal{R}^u(x) + |x|^{l_u} \sum_{1 \le k \le \theta} \mathcal{W}^{u,k}\Big(\frac{\tilde{x}}{|x|}\Big)$$

for $1 \le u \le \mathcal{N}, 0 \le \theta \le M(u) - M(u-1)$ and $\Gamma_{M(m)}(x) = \Phi(x)$. Let $d(u) = \dim(V_{n,l_u})$. For each $u \in \{1, \dots, \mathcal{N}\}$, we write

$$\{\beta \in \mathbb{N}^n : |\beta| = l_u\} := \{\beta(u, 1), \dots, \beta(u, d(u))\}.$$

Hence we can write $P_j^u(x) = \sum_{s=1}^{d(u)} b'_{ujs} x^{\beta(u,s)}$, where $b'_{ujs} = b_{uj\beta(u,s)}$. Denote by $d(u, \eta)$ the number of distinct elements in $\{\varpi \in \mathbb{N}^{n-1} : |\varpi| = h_{u,\eta}\}$. For $1 \le u \le N_3$, $1 \le \eta \le v(u)$ and $1 \le j \le d$, write

$$\{\varpi : |\varpi| = h_{u,\eta}\} = \{\varpi(u, \eta, 1), \dots, \varpi(u, \eta, d(u, \eta))\},$$

and

$$W_{j\eta}^u(\tilde{x}) = \sum_{s=1}^{d(u,\eta)} w_{u,j,\eta,s} \tilde{x}^{\varpi(u,\eta,s)}.$$

For $1 \le u \le N_3$, we define $\Lambda_1, \dots, \Lambda_{M(N_3)} \in \mathbb{N}$ by

$$\Lambda_{M(u-1)+\theta} = \begin{cases} d(u, \theta) & \text{if } 1 \le \theta < M(u) - M(u-1), \\ d(u) & \text{if } \theta = M(u) - M(u-1). \end{cases}$$

Also, we define linear transformations $L_i : \mathbb{R}^d \to \mathbb{R}^{\Lambda_i}$ for $1 \le i \le M(N_3)$ by

$$L_{M(u-1)+\theta}(\xi) = \begin{cases} \Big(\sum_{j=1}^d w_{u,j,\theta,s}\xi_j, \dots, \sum_{j=1}^d w_{u,j,\theta,d(u,\theta)}\xi_j\Big) & \text{if } 1 \le \theta < M(u) - M(u-1), \\ \Big(\sum_{j=1}^d b'_{uj1}\xi_j, \dots, \sum_{j=1}^d b'_{ujd(u)}\xi_j\Big) & \text{if } \theta = M(u) - M(u-1). \end{cases}$$

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For $s = 1, \dots, M(N_3)$, we set

$$\begin{cases} l(s) = l_u, \quad \delta(s) = h_{u,\theta}, \\ \quad \gamma(s) = \dfrac{1}{4h_{u,\theta} l_u \gamma'} & \text{if } \theta = s - M(u-1) \in [1, M(u) - M(u-1)), \\ l(s) = l_u, \quad \delta(s) = 4l_u(n-1), \\ \quad \gamma(s) = \dfrac{1}{8 l_u \gamma'} & \text{if } s = M(u). \end{cases}$$

For $t \in \mathbb{R}_+$ and $0 \le s \le M(N_3)$, we denote $\omega^{\kappa}_{s,t}$ by $\sigma_{h,\Omega_\kappa,\Gamma,t}$ and $|\omega^{\kappa}_{s,t}|$ by $|\sigma_{h,\Omega_\kappa,\Gamma,t}|$ with $\Gamma(y) = (\Gamma_s(\varphi(|y|)y'), \Psi(\varphi(|y|)))$. One can easily get by (1.6) that

$$\omega^{\kappa}_{s,t} = 0. \tag{4.2}$$

We also easily obtain that

$$|\widehat{\omega^{\kappa}_{s,t}}(\xi,\eta)| \le C; \tag{4.3}$$

$$|\widehat{\omega^{\kappa}_{s,t}}(\xi,\eta) - \widehat{\omega^{\kappa}_{s-1,t}}(\xi,\eta)| \le C\varphi(t)^{l(s)} \varsigma^{\delta(s)} |L_s(\xi)|. \tag{4.4}$$

On the other hand, by a change of variable and the same argument as for [23, (3.2)],

$$\begin{aligned} |\widehat{\omega^{\kappa}_{s,t}}(\xi,\eta)| &= \Big| \frac{1}{t^\rho} \int_{t/2}^{t} \int_{S^{n-1}} \Omega(y') e^{-2\pi i(\xi\cdot\Gamma_s(\varphi(r)y') + \eta\cdot\Psi(\varphi(r)))}\, d\sigma(y') h(r) \frac{dr}{r^{1-\rho}} \Big| \\ &\le \int_{t/2}^{t} \Big| \int_{S^{n-1}} \Omega(y') e^{-2\pi i \xi\cdot\Gamma_s(\varphi(r)y')}\, d\sigma(y') \Big| |h(r)| \frac{dr}{r} \\ &\le C(\varphi(t)^{l(s)} \varsigma^{\delta(s)} |L_s(\xi)|)^{-\gamma(s)}. \end{aligned} \tag{4.5}$$

Using Lemma 2.3 with $v = 1$ and Hölder's inequality we have

$$\begin{aligned} &\Big\| \Big(\sum_{j\in\mathbb{Z}} \Big(\int_{\Re_{d+m}} \Big(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \| \omega^{\kappa}_{s,t}| * g_{j,\zeta,k}|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^{d+m})} \\ &\quad \le C \Big\| \Big(\sum_{j\in\mathbb{Z}} \Big\| \Big(\sum_{k\in\mathbb{Z}} |g_{j,\zeta,k}|^2 \Big)^{1/2} \Big\|^q_{L^r(\Re_{d+m})} \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^{d+m})} \end{aligned} \tag{4.6}$$

for all $1 \le s \le M(N_3)$ and $(1/p, 1/q, 1/r)$ belonging to the interior of the convex hull of three cubes $(\frac{1}{2}, \frac{1}{2} + 1/\max\{2,\gamma'\})^3$, $(\frac{1}{2} - 1/\max\{2,\gamma'\}, \frac{1}{2})^3$ and $(1/2\gamma, 1 - 1/2\gamma)^3$.

For $1 \le s \le M(N_3)$, define the linear transformation $\mathcal{L}_s : \mathbb{R}^{d+m} \to \mathbb{R}^{\Lambda_s}$ by $\mathcal{L}_s(\xi,\eta) = \varsigma^{\delta(s)} L_s(\xi)$. Then by (4.2) to (4.6) and Lemma 2.5 we have

$$\Big\| \Big(\sum_{l\in\mathbb{Z}} 2^{lq\alpha} \Big(\int_{\Re_{d+m}} \Big(\int_0^\infty |\omega^{\kappa}_{M(N_3),t} * \Delta_{2^{-l}\zeta}(f)|^2 \frac{dt}{t} \Big)^{1/2} d\zeta \Big)^q \Big)^{1/q} \Big\|_{L^p(\mathbb{R}^{d+m})} \le C \|f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^{d+m})} \tag{4.7}$$

for any $\alpha \in (0,1)$ and $(1/p, 1/q) \in \mathscr{R}_\gamma$. By an argument similar to that for getting (3.13),

$$\mathscr{M}_{h,\Omega_\kappa,\Phi,\Psi,\rho} f(x,y) \le \frac{1}{1 - 2^{-\tau}} \Big(\int_0^\infty |\omega^{\kappa}_{M(N_3),t} * f(x,y)|^2 \frac{dt}{t} \Big)^{1/2}. \tag{4.8}$$

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By (1.9), (4.1), (4.7), (4.8), (i) of Lemma 2.4 and Minkowski's inequality, we obtain

$$\begin{aligned}
&\|\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^{d+m})}\\
&\quad\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_{d+m}}\triangle_{2^{-l}\zeta}(\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}f)\,d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\le C\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_{d+m}}\mathscr{M}_{h,\Omega,\Phi,\Psi,\rho}(\triangle_{2^{-l}\zeta}(f))\,d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\le C\sum_{\kappa}|c_\kappa|\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_{d+m}}\mathscr{M}_{h,\Omega_\kappa,\Phi,\Psi,\rho}(\triangle_{2^{-l}\zeta}(f))\,d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\le C\sum_{\kappa}|c_\kappa|\Big\|\Big(\sum_{l\in\mathbb{Z}}2^{lq\alpha}\Big(\int_{\mathfrak{R}_{d+m}}\Big(\int_0^\infty|\omega^{\kappa}_{M(N_3),t}*\triangle_{2^{-l}\zeta}(f)|^2\,\frac{dt}{t}\Big)^{1/2}d\zeta\Big)^{q}\Big)^{1/q}\Big\|_{L^p(\mathbb{R}^{d+m})}\\
&\quad\le C\|\Omega\|_{H^1(\mathrm{S}^{n-1})}\|f\|_{\dot{F}^{p,q}_{\alpha}(\mathbb{R}^{d+m})}
\end{aligned}$$

for $\alpha\in(0,1)$ and $(1/p,1/q)\in\mathscr{R}_\gamma$. This finishes the proof of Theorem 1.2. □

Acknowledgement

The author would like to thank the referee for his/her invaluable comments and suggestions.

References

[1] A. Al-Salman, H. Al-Qassem, L. C. Cheng and Y. Pan, 'L^p bounds for the function of Marcinkiewicz', *Math. Res. Lett.* **9** (2002), 697–700.
[2] A. Al-Salman and Y. Pan, 'Singular integrals with rough kernels in $L\log^+L(\mathrm{S}^{n-1})$', *J. Lond. Math. Soc.* **66** (2002), 153–174.
[3] H. M. Al-Qassem and A. Al-Salman, 'A note on Marcinkiewicz integral operators', *J. Math. Anal. Appl.* **282** (2003), 698–710.
[4] H. M. Al-Qassem and Y. Pan, 'On certain estimates for Marcinkiewicz integrals and extrapolation', *Collect. Math.* **60** (2009), 123–145.
[5] J. Chen, D. Fan and Y. Pan, 'A note on a Marcinkiewicz integral operator', *Math. Nachr.* **227** (2001), 33–42.
[6] L. Colzani, 'Hardy spaces on spheres', PhD Thesis, Washington University in St. Louis, 1982.
[7] L. Colzani, M. Taibleson and G. Weiss, 'Maximal estimates for Cesàro and Riesz means on sphere', *Indiana Univ. Math. J.* **33** (1984), 873–889.
[8] Y. Ding, D. Fan and Y. Pan, 'L^p-boundedness of Marcinkiewicz integrals with Hardy space function kernel', *Acta Math. Sin. (Engl. Ser.)* **16** (2000), 593–600.
[9] Y. Ding, D. Fan and Y. Pan, 'On the L^p boundedness of Marcinkiewicz integrals', *Michigan Math. J.* **50** (2002), 17–26.
[10] Y. Ding, S. Lu and K. Yabuta, 'A problem on rough parametric Marcinkiewicz functions', *J. Aust. Math. Soc.* **72** (2002), 13–21.
[11] Y. Ding and Y. Pan, 'L^p bounds for Marcinkiewicz integrals', *Proc. Edinb. Math. Soc.* **46** (2003), 669–677.
[12] D. Fan and Y. Pan, 'Singular integral operators with rough kernels supported by subvarieties', *Amer. J. Math.* **119** (1997), 799–839.
[13] D. Fan and S. Sato, 'Remarks on Littlewood–Paley functions and singular integrals', *J. Math. Soc. Japan* **54** (2002), 565–585.

[14] M. Frazier, B. Jawerth and G. Weiss, *Littlewood–Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, 79 (American Mathematical Society, Providence, RI, 1991).
[15] L. Grafakos, *Classical and Modern Fourier Analysis* (Prentice Hall, Upper Saddle River, NJ, 2003).
[16] L. Grafakos and A. Stefanov, 'L^p bounds for singular integrals and maximal singular integrals with rough kernels', *Indiana Univ. Math. J.* **47** (1998), 455–469.
[17] F. Gürbüz, 'Parabolic sublinear operators with rough kernel generated by parabolic Calderón-Zygmund operators and parabolic local Campanato space estimates for their commutators on the parabolic generalized local Morrey spaces', *Open Math.* **14** (2016), 300–323.
[18] F. Gürbüz, 'Some estimates for generalized commutators of rough fractional maximal and integral operators on generalized weighted Morrey spaces', *Canad. Math. Bull.* **60**(1) (2017), 131–145.
[19] F. Liu, 'Integral operators of Marcinkiewicz type on Triebel–Lizorkin spaces', *Math. Nachr.* **291**(1) (2017), 75–96.
[20] F. Liu, 'Rough singular integrals associated to surfaces of revolution on Triebel–Lizorkin spaces', *Rocky Mountain J. Math.*, to appear.
[21] F. Liu, Z. Fu, Y. Zheng and Q. Yuan, 'A rough Marcinkiewicz integral along smooth curves', *J. Nonlinear Sci. Appl.* **9** (2016), 4450–4464.
[22] F. Liu and H. Wu, 'Multiple singular integrals and Marcinkiewicz integrals with mixed homogeneity along surfaces', *J. Inequal. Appl.* **2012** (2012), 1–23.
[23] F. Liu and H. Wu, 'On Marcinkiewicz integrals associated to compound mappings with rough kernels', *Acta Math. Sin. (Engl. Ser.)* **30** (2014), 1210–1230.
[24] F. Liu and H. Wu, 'L^p bounds for Marcinkiewicz integrals associated to homogeneous mappings', *Monatsh. Math.* **181**(4) (2016), 875–906.
[25] R. Ricci and E. M. Stein, 'Harmonic analysis on nilpotent groups and singular integrals I: oscillatory integrals', *J. Funct. Anal.* **73** (1987), 179–184.
[26] F. Ricci and G. Weiss, *A Characterization of* $H^1(S^{n-1})$, Proceedings of Symposia in Pure Mathematics, 35 (eds. S. Wainger and G. Weiss) (American Mathematical Society, Providence, RI, 1979), 289–294.
[27] H. Triebel, *Theory of Function Spaces*, Monographs in Mathematics, 78 (Birkhäser Verlag, Basel, 1983).
[28] H. Wu, 'L^p bounds for Marcinkiewicz integrals associates to surfaces of revolution', *J. Math. Anal. Appl.* **321** (2006), 811–827.
[29] K. Yabuta, 'Triebel–Lizorkin space boundedness of Marcinkiewicz integrals associated to surfaces', *Appl. Math. J. Chinese Univ. Ser.* B **30** (2015), 418–446.
[30] C. Zhang and J. Chen, 'Boundedness of g-functions on Triebel–Lizorkin spaces', *Taiwanese J. Math.* **13** (2009), 973–981.
[31] C. Zhang and J. Chen, 'Boundedness of Marcinkiewicz integral on Triebel–Lizorkin spaces', *Appl. Math. J. Chinese Univ. Ser.* B **25** (2010), 48–54.

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https://doi.org/10.1017/S1446788717000143 Published online by Cambridge University Press