Can. J. Math., Vol. XXIV, No. 6, 1972, pp. 1114–1121

# SOME PROPERTIES OF ANOSOV FLOWS

W. BYERS

**Introduction.** Anosov flows are a generalization of geodesic flows on the unit tangent bundles of compact manifolds of negative sectional curvature. They were introduced and are dealt with at length by Anosov in [**2**]. Moreover they form an important class of examples of flows satisfying Smale's axioms A and B (see [**15**]). In that paper Smale poses the problem of determining which manifolds admit Anosov flows. In this paper we obtain information about the fundamental groups of such manifolds. These generalize results which have been obtained for the fundamental groups of manifolds of negative curvature (see Preissmann [**13**], Byers [**6**]). In Theorem 9 we show that if the flow is semi-splitting (i.e. stable and unstable manifolds in the universal cover intersect in at most one orbit) then $\pi_1(M)$ cannot be abelian. If the flow satisfies the more restrictive Condition A (which is satisfied by most known examples of Anosov flows) then $\pi_1(M)$ cannot even have a centre. Finally under this same condition Theorem 12 shows that for certain subgroups of $\pi_1(M)$ solvability implies that they are infinite cyclic.

It is interesting to contrast the situation for Anosov flows with the one for Anosov diffeomorphisms. The work of Avez [**4**], Franks [**8**], and Newhouse [**11**] shows that all codimension one Anosov diffeomorphisms exist on the $n$-torus for some $n$ (although this is not true in general as shown by Smale's nontoral example [**15**]). In comparison, even in dimension three Anosov flows exist on infinitely many different manifolds. Theorem 9 indicates that Anosov flows are unlikely to exist on the $n$-torus and in fact there are no known examples of compact manifolds which admit both an Anosov diffeomorphism and an Anosov flow.

Let $M$ be a complete Riemannian manifold and $\phi_t : M \to M$ a flow on $M$ generated by the non-singular vector field $X$. The flow is said to be an *Anosov flow* if there is a (continuous) splitting of the tangent bundle

$$TM = E^s + E^u + \dot{X},$$

where $\dot{X}$ is the one-dimensional distribution generated by the flow, satisfying:

(i) the splitting is preserved by the induced flow on the tangent bundle $D\phi_t : TM \to TM$;

(ii) there exists constants $a$, $\lambda > 0$ such that for all $t > 0$,

$$||D\phi_t(v)|| \leqq ae^{-\lambda t}||v|| \text{ for } v \in E^s$$

Received October 20, 1971 and in revised form, November 5, 1971. This research constitutes part of a Ph.D. thesis written at the University of California in 1969, under the direction of Professor S. Smale.

and

$$||D\phi_t(v)|| \geqq ae^{\lambda t}||v|| \quad \text{for } v \in E^u.$$

If $M$ is compact then condition (ii) is satisfied for one metric if and only if it is satisfied for all metrices. We shall always take the manifold denoted by $M$ to be compact.

The most important tool for working with Anosov flows is the Stable Manifold Theorem. Proofs can be found in Anosov [**2**], Arnold and Avez [**3**] and in Hirsch, Pugh, and Shub [**8**].

THEOREM 1. *Let $\phi_t : M \to M$ be an Anosov flow. $E^s$ and $E^u$, the stable and unstable distributions, are integrable.*

Denote the leaf of the stable foliation which passes through the point $x \in M$ by $s(x)$ and the unstable leaf through $x$ by $u(x)$. Each of these leaves is diffeomorphic to the Euclidean space of the appropriate dimension. Moreover the foliations are invariant under the flow, i.e.,

$$\phi_t(s(x)) = s(\phi_t(x)) \quad \text{and} \quad \phi_t(u(x)) = u(\phi_t(x)) \quad \text{for} \quad x \in M.$$

For every $x \in M$ we define the *stable manifold* through $x$, $W^s(x)$, to be the union over all $t \in \mathbf{R}$ of the stable leaves $s(\phi_t(x))$. Similarly the unstable manifold through $x$, $W^u(x) = U\{\phi_t(s(x)) : t \in \mathbf{R}\}$. In this way we obtain two new foliations tangent to the distributions $E^s + \dot{X}$ and $E^u + \dot{X}$ respectively.

Now let $\pi : \bar{M} \to M$ be a universal cover for $M$ where $\bar{M}$ is endowed with the Riemannian metric induced from $M$. We shall always identify the fundamental group $\pi_1(M)$ with the group of covering transformations of this cover.

It is easy to see that an Anosov flow $\phi_t : M \to M$ lifts to an Anosov flow on the universal cover, $\bar{\phi}_t : \bar{M} \to \bar{M}$. In fact the inequalities mentioned in the definition are given by the same constants as those for $\phi_t$. All four foliations $\{u(x)\}$, $\{s(x)\}$, $\{W^u(x)\}$, $\{W^s(x)\}$ on $M$ lift to the corresponding foliations for $\bar{\phi}_t$ on $\bar{M}$ and we denote these by $\{\bar{u}(x)\}$, $\{\bar{s}(x)\}$, $\{\bar{W}^u(x)\}$, $\{\bar{W}^s(x)\}$ respectively. It is then straightforward to verify that:

LEMMA 2. *If $\alpha \in \pi_1(M)$ then $\alpha$ permutes the leaves of each lifted foliation and $\alpha \circ \bar{\phi}_t = \bar{\phi}_t \circ \alpha$ for $t \in \mathbf{R}$.*

Now we can define a complete metric on each stable leaf $\bar{s}(x)$ by setting $d(\bar{s}(x); y, z)$ equal to the infimum of the lengths of all piecewise smooth curves in $\bar{s}(x)$ which join $y$ to $z$. Similarly we define the metric $d(\bar{u}(x); \cdot, \cdot)$.

LEMMA 3. *If $s(x) = \bar{s}(y)$ then*

(i) $d(\bar{s}(\alpha x); \alpha x, \alpha y) = d(\bar{s}(x); x, y)$ *for* $\alpha \in \pi_1(M)$.

(ii) $d(\bar{s}(\bar{\phi}_t(x)); \bar{\phi}_t(x), \bar{\phi}_t(y)) \leqq ae^{-\lambda t}d(\bar{s}(x); x, y)$ *for* $t > 0$.

(iii) $d(\bar{u}(\bar{\phi}_t(x)); \bar{\phi}_t(x), \bar{\phi}_t(y)) \geqq ae^{\lambda t}d(\bar{u}(x); x, y)$ *for* $t > 0$.

*Proof.* (i) follows from Lemma 2 and the fact that $\alpha \in \pi_1(M)$ is an isometry. The proofs of (i) and (ii) are similar to the proofs of the analogous facts for Anosov diffeomorphisms which can be found in [**8**].

*Definition.* An Anosov flow is *semi-splitting* if for every pair of points $x, y \in \bar{M}$, $\bar{W}^u(x)$ intersects $\bar{W}^s(y)$ in at most one orbit of the flow $\bar{\Phi}_t : \bar{M} \to \bar{M}$. (It is easy to verify that this condition is equivalent to requiring that $W^u(x)$ intersects $\bar{s}(y)$ in at most one point or that $\bar{W}^s(x)$ intersects $\bar{u}(y)$ in at most one point for all $x, y \in \bar{M}$.)

To the best of my knowledge there are no known examples of Anosov flows which are not semi-splitting. Suspensions of known examples of Anosov diffeomorphisms are even splitting, i.e., $\bar{W}^u(x)$ intersects $\bar{W}^s(y)$ in exactly one orbit for all $x, y \in \bar{M}$. (cf. Franks [**7**, p. 81]). For a different class of examples we have:

PROPOSITION 4. *The geodesic flow on the unit tangent bundle of a manifold of negative curvature is semisplitting.*

*Proof.* Let $N$ be a compact manifold of negative sectional curvature and $\pi : \bar{N} \to N$ its universal cover. If $\dim N > 2$ then $D\pi : T_1\bar{N} \to T_1N$ is a universal cover of the unit tangent bundle $T_1N$. Even if $\dim N = 2$ this map is still a covering and so it suffices to show that stable and unstable manifolds in $T_1\bar{N}$ intersect in at most one orbit.

If we identify orbits of the geodesic flow on $T_1\bar{N}$ with oriented geodesics on $\bar{N}$ then it is well-known (see Arnold-Avez [**3**]) that the stable (unstable) manifolds are unions of geodesics which are positively (negatively) asymptotic to one another. Now if $\sigma$, $\tau$ are two oriented geodesics in $\bar{N}$ they are positively (negatively) asymptotic if and only if the convex function $t \mapsto d(\sigma, \tau(t))$ is bounded as $t \to +\infty$ $(t \to -\infty)$, where $d(\cdot\,, \cdot)$ is the Riemannian distance on $\bar{N}$ (Busemann [**5**]). If $\sigma$ and $\tau$ happen to be in the same stable and unstable manifold, the above convex function is bounded as $t \to \pm\infty$ and thus reduces to a constant. This is impossible since curvature of $\bar{N}$ is bounded from above by a constant less than zero [**4**, Lemma 10.7].

We also have semi-splitting in the following situation:

PROPOSITION 5. *Let $\phi_t : M \to M$ be an Anosov flow whose unstable manifolds (or stable manifolds), $\{W^u(x) : x \in M\}$, form a* $\mathbf{C}^2$ *foliation of the codimension one, i.e. the dimension of $W^u(x)$ is $(n - 1)$. Then the flow is semi-splitting.*

*Proof.* Assume that there exist points $x, y \in \bar{M}$ such that $\bar{W}^u(x) = \bar{W}^u(y)$ and $\bar{s}(x) = \bar{s}(y)$, and that $\bar{s}(x)$ is one dimensional. Let $h : [0, 2] \to \bar{s}(x)$ be a map such that $h|[0, 1]$ is a diffeomorphism onto the arc in $\bar{s}(y)$ joining $x$ and $y$ with $h(0) = x$ and $h(1) = y$ and $h|[1, 2]$ is a diffeomorphism onto an arc in $\bar{W}^u(x)$ joining $y$ to $x$. By constructing a finite number of (overlapping) coordinate neighbourhoods for the unstable foliation along the arc $h([1, 2])$ it is easy to see that the closed loop $h$ can be approximated arbitrarily closely by a closed loop which is transverse to the unstable foliation and passes through $\bar{W}^u(x)$ (see Novikov [**10**, pp. 269–271]).

Now $\bar{M}$ is simply connected, so the transverse loop is homotopic to a constant map. The following result of Haefliger is now relevant [**9**, p. 390]:

PROPOSITION. *Let $\mathscr{F}$ be a $\mathbf{C}^2$ doliation of codimension one on a manifold V. Suppose that there exists a transversal to $\mathscr{F}$ which is closed and homotopic to a constant. Then there exists a loop L on a leaf F such that the germ of the homeomorphism of $\mathbf{R}$ at 0 which corresponds to L in the holonomy group of F is not that of the identity but is the germ of a homeomorphism which is the identity on* $(-\infty, 0]$ *or* $[0, \infty)$.

*Note.* Definitions and basic ideals about foliations may be found in [**9**].

For the unstable foliation associated with an Anosov flow the unstable manifolds must either be simply connected or else have fundamental group isomorphic to the integers and exactly one closed orbit (see Abraham and Robbin [**1**, pp. 93–94]). In the former case the holonomy group is trivial. In the latter all loops are homotopic to powers of the closed orbit. In this case let $t_0$ be a point on the closed orbit which we assume to have period $t_0$. Now $\bar{s}(p)$ is transverse to the unstable foliation and by Lemma 3 $\bar{\phi}_{t_0}|\bar{s}(p)$ is a contraction. Thus either case contradicts Haefliger's proposition and this establishes the result.

The following are simple geometrical consequences of the semi-splitting condition:

PROPOSITION 6. *If $\phi_t : M \to M$ is semi-splitting then*
(1) *$\bar{W}^u(x)$ and $\bar{W}^s(x)$ are closed for all $x \in \bar{M}$.*
(2) *$\bar{W}^u(x)$ and $\bar{W}^s(x)$ are simply connected for all $x \in \bar{M}$.*

*Proof.* If $\bar{W}^u(x)$ is not closed it must accumulate on some unstable manifold $\bar{W}^u$ (Reeb [**12**]). Let $y \in \bar{W}^u$. Since $\bar{s}(y)$ is transverse to the unstable foliation it must intersect $\bar{W}^u(x)$ infinitely often, contradicting the semi-splitting assumption.

To prove (2) suppose $\bar{W}^u(x)$ contains a closed orbit $\theta$ of period $t_0$. Let $p \in \theta$ and $x \in \bar{s}(p)$, $x \neq p$. Then $\bar{\phi}_{t_0}(x) \in \bar{s}(p) \cap \bar{W}^u(x)$, where $\bar{\phi}_{t_0}(x) \neq x$ by Lemma 3. Again this contradicts the semi-splitting of $\bar{\phi}_t$.

LEMMA 7. *Let $\phi_t : M \to M$ be a semi-splitting Anosov flow and $\alpha \in \pi_1(M)$ a covering transformation which preserves the orbit of $\bar{\phi}_t : M \to M$ through the point $x_0$. If $A = \bigcup \{\bar{W}^u(x) : x \in \bar{s}(x_0)\}$ and $B = \bigcup \{\bar{W}^s(x) : x \in \bar{u}(x_0)\}$ then $\alpha$ preserves exactly one orbit of $\bar{\phi}_t$ in $A \cup B$.*

*Proof.* We prove the statement for $A$, the proof for $B$ being similar. $A$ is the union of a set of unstable manifolds. We define a metric on $A^u$, the set of unstable manifolds in $A$, by setting

$$d(\bar{W}_1^u, \bar{W}_2^u) = d(\bar{s}(x_0); \bar{W}_1^u \cap \bar{s}(x_u), \bar{W}_2^u \cap \bar{s}(x_u)).$$

This makes $A^u$ into a metric space isometric to $\bar{s}(x_0)$ and hence complete. Now suppose $\alpha(x_0) = \bar{\phi}_{t_0}(x_0)$ where $t_0 < 0$. Then $\alpha\bar{\phi}_{-t_0}(\bar{s}(x_0)) = \bar{s}(x_0)$ and $\alpha\bar{W}^u \cap \bar{s}(x_0) = \alpha\phi_{-t_0}(\bar{W}^u \cap \bar{s}(x_0))$. Thus $\alpha$ induces a map from $A^u$ into itself. Furthermore, using Lemma 3

$$\begin{aligned} d(\alpha^n\bar{W}_1{}^u, \alpha^n\bar{W}_2{}^u) &= d(\bar{s}(x_0); \alpha^n\bar{W}_1{}^u \cap \bar{s}(x_0), \alpha^n\bar{W}_2{}^u \cap \bar{s}(x_0)) \\ &= d(\bar{s}(x_0); \alpha^n\phi_{-n t_0}(\bar{W}_1{}^u \cap \bar{s}(x_0)), \alpha^n\phi_{-n t_0}(\bar{W}_2{}^u \cap \bar{s}(x_0)) \\ &= d(\bar{s}(\alpha^{-n}x_0); \phi_{-n t_0}(\bar{W}_1{}^u \cap \bar{s}(x_0)), \phi_{-n t_0}(\bar{W}_2{}^u \cap \bar{s}(x_0)) \\ &\leqq ae^{-\lambda n t_0} d(\bar{s}(x_0); \bar{W}_1{}^u \cap \bar{s}(x_0), \bar{W}_2{}^u \cap \bar{s}(x_0)) \\ &= ae^{-\lambda n t_0} d(\bar{W}_1{}^u, \bar{W}_2{}^u) \end{aligned}$$

for any points $\bar{W}_1{}^u$, $\bar{W}_2{}^u$ in $A^u$. Thus $\alpha$ is a contraction of $A^u$. By the contraction mapping theorem for complete metric spaces $\alpha$ has exactly one fixed point, i.e. it preserves exactly one unstable manifold, $\bar{W}^u(x_0)$, in $A$.

Since $\bar{W}^u(x_0)$ is simply connected it is not difficult to see that for $x, y \in \bar{W}^u(x_0)$, the orbit through $x$ intersects $\bar{u}(y)$ in exactly one point. We can now consider the space of orbits of $\bar{\phi}_t$ restricted to the unstable manifold $\bar{W}^u(x_0)$. $\alpha$ induces a map on this space and if we define a metric on the space by setting

$$d(\theta_1, \theta_2) = d(\bar{u}(x_0); \theta_1 \cap \bar{u}(x_0), \theta_2 \cap \bar{u}(x_0))$$

we can show as above that $\alpha^{-1}$ is a contraction and so it preserves exactly one orbit in $\bar{W}^u(x_0)$ and thus one orbit in $A$, namely, the orbit through $x_0$.

LEMMA 8 (Local Product Structure). *Let $\phi_t : M \to M$ be an Anosov flow. For every $x_0 \in M$ there exists a neighbourhood $U$ of $x_0$ such that for all $x, y \in U$ the connected components of $U \cap s(x)$ and $U \cap W^u(y)$ which contain $x$ and $y$ respectively intersect in exactly one point.*

*Proof.* The proof follows from the transversality of the foliations and the fact that the leaves of the foliations vary continuously (cf. Franks [**7**]).

THEOREM 9. *A compact manifold with abelian fundamental group does not admit any semi-splitting Anosov flows.*

*Proof.* Anosov [**2**, Theorem 9] proves that there exists a finite number of closed orbits of the flow, $\gamma_1, \ldots, \gamma_n$, the union of whose unstable manifolds is dense in $M$. We claim that one of these unstable manifolds must accumulate on itself. Assume that this does not happen and set $i_1 = 1$. Since $\bigcup_{j=1}^n W^u(\gamma_j)$ is dense there exists $i_2 \neq 1$ such that $W^u(\gamma_{i_2})$ accumulates on $W^u(\gamma_1)$. If $W^u(\gamma_1)$ in turn accumulates on $W^u(\gamma_{i_2})$ it must accumulate on itself. Thus there exists $i_3 \neq 1, i_2$ such that $W^u(\gamma_{i_3})$ accumulates on $W^u(\gamma_{i_2})$. Continuing in this way we arrange the integers less than or equal to $n$ in a sequence $\{i_k\}_{k=1}^n$. Consider $W^u(\gamma_{i_n})$. If it does not accumulate on itself there exists $m < n$ such that $W^u(\gamma_{i_m})$ accumulates on $W^u(\gamma_{i_n})$ which in turn accumulates on $W^u(\gamma_{i_m})$ by construction. Thus $W^u(\gamma_{i_m})$ accumulates on itself.

Now suppose $\gamma$ is the closed orbit such that $W^u(\gamma)$ accumulates on itself and let $x_0 \in \gamma$. Then if $\pi(\bar{x}_0) = x_0$ there exists $\alpha \in \pi_1(M)$, $\alpha \neq$ identity, such that $\alpha(\bar{W}^u(\bar{x}_0)) = \bar{W}^u(\bar{x}_0)$. Let $U$ be a product neighbourhood of $x_0$ which is evenly covered by the projection $\pi$ and lift $U$ homemorphically to a product neighbourhood $\bar{U}$ of $\bar{x}_0$. Since $W^u(x_0)$ accumulates on itself there exists a connected component $V$ of $U \cap W^u(x_0)$ such that $x_0 \notin V$. Let $\bar{V} = (\pi|\bar{U})^{-1}(V)$ and let $\bar{W}^u$ be the unstable manifold which contains $\bar{V}$. Since $\pi(\bar{W}^u) = \pi(\bar{W}^u(\bar{x}_0)) = W^u(x_0)$ there exists $\beta \in \pi_1(M)$ such that $\beta(\bar{W}^u(\bar{x}_0)) = \bar{W}^u$. If $\beta = \alpha^n$ then $\bar{W}^u(\bar{x}_0) = \bar{W}^u$. However since $\bar{U}$ is a product neighbourhood $\bar{s}(x_0)$ then intersects $\bar{W}^u(\bar{x}_0)$ at some point of $\bar{V}$ as well as at $\bar{x}_0$, contradicting the semi-splitting of the flow. Thus $\bar{W}^u \neq \bar{W}^u(x_0)$. Since $\pi_1(M)$ is abelian we have

$$\alpha(\bar{W}^u) = \alpha\beta(\bar{W}^u(\bar{x}_0)) = \beta\alpha(\bar{W}^u(\bar{x}_0)) = \beta(\bar{W}^u(\bar{x}_0)) = \bar{W}^u.$$

Thus $\alpha$ preserves two distinct unstable manifolds which both intersect $\bar{s}(x_0)$. This is impossible by Lemma 7 which proves the theorem.

If the Anosov flow under consideration is semi-splitting let

$$U(x_0) = U\{\bar{W}^u(x) : x \in \bar{s}(x_0)\}, \text{ for } x_0 \in \bar{M}.$$

Then the flow is splitting if $U(x) = M$ for all $x \in \bar{M}$. Since the negative curvature examples are not splitting we consider the following more general situation:

*Condition* A. Let $\phi_t : M \to M$ be a semi-splitting Anosov flow. Then for $x \in \bar{M}$, $\bar{M} - U(x)$ consists of at most one unstable manifold.

PROPOSITION 10. *Let $N$ be a compact manifold of negative sectional curvature and dimension greater than two. Then the geodesic flow on $T_1N$ satisfies Condition* A.

*Proof*. We make the same identifications as in Proposition 4 and consider the geodesic flow in $T_1\bar{N}$. Notice that in general

$$U(x_0) = U\{\bar{W}^u(x) : x \in \bar{W}^s(x_0)\}.$$

Now let $W^s$ be any unstable manifold made up of a family of positively asympototic (oriented) geodesics. Let $W^u(W^s)$ be the stable manifold obtained by reversing the orientations of all the geodesics in $W^u$. It is clear that $W^s \cap W^u(W^s) = \emptyset$.

Let $W^u$ be any unstable manifold. If $x \in \bar{N}$ there exist unique oriented geodesics $\gamma$ and $\delta$ which pass through $x$ and are contained in $W^s$ and $W^u$ respectively. If $\gamma$ and $\delta$ are the same geodesic with opposite orientations then $W^u = W^u(W^s)$ and $W^s \cap W^u = \emptyset$. However if $\gamma = \delta$ or $\gamma$ and $\delta$ are distinct (except for the point $x$) then it follows from Lemma 9.10 in [**5**] that there exists a geodesic $\sigma$ which is in both $W^u$ and $W^s$. Thus $W^s$ intersects every stable manifold except $W^s(W^u)$ in exactly one orbit of the geodesic flow.

THEOREM 11. *If the Anosov flow $\phi_t : M \to M$ satisfies Condition A then $\pi_1(M)$ has no center.*

*Proof.* There exists a closed orbit of the flow whose unstable manifold, $W_0^u$, accumulates on itself as in Theorem 9. Let $x_0$ be any point on this closed orbit and let $\bar{W}_0^u$ be an unstable manifold of $\bar{\phi}_t : \bar{M} \to \bar{M}$ such that $\pi(\bar{W}_0^u) = W_0^u$. If $\bar{x}_0 \in \bar{W}_0^u$ and $\pi(\bar{x}_0) = x_0$ then $M - U(\bar{x}_0)$ is either empty or a single unstable manifold, $\bar{W}_1^u$, because of Condition A. The same arguments as in Theorem 9 imply that there exist covering transformations $\alpha, \beta \in \pi_1(M)$ with $\beta \neq \alpha^n$ for any $n \in \mathbf{Z}$ such that $\alpha(\bar{W}_0^u) = \bar{W}_0^u$ and $\beta(\bar{W}_0^u)$ is unequal to either $\bar{W}_0^u$ or $\bar{W}_1^u$.

Notice that Condition A and Lemma 7 imply that any covering transformation can preserve at most two unstable manifolds, in fact at most two orbits of $\bar{\phi}_t$. In particular $\alpha$ preserves no unstable manifolds besides $\bar{W}_0^u$ and $\bar{W}_1^u$. Now let $\gamma$ be in the centre of $\pi_1(M)$. Then $\alpha\gamma(\bar{W}_0^u) = \gamma\alpha(\bar{W}_0^u) = \gamma(\bar{W}_0^u)$. By the previous remark $\gamma(\bar{W}_0^u)$ is either $\bar{W}_0^u$ or $\bar{W}_1^u$. In either case we must have $\gamma^2(\bar{W}_0^u) = \bar{W}_0^u$.

Now let $\sigma$ be an orbiting element of $\pi_1(M)$. Then as above $\gamma^2\sigma(\bar{W}_0^u) = \sigma(\bar{W}_0^u)$. Thus $\sigma(\bar{W}_0^u)$ must equal $\bar{W}_0^u$ or $\bar{W}_1^u$ for all $\sigma \in \pi_1(M)$. However the transformation $\beta$ mentioned above does not satisfy this condition and this contradiction establishes the theorem.

In the next result we consider covering transformations which preserve some orbit of the flow $\bar{\phi}_t : \bar{M} \to \bar{M}$, i.e. are generated by some closed orbit in $M$. We remark that for geodesic flows on the unit tangent bundle of a compact manifold (dimension greater than 2) of negative curvature, every covering transformation has this property.

THEOREM 12. *Let $\phi_t : M \to M$ be an Anosov flow satisfying Condition* A. *Suppose $H$ is a solvable subgroup of $\pi_1(M)$ each of whose elements preserves some orbit of the induced flow $\bar{\phi}_t : \bar{M} \to \bar{M}$. Then $H$ is infinite cyclic.*

*Proof.* Let $H = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_{k-1} \supset H_k = \{1\}$ be the derived series for $H$ where $H_i = [H_{i-1}, H_{i-1}]$. Let $\alpha$ and $\beta$ be arbitrary elements of $H_{k-1}$ and $\theta_0$ an orbit of $\bar{\phi}_t$ preserved by $\alpha$. Since $H_{k-1}$ is abelian we have $\alpha\beta(\theta_0) = \beta\alpha(\theta_0) = \beta(\theta_0)$. Now $\alpha$ can preserve at most two orbits, $\theta_0$ and $\theta_1$, say, by Condition A and Lemma 7. Thus $\beta(\theta_0) = \theta_i$, $i = 0$ or 1; and $\beta^2(\theta_i) = \theta_i$, $i = 0$ and 1. However since $\beta$ itself preserves some orbit of $\bar{\phi}_t$ and since $\beta^2$ preserves at most two orbits of $\bar{\phi}_t$ it follows that $\beta(\theta_i) = \theta_i$, $i = 0$ and 1. (Of course $\theta_1$ may not exist, in which case the argument is simpler.) Let $\gamma$ be the covering transformation which preserves $\theta_0$ and is generated by going around the closed orbit $\pi(\theta_0)$ exactly once. We have shown that $H_{k-1} \subseteq \{\gamma^n\}_{n\in\mathbf{Z}}$.

Now let $\delta$ be in $H_{k-2}$. Then $[\delta, \alpha^{-1}] = \delta^{-1}\alpha\delta\alpha^{-1}$ is in $H_{k-1}$ and so $\delta^{-1}\alpha\delta\alpha^{-1} = \gamma^n$ for some $n$. Thus $\delta^{-1}\alpha\delta(\theta_0) = \delta^{-1}\alpha\delta\alpha^{-1}(\theta_0) = \gamma^n(\theta_0) = \theta_0$ and $\alpha\delta(\theta_0) = \delta(\theta_0)$. Reasoning as above it follows that $\delta(\theta_0) = \theta_0$. Thus $G_{k-2} \subseteq \{\gamma^n\}_{n\in\mathbf{Z}}$. Proceeding in this way $H \subseteq \{\gamma^n\}_{n\in\mathbf{Z}}$ and thus $H$ is infinite cyclic.

*Added in proof.* J. F. Plante and W. P. Thurston have shown that Anosov flows of codimension one are only found on manifolds whose fundamental groups have exponential growth (*Anosov flows and the fundamental group*, Topology *11* (1972), 147–150).

REFERENCES

**1.** R. Abraham and J. Robbin, *Transversal mappings and flows* (Benjamin, New York, 1967).
**2.** D. V. Anosov, *Geodesic flows on closed Riemannian manifolds of negative curvature*, Proc. Steklov Inst. of Math. *90* (1967) (A.M.S. translation 1969).
**3.** V. I. Arnold and A. Avez, *Ergodic problems of classical mechanics* (Benjamin, New York, 1968).
**4.** A. Avez, *Anosov diffeomorphisms*, *Topological dynamics* (ed. Auslander, Gottschalk, Benjamin, 1968).
**5.** R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. *145* (1969), 1–49.
**6.** H. Busemann, *Spaces with non-positive curvature*, Acta Math. *80* (1948), 259–310.
**7.** W. Byers, *On a theorem of Preissmann*, Proc. Amer. Math. Soc. *24* (1970), 50–51.
**8.** J. Franks, *Anosov diffeomorphisms*, Global Analysis, Proc. Symp. Pure Math., Vol. XIV, Amer. Math. Soc. (1970), 61–93.
**9.** M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds* (to appear).
**10.** A. Haefliger, *Varietées Feuilletées*, Ann. Scoula Norm. Sup. Pisa *16* (1962), 367–397.
**11.** S. E. Newhouse, *Anosov diffeomorphisms*, Amer. J. Math. *92* (1970), 761–770.
**12.** S. P. Novikov, *Topology of foliations*, Trudy Moskov. Mat. Obšč. *14* (1965), 248–278.
**13.** A. Preissmann, *Quelques propriétés globales des expaces de Riemann*, Comment. Math. Helv. *15* (1942–43), 175–216.
**14.** G. Reeb, *Sur certaines propriétés topologiques des variétés feuillitées*, Act. Sc. et Ind. (Hermann, Paris, 1952).
**15.** S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. *73* (1967), 747–817.

*Sir George Williams University,*
*Montreal, Quebec*